Gromov-Witten invariants of blow-ups

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In the first part of the paper, we give an explicit algorithm to compute the (genus zero) Gromov-Witten invariants of blow-ups of an arbitrary convex projective variety in some points if one knows the Gromov-Witten invariants of the original variety. In the second part, we specialize to blow-ups of $\mathbb{P}^r$ and show that many invariants of these blow-ups can be interpreted as numbers of rational curves on $\mathbb{P}^r$ having specified global multiplicities or tangent directions in the blown-up points. We give various numerical examples, including a new easy way to determine the famous multiplicity d^{-3} for d-fold coverings of rational curves on the quintic threefold, and, as an outlook, two examples of blow-ups along subvarieties, whose Gromov-Witten invariants lead to classical multisecant formulas.

Over the last few years, Gromov-Witten invariants of smooth projective varieties have become a powerful tool in enumerative geometry. Originally applicable only to convex varieties where the spaces of stable maps have the expected dimension, the theory is now well-developed for all varieties using virtual fundamental classes [LT], [BF], [B].

There are at least two motivations to look at Gromov-Witten invariants of blow-ups. Firstly, a blow-up $\tilde{X}$ of a convex variety X provides an easy example for a non-convex variety, in the sense that one has reasonably good control over the stable maps with $h^1(C, f^*T_{\tilde{X}}) \neq 0$ since they all must be such that they intersect the exceptional divisor. Hence this gives a good class of examples where one can study the effects of virtual fundamental classes on Gromov-Witten theory. Secondly, curves on the blowup $\tilde{X}$ of a variety X are closely related to curves on X. At least for irreducible curves not contained in the exceptional divisor, the strict transform of curves gives a correspondence between curves in $\tilde{X}$ of specified homology class and curves in X intersecting the blown-up variety with a given (global) multiplicity. Hence, being able to calculate Gromov-Witten invariants of blow-ups, one can hope to solve enumerative problems on X involving multiplicity conditions at the blown-up variety.

Apart from the last section of this chapter, we will only be concerned with blow-ups of points, since both the calculation and the question of enumerative significance get very complicated in the case of blow-ups of general subvarieties. Everything will be done over $\mathbb{C}$ and for curves of genus zero.

We first address the question of how one can compute the Gromov-Witten invariants of blow-ups. For any convex variety X, we state and prove an explicit algorithm to reconstruct all invariants of $\tilde{X}$ from those of X in section 2. Directly from the algorithm, many of the invariants of $\tilde{X}$ can be seen to vanish or to coincide with others of X. This is done in section 3. For example, we will show in corollary 3.2 that the equality

$$I^X_\beta(\gamma_1 \otimes \ldots \otimes \gamma_n \otimes pt) = I^{\tilde{X}}_{p^*\beta - E'}(p^*\gamma_1 \otimes \ldots \otimes p^*\gamma_n)$$

holds for $\beta \in A_1(X)$ and $\gamma_i \in A^*(X)$, where $p : \tilde{X} \to X$ is the blow-up and E' the class of a line in the exceptional divisor. As curves in $\tilde{X}$ with homology class $p^*\beta - E'$ correspond to curves in X with homology class β intersecting the blown-up point with multiplicity one, both these invariants are supposed to count curves on X of class β intersecting generic subvarieties representing the γ_i and one additional point in X. If the left invariant in fact counts these curves (which is the case e.g. for $X = \mathbb{P}^r$ by the Bertini lemma), then the right invariant also does, and we call this invariant on $\tilde{X}$ *enumerative* as it has the expected geometric meaning.

In general, if $\tilde{X} = \tilde{X}(s)$ is the blow-up of X at s generic points $P_1, \dots, P_s$, we will call an invariant on $\tilde{X}$ of the form

$$I^{\tilde{X}}_{p^*\beta + e_1 E'_1 + \dots + e_s E'_s}(p^*\gamma_1 \otimes \dots \otimes p^*\gamma_n)$$

with all $e_i \leq 0$ enumerative if it counts the number of curves on X of class β intersecting generic subvarieties representing the γ_i, and in addition passing through each P_i with global multiplicity $-e_i$ (see definition 4.1). One would then expect these curves to have $-e_i$ smooth local branches at every point P_i.

The question whether such a given invariant on $\tilde{X}$ is enumerative or not is in general very difficult. We will discuss this question in the case $\tilde{X} = \tilde{\mathbb{P}}^r(s)$ in sections 4 to 6. The results are as follows:

- If $s = 1$ then all invariants on $\tilde{X}$ are enumerative. This is shown in theorem 5.3.
- If $r = 3$, $s \leq 4$, and the invariant contains only point classes as incidence conditions, then this invariant is enumerative, except for some few cases discussed below. This is shown in theorem 6.4.
- If $r = 3$ and the invariant contains not only point classes, then it is in general not enumerative. This is discussed in section 4.
- If $r \geq 4$ and $s \geq 2$, then the invariants are "almost never" enumerative. This is discussed in section 4.

In addition, Göttsche and Pandharipande [GP] showed independently that almost all invariants are enumerative if $r = 2$. Taking all these results together, the main point left open is the case $r = 3$ and $s \geq 5$.

In section 7 we show that Gromov-Witten invariants of blow-ups can also be used to count numbers of curves in $X = \mathbb{P}^r$ satisfying certain tangency conditions: the number of curves in X of class β intersecting generic representatives of classes $\gamma_i \in A^*(X)$, and passing in addition through a given point $P \in X$ with tangent direction in a given k-codimensional subspace of $T_{X,P}$ is equal to

$$\begin{aligned} I^{\tilde{X}}_{p^*\beta - E'}(p^*\gamma_1 \otimes \dots \otimes p^*\gamma_n \otimes -(-E)^{k+1}) \qquad & \text{if } k < r-1, \\ I^X_\beta(\gamma_1 \otimes \dots \otimes \gamma_n \otimes pt^{\otimes 2}) - 2 I^{\tilde{X}}_{p^*\beta - 2E'}(p^*\gamma_1 \otimes \dots \otimes p^*\gamma_n) \qquad & \text{if } k = r-1, \end{aligned}$$

see theorem 7.1. Various numerical examples of our results can be found in section 8. This also includes a very interesting case of non-enumerative invariants in example 8.5, namely

$$I^{\tilde{\mathbb{P}}^3(2)}_{dp^*H'-dE'_1-dE'_2}(1) = d^{-3}$$

where H' is the class of a line in $\mathbb{P}^3$ and the notation $1 \in A^*(X)^{\otimes 0}$ means that there are no cohomology classes in the invariant. This invariant can be shown to coincide with the famous multiplicity with which multiple coverings get counted in the Gromov-Witten invariants of the quintic threefold. Thus our algorithm to compute Gromov-Witten invariants of blow-ups gives a new easy way to reproduce this result.

We conclude our work with two easy examples of Gromov-Witten invariants of blow-ups of subvarieties in section 9. In the case of the blow-up of a space curve $Y \subset \mathbb{P}^3$, we reproduce the well-known (possibly virtual) number of 3-secants of Y intersecting a fixed line, and the number of 4-secants of Y. In the case of the blow-up of an abelian surface in $\mathbb{P}^4$, we reproduce the well-known result that the generic abelian surface in $\mathbb{P}^4$ has 25 6-secants.

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1 Preliminaries

We start by describing the setup and the notation that will be used throughout the work. For a complex smooth projective variety X of dimension r, we denote by $\boldsymbol{A_i(X)}$ the algebraic part of $H_{2i}(X)$ modulo torsion and by $\boldsymbol{A^i(X)}$ the algebraic part of $H^{2i}(X)$ modulo torsion. These are finitely generated abelian groups. The classes in $A^i(X)$ will be said to have **codimension** i. By abuse of notation, we will often denote a subvariety of X and its fundamental class in $A_*(X)$ or $A^*(X)$ (via Poincaré duality) by the same symbol if no confusion can result. The intersection product of two elements γ, γ' in $A^*(X)$ (or $A_*(X)$ via Poincaré duality) will be denoted $\gamma \cdot \gamma'$. The class of a point will be denoted pt. If $X = \mathbb{P}^r$, the hyperplane class will be called $\boldsymbol{H} \in A^1(X)$, and the class of a line will be called $\boldsymbol{H'} \in A_1(X)$.

For $\beta \in A_1(X)$ an effective homology class and $n \geq 0$, we denote as usual by $\bar{M}_{0,n}(X,\beta)$ the moduli spaces of stable maps of genus zero to X [BM], and by $ev_i : \bar{M}_{0,n}(X,\beta) \to X$ the evaluation maps. We will sometimes associate to a stable map $(C, x_1, \dots, x_n, f) \in$

$\bar{M}_{0,n}(X,\beta)$ a **topology** τ, by which we mean the homeomorphism class of the n-pointed topological space $(C,x_1,\dots,x_n)$ together with the data of the homology classes $f_*[C_i] \in A_1(X)$ on each irreducible component C_i of C. This definition can be made much more precise and formal using the language of graphs [BM], however then the notation is likely to get very messy, so we will not make use of it.

These moduli spaces of stable maps possess an expected dimension

$$\textbf{vdim } \bar{M}_{0,n}(X,\beta) := -K_X \cdot \beta + r + n - 3$$

and a **virtual fundamental class** $[\bar{M}_{0,n}(X,\beta)]^{virt} \in A_{\text{vdim } \bar{M}_{0,n}(X,\beta)}(\bar{M}_{0,n}(X,\beta))$ [LT], [BF], [B]. This class is constructed using the obstructions $H^1(C, f^*T_X)$ for stable maps $(C,x_1,\dots,x_n,f) \in \bar{M}_{0,n}(X,\beta)$. In particular, if these obstructions vanish for all stable maps in the moduli space, then the virtual fundamental class coincides with the usual one. There exists a local version of this property too, which follows immediately from the construction:

Lemma 1.1 *Let* $(C,x_1,\dots,x_n,f) \in \bar{M}_{0,n}(X,\beta)$ *be a stable map with* $h^1(C, f^*T_X) = 0$. *Then* $(C,x_1,\dots,x_n,f)$ *lies in a unique irreducible component* Z *of* $\bar{M}_{0,n}(X,\beta)$ *of dimension* vdim $\bar{M}_{0,n}(X,\beta)$*, and if* R *denotes the union of all the other irreducible components, then*

$$[\bar{M}_{0,n}(X,\beta)]^{virt} = [Z] + \textit{some cycle supported on } R. \qquad \square$$

We now come to Gromov-Witten invariants. If $\gamma_1,\dots,\gamma_n \in A^*(X)$ are classes on X, the associated Gromov-Witten invariant is

$$\boldsymbol{I^X_\beta(\gamma_1 \otimes \dots \otimes \gamma_n)} := (ev_1^*\gamma_1 \cdot \ldots \cdot ev_n^*\gamma_n) \cdot [\bar{M}_{0,n}(X,\beta)]^{virt} \in \mathbb{Q}$$

if $\sum_{i=1}^n \text{codim } \gamma_i = \text{vdim } \bar{M}_{0,n}(X,\beta)$, and zero otherwise.

Concerning the notation, we will often drop the superscript X. To shorten notation, we will often write $\mathcal{T} = \gamma_1 \otimes \dots \otimes \gamma_n$ and call $\mathcal{T} \in (A^*(X))^{\otimes n}$ a **collection of classes**. Correspondingly, we write $ev^*\mathcal{T}$ for $ev_1^*\gamma_1 \cdot \ldots \cdot ev_n^*\gamma_n$. If $X = \mathbb{P}^r$, the invariant $I_\beta(\mathcal{T})$ is also denoted by $I_d(\mathcal{T})$, where $\beta = d\,H'$.

We now review briefly the relations among these invariants (see e.g. [FP]), mainly to fix notation for the splitting axiom.

Proposition 1.2 *Properties of Gromov-Witten invariants*

(i) ***(Mapping to a point)*** *If* $\beta = 0$*, then the invariant is equal to the triple intersection product:*

$$I_0(\gamma_1 \otimes \dots \otimes \gamma_n) = \begin{cases} \gamma_1 \cdot \gamma_2 \cdot \gamma_3 & \textit{if } n = 3 \textit{ and } \sum_i \text{codim } \gamma_i = r, \\ 0 & \textit{otherwise.} \end{cases}$$

(ii) ***(Fundamental class)*** *If $\beta \neq 0$ and the invariant contains the fundamental class of X, then the invariant is zero:*

$$I_\beta(X \otimes \mathcal{T}) = 0 \quad \textit{for all } \mathcal{T} \textit{ and all } \beta \neq 0.$$

(iii) ***(Divisor axiom)*** *If $\beta \neq 0$ and $\gamma \in A^1(X)$ is a divisor, then*

$$I_\beta(\gamma \otimes \mathcal{T}) = (\gamma \cdot \beta)\, I_\beta(\mathcal{T}) \quad \textit{for all } \mathcal{T}.$$

(iv) ***(Splitting axiom)*** *Choose a homogeneous basis $\mathcal{B} = \{T_0, \dots, T_q\}$ of $A^*(X)$, define $g = (g_{ij})$ to be the intersection matrix*

$$\boldsymbol{g_{ij}} = \begin{cases} T_i \cdot T_j & \textit{if } \mathrm{codim}\ T_i + \mathrm{codim}\ T_j = r, \\ 0 & \textit{otherwise,} \end{cases}$$

and let $g^{-1} = (\boldsymbol{g^{ij}})$ be the inverse matrix. Choose $\beta \in A_1(X)$, four classes $\mu_1, \dots, \mu_4 \in A^*(X)$ and a collection $\mathcal{T} = \gamma_1 \otimes \dots \otimes \gamma_n$ of classes such that

$$\sum_{i=1}^{n} \mathrm{codim}\ \gamma_i + \sum_{i=1}^{4} \mathrm{codim}\ \mu_i = -K_X \cdot \beta + r + n.$$

Then we have the equation

$$\begin{aligned} 0 = {} & I_\beta(\mathcal{T} \otimes \mu_1 \otimes \mu_2 \otimes \mu_3 \cdot \mu_4) + I_\beta(\mathcal{T} \otimes \mu_3 \otimes \mu_4 \otimes \mu_1 \cdot \mu_2) \\ & - I_\beta(\mathcal{T} \otimes \mu_1 \otimes \mu_3 \otimes \mu_2 \cdot \mu_4) - I_\beta(\mathcal{T} \otimes \mu_2 \otimes \mu_4 \otimes \mu_1 \cdot \mu_3) \\ & + \sum_{\beta_1, \beta_2 \neq 0} \sum_{\mathcal{T}_1, \mathcal{T}_2} \sum_{i,j} g^{ij} \Big(I_{\beta_1}(\mathcal{T}_1 \otimes \mu_1 \otimes \mu_2 \otimes T_i)\, I_{\beta_2}(\mathcal{T}_2 \otimes \mu_3 \otimes \mu_4 \otimes T_j) \\ & \qquad - I_{\beta_1}(\mathcal{T}_1 \otimes \mu_1 \otimes \mu_3 \otimes T_i)\, I_{\beta_2}(\mathcal{T}_2 \otimes \mu_2 \otimes \mu_4 \otimes T_j) \Big) \end{aligned}$$

where the sum is taken over

- *all effective classes $\beta_1, \beta_2 \in A_1(X)$ with $\beta_1 + \beta_2 = \beta$,*
- *all $\mathcal{T}_1 = \gamma_{i_1} \otimes \dots \otimes \gamma_{i_{n_1}}$ and $\mathcal{T}_2 = \gamma_{j_1} \otimes \dots \otimes \gamma_{j_{n_2}}$ such that $i_1 < \dots < i_{n_1}$, $j_1 < \dots < j_{n_2}$, and $\{i_1, \dots, i_{n_1}\} \dot\cup \{j_1, \dots, j_{n_2}\} = \{1, \dots, n\}$ (i.e. "the classes of $\mathcal{T}$ get distributed in all possible ways onto the two factors"),*
- *all $0 \le i, j \le q$.*

In the sequel we will call this equation $\boldsymbol{\mathcal{E}_\beta(\mathcal{T}\, ;\, \mu_1, \mu_2 \mid \mu_3, \mu_4)}$.

Now let $p : \tilde{X} = \tilde{X}(s) \to X$ be the blow-up of X at s generic points $P_1, \dots, P_s \in X$, and let E_i be the exceptional divisors. Fix a homogeneous basis $\boldsymbol{\mathcal{B}} = \{T_0, \dots, T_q\}$ of $A^*(X)$

of increasing codimension such that $T_0 = X$ is the fundamental class and $T_q = pt$. If we define $T_{q+1}, \dots, T_{\tilde{q}}$ with $\tilde{q} = q + s(r-1)$ to be the classes

$$E_i^k \in A^*(\tilde{X}) \quad \text{where } 1 \le i \le s, 1 \le k \le r-1$$

(in any order), then

$$\tilde{\boldsymbol{\mathcal{B}}} = \{p^*T_1, \dots, p^*T_q, T_{q+1}, \dots, T_{\tilde{q}}\}$$

is a homogeneous basis of $A^*(\tilde{X})$. We call the classes $p^*T_1, \dots, p^*T_q$ **non-exceptional** and $T_{q+1}, \dots, T_{\tilde{q}}$ **exceptional**. A collection of classes $\mathcal{T}$ will be called non-exceptional if all its classes are non-exceptional. Since the Gromov-Witten invariants are multilinear in the cohomology classes, we will for computational purposes only consider invariants of the form $I_\beta(\mathcal{T})$ where $\mathcal{T}$ is of the form $\mathcal{T} = T_{j_1} \otimes \dots \otimes T_{j_n}$.

In terms of the basis $\tilde{\mathcal{B}}$, the intersection theory on $\tilde{X}$ is given by

$$\begin{aligned} p^*T_j \cdot p^*T_{j'} &= p^*(T_j \cdot T_{j'}) \\ p^*T_j \cdot E_i^k &= 0 \\ E_i^k \cdot E_{i'}^{k'} &= \delta_{i,i'} E_i^{k+k'} \\ E_i^r &= (-1)^{r-1} pt \end{aligned}$$

for $1 \le j, j' \le q$; $1 \le i, i' \le s$; $1 \le k, k' \le r-1$. If there is no danger of confusion, we will write the classes $p^*T_1, \dots, p^*T_q$ simply as $T_1, \dots, T_q$.

The homology group $A_1(\tilde{X})$ has a canonical decomposition

$$A_1(\tilde{X}) = A_1(X) \oplus \mathbb{Z} E_1' \oplus \dots \oplus \mathbb{Z} E_s'$$

where $\boldsymbol{E_i'}$ denotes the class of a line in the exceptional divisor $E_i \cong \mathbb{P}^{r-1}$, such that $E_i' = -(-E_i)^{r-1}$ via Poincaré duality. We denote the $s+1$ projections onto the summands of the above decomposition by $\boldsymbol{d : A_1(\tilde{X}) \to A_1(X)}$ and $\boldsymbol{e_1, \dots, e_s : A_1(\tilde{X}) \to \mathbb{Z}}$, and we set $\boldsymbol{e = e_1 + \dots + e_s}$. If $X = \mathbb{P}^r$, we will identify $A_1(X)$ with $\mathbb{Z}$ in the obvious way and consider d as a function $d : A_1(\tilde{X}) \to \mathbb{Z}$.

For a homology class $\beta \in A_1(\tilde{X})$, we call $d(\beta)$ the **non-exceptional part** and $e(\beta)$ the **exceptional part**. The class β is called a **non-exceptional class** if $e_i(\beta) = 0$ for all i and a **purely exceptional class** if $d(\beta) = 0$ and $e_i(\beta) \neq 0$ for at least one i. For a homology class $\beta \in A_1(X)$, we will denote the corresponding non-exceptional class in $A_1(\tilde{X})$ also by β.

The canonical divisor on $\tilde{X}$ is given by $K_{\tilde{X}} = p^*K_X + (r-1)E$ (see [GH] section 1.4), hence the virtual dimension of the moduli space $\bar{M}_{0,n}(X, \beta)$ is

$$\begin{aligned} \operatorname{vdim} \bar{M}_{0,n}(\tilde{X}, \beta) &= -K_{\tilde{X}} \cdot \beta + n + r - 3 \\ &= \operatorname{vdim} \bar{M}_{0,n}(X, d(\beta)) + (r-1)\, e(\beta). \end{aligned}$$

2 Calculation of the invariants

The aim of this section is to prove the following.

Theorem 2.1 *Let X be a convex variety and $\tilde{X}$ the blow-up of X at some points. Then there exists an explicit algorithm to compute the Gromov-Witten invariants of $\tilde{X}$ from those of X.*

The computation is done in three steps. Firstly, we show in lemma 2.2 that all invariants $I_\beta^{\tilde{X}}(\mathcal{T})$ with β and $\mathcal{T}$ non-exceptional are actually equal to the corresponding invariants on X. Secondly, in lemma 2.4 we compute the invariants $I_\beta^{\tilde{X}}(\mathcal{T})$ with β purely exceptional using a technique similar to the First Reconstruction Theorem of Kontsevich and Manin. Thirdly, we state and prove an algorithm that allows one to compute all Gromov-Witten invariants on $\tilde{X}$ recursively from those obtained in the first two steps.

Lemma 2.2 *Let $\mathcal{T} = T_{j_1} \otimes \ldots \otimes T_{j_n}$ be a collection of non-exceptional classes and let $\beta \in A_1(X)$ be a non-exceptional homology class. Then*

$$I_\beta^{\tilde{X}}(\mathcal{T}) = I_\beta^X(\mathcal{T}).$$

In this case we will say that the invariant $I_\beta^{\tilde{X}}(\mathcal{T})$ is ***induced by*** *X.*

Proof Consider the commutative diagram

$$\begin{array}{ccc} \bar{M}_{0,n}(\tilde{X},\beta) & \xrightarrow{\phi} & \bar{M}_{0,n}(X,\beta) \\ {\scriptstyle ev_i}\downarrow & & \downarrow{\scriptstyle ev_i} \\ \tilde{X} & \xrightarrow{p} & X \end{array}$$

for $1 \le i \le n$. First we show that $\phi_*[\bar{M}_{0,n}(\tilde{X},\beta)]^{virt} = [\bar{M}_{0,n}(X,\beta)]^{virt}$: since X is convex, $\bar{M}_{0,n}(X,\beta)$ is a smooth stack of the expected dimension $d = \text{vdim } \bar{M}_{0,n}(X,\beta)$. Let $Z_1,\ldots,Z_k$ be the connected components of $\bar{M}_{0,n}(X,\beta)$, so that $A_d(\bar{M}_{0,n}(X,\beta)) = \mathbb{Q}[Z_1] \oplus \cdots \oplus \mathbb{Q}[Z_k]$. Since $\text{vdim } \bar{M}_{0,n}(\tilde{X},\beta) = d$, we must therefore have

$$\phi_*[\bar{M}_{0,n}(\tilde{X},\beta)]^{virt} = \alpha_1[Z_1] + \cdots + \alpha_k[Z_k]$$

for some $\alpha_i \in \mathbb{Q}$.

To see that all $\alpha_i = 1$, pick a stable map $\mathcal{C}_i \in Z_i$ whose image does not intersect the blown-up points. Then $\phi^{-1}(\mathcal{C}_i)$ consists of exactly one stable map $\tilde{\mathcal{C}}_i$, and the map $\phi : \bar{M}_{0,n}(\tilde{X},\beta) \to \bar{M}_{0,n}(X,\beta)$ is a local isomorphism around the point $\tilde{\mathcal{C}}_i$. Hence $\tilde{\mathcal{C}}_i$ is a smooth point of an irreducible component $\tilde{Z}_i$ of $\bar{M}_{0,n}(\tilde{X},\beta)$. Denote by $\tilde{R}_i$ the union of the other irreducible components of $\bar{M}_{0,n}(\tilde{X},\beta)$. Then, by lemma 1.1,

$$[\bar{M}_{0,n}(\tilde{X},\beta)]^{virt} = [\tilde{Z}_i] + \text{some cycle supported on } \tilde{R}_i.$$

Now, since $\phi : \tilde{Z}_i \to Z_i$ is a local isomorphism around $\tilde{\mathcal{C}}_i$, we have $\phi_*[\tilde{Z}_i] = [Z_i]$. However, the pushforward of a d-cycle supported on $\tilde{R}_i$ will give no contribution to α_i since $\mathcal{C}_i$ and therefore Z_i is not contained in the image of $\tilde{R}_i$ under ϕ. We conclude that all $\alpha_i = 1$ and that therefore

$$\begin{aligned}\phi_*[\bar{M}_{0,n}(\tilde{X},\beta)]^{virt} &= [Z_1] + \cdots + [Z_k] \\ &= [\bar{M}_{0,n}(X,\beta)] \\ &= [\bar{M}_{0,n}(X,\beta)]^{virt}.\end{aligned}$$

To complete the proof, note that by the projection formula

$$\begin{aligned}I_\beta^{\tilde{X}}(\mathcal{T}) &= (\prod_i ev_i^* p^* T_{j_i}) \cdot [\bar{M}_{0,n}(\tilde{X},\beta)]^{virt} \\ &= (\prod_i \phi^* ev_i^* T_{j_i}) \cdot [\bar{M}_{0,n}(\tilde{X},\beta)]^{virt} \\ &= (\prod_i ev_i^* T_{j_i}) \cdot \phi_*[\bar{M}_{0,n}(\tilde{X},\beta)]^{virt} \\ &= (\prod_i ev_i^* T_{j_i}) \cdot [\bar{M}_{0,n}(X,\beta)]^{virt} \\ &= I_\beta^X(\mathcal{T}).\end{aligned}$$

□

Remark 2.3 This lemma is actually the only point in the proof of theorem 2.1 where the convexity of X is needed. Hence, one can formulate the theorem also in the following, more general way:

Let X be a smooth projective variety and $\tilde{X}$ the blow-up of X at some points. There exists an explicit algorithm to compute all Gromov-Witten invariants $I_\beta^{\tilde{X}}(\mathcal{T})$ of $\tilde{X}$ from those where β and $\mathcal{T}$ are non-exceptional.

The proof would be literally the same, just skipping lemma 2.2. In fact, it may even be that lemma 2.2 also holds for non-convex X, but I do not know how to prove it in this case.

Lemma 2.4 *Let $\mathcal{T} = T_{j_1} \otimes \ldots \otimes T_{j_n}$ with $T_{j_i} \in \tilde{\mathcal{B}}$ be a collection of classes and let $\beta \in A_1(\tilde{X})$ be a purely exceptional homology class. Then*

(i) If β is not of the form $d \cdot E_i'$ for $d > 0$ and some $1 \le i \le s$, then $I_\beta^{\tilde{X}}(\mathcal{T}) = 0$. Moreover, the invariant can only be non-zero if all classes in $\mathcal{T}$ are exceptional with support in the exceptional divisor E_i.

(ii) $I_{E_i'}^{\tilde{X}}(E_i^{r-1} \otimes E_i^{r-1}) = 1$ for all $1 \le i \le s$.

(iii) All other invariants with purely exceptional homology class can be computed recursively.

Proof

(i) This follows easily from the fact that a Gromov-Witten invariant $I_\beta^{\tilde{X}}(\mathcal{T})$ is always zero if there is no stable map in $\bar{M}_{0,n}(\tilde{X},\beta)$ satisfying the conditions given by $\mathcal{T}$.

(ii) Note that $\bar{M}_{0,2}(\tilde{X},E_i') \cong \bar{M}_{0,2}(\mathbb{P}^{r-1},1)$ and that this space is of the expected dimension (which is $2r-2$), hence we do not need virtual fundamental classes to compute this invariant. Choose two curves $Y_1,Y_2 \subset X$ intersecting transversally at the blown-up point P_i, and let $\gamma_1,\gamma_2 \in A^{r-1}(X)$ be their cohomology classes. Let $\tilde{Y}_k$ be the strict transform of Y_k for $k=1,2$. Then $\tilde{Y}_1$ and $\tilde{Y}_2$ intersect E_i transversally at different points, so the invariant

$$I_{E_i'}^{\tilde{X}}([\tilde{Y}_1]\otimes[\tilde{Y}_2]) = I_{E_i'}^{\tilde{X}}((\gamma_1+(-E_i)^{r-1})\otimes(\gamma_2+(-E_i)^{r-1}))$$

simply counts the number of lines in E_i through two points in E_i, which is 1. Therefore, by the multilinearity of the Gromov-Witten invariants and by (i) we conclude that

$$\begin{aligned} I_{E_i'}^{\tilde{X}}(E_i^{r-1}\otimes E_i^{r-1}) &= I_{E_i'}^{\tilde{X}}((\gamma_1+(-E_i)^{r-1})\otimes(\gamma_2+(-E_i)^{r-1})) \\ &= 1. \end{aligned}$$

(iii) (This is essentially the First Reconstruction Theorem of Kontsevich and Manin, see [KM].) As in (ii) we assume that $\tilde{X}=\tilde{\mathbb{P}}^r(1)$ and that we want to compute the invariant $I_{dE'}(E^{j_1}\otimes\ldots\otimes E^{j_n})$ for some d and some j_i. Consider the equation $\mathcal{E}_{dE'}(\mathcal{T}\,;\,E^a,E^b\mid E^c,E)$ for some $\mathcal{T}$ consisting of exceptional classes and for some $2\le a\le r-1$, $2\le b\le r-1$, $1\le c\le r-1$:

$$\begin{aligned} 0 = {} & I_{dE'}(\mathcal{T}\otimes E^a\otimes E^b\otimes E^c\cdot E) && (1)\\ & + I_{dE'}(\mathcal{T}\otimes E^c\otimes E\otimes E^a\cdot E^b) && (2)\\ & - I_{dE'}(\mathcal{T}\otimes E^a\otimes E^c\otimes E^b\cdot E) && (3)\\ & - I_{dE'}(\mathcal{T}\otimes E^b\otimes E\otimes E^a\cdot E^c) && (4)\\ & + (\text{terms with homology classes } d'E' \text{ with } d'<d). && (5) \end{aligned}$$

We want to compute the invariants by induction on the degree d and on the number of non-divisorial classes in the invariant. Obviously, the terms in (5) have lower degree and those in (2) and (4) have same degree but a smaller number of non-divisorial classes than (1). The degree of (3) is equal to that of (1), and its number of non-divisorial classes is not bigger than that of (1). In any case, we can write

$$\begin{aligned} I_{dE'}(\mathcal{T}\otimes E^a\otimes E^b\otimes E^{c+1}) = {} & I_{dE'}(\mathcal{T}\otimes E^a\otimes E^{b+1}\otimes E^c) \\ & + (\text{recursively known terms}). \end{aligned}$$

Thus if a Gromov-Witten invariant contains at least three non-divisorial classes, we can use this equation repeatedly to express $I_{dE'}(\mathcal{T} \otimes E^a \otimes E^b \otimes E^{c+1})$ in terms of $I_{dE'}(\mathcal{T} \otimes E^a \otimes E^{b+c} \otimes E)$ (and recursively known terms), which again has fewer non-divisorial classes. This makes the induction work and reduces everything to invariants with at most two non-divisorial classes. However, since vdim $\bar{M}_{0,n}(\tilde{X}, dE') = (r-1)d + r + n - 3$ and each class has codimension at most r, it is easy to check that the only such invariant is the one calculated in (ii).

□

We now come to the main part of the proof of theorem 2.1, namely the algorithm to compute all invariants on $\tilde{X}$ from those calculated so far. We will first state the algorithm in such a way that it can be programmed easily on a computer, and afterwards give the proof that it really does the job. Many numbers computed using this algorithm can be found in section 8.

From now on, Gromov-Witten invariants will always be on $\tilde{X}$ unless otherwise stated, so we will often write them as $I_\beta(\mathcal{T})$ instead of $I_\beta^{\tilde{X}}(\mathcal{T})$.

Algorithm 2.5 *Suppose one wants to calculate an invariant $I_\beta^{\tilde{X}}(\mathcal{T})$. Assume that the invariant is not induced by X and that β is not purely exceptional. We may assume without loss of generality that the sum of the codimensions of the non-exceptional classes in $\mathcal{T}$ is at least $r+1$ (hence in particular that there are at least two non-exceptional classes) — otherwise choose a divisor $\rho \in \mathcal{B}$ with $\rho \cdot \beta \neq 0$ (such a ρ exists because β is not purely exceptional) and use $\mathcal{T} \otimes \rho^{\otimes(r+1)}$ instead of $\mathcal{T}$, which gives essentially the same invariant by the divisor axiom.*

We can further assume without loss of generality that $\mathcal{T}$ contains no exceptional divisor class and that the classes $T_{j_1}, \dots, T_{j_n}$ in $\mathcal{T}$ are ordered such that the non-exceptional classes are exactly $T_{j_1}, \dots, T_{j_m}$, where codim $T_{j_1} \geq \dots \geq$ codim T_{j_m}. *In particular, T_{j_1} and T_{j_2} are two non-exceptional classes with maximal codimension in $\mathcal{T}$.*

We now distinguish the following three cases.

(A) $n > m$, i.e. $T_{j_n} = E_i^k$ (for some $1 \leq i \leq s$, $2 \leq k \leq r-1$) is an exceptional class. Then use the equation

$$\mathcal{E}_\beta(\mathcal{T}' \,;\, T_{j_1}, T_{j_2} \mid E_i, E_i^{k-1}) \qquad \text{where } \mathcal{T}' = T_{j_3} \otimes \ldots \otimes T_{j_{n-1}}.$$

(B) $n = m$ (i.e. there is no exceptional class in $\mathcal{T}$), $T_{j_1} = pt$ and codim $T_{j_2} \geq 2$. *Then choose $\mu, \nu \in \mathcal{B}$ such that* codim $\mu = 1$, codim $\nu = r-1$, *and $\mu \cdot \nu \neq 0$. Since the invariant to be computed is not induced by X, there is an $i \in \{1, \dots, s\}$ such that $E_i \cdot \beta \neq 0$. Use the equation*

$$\mathcal{E}_\beta(\mathcal{T}' \,;\, \mu, \nu \mid E_i, T_{j_2}) \qquad \text{where } \mathcal{T}' = T_{j_3} \otimes \ldots \otimes T_{j_n}.$$

(C) *$n = m$, and it is not true that $T_{j_1} = pt$ and* codim $T_{j_2} \geq 2$. *Then again there is an $i \in \{1, \ldots, s\}$ such that $E_i \cdot \beta \neq 0$. Use the equation*

$$\mathcal{E}_{\beta+E_i'}(\mathcal{T}' \,;\, T_{j_1}, T_{j_2} \mid E_i, E_i^{r-1}) \qquad \textit{where } \mathcal{T}' = T_{j_3} \otimes \ldots \otimes T_{j_n}.$$

Here, "use equation $\mathcal{E}$" means: the Gromov-Witten invariant $I_\beta(\mathcal{T})$ to be calculated appears in $\mathcal{E}$ linearly with non-zero coefficient. Solve this equation for $I_\beta(\mathcal{T})$ and compute recursively with the same rules all other invariants in this equation that are not already known.

Proof (of theorem 2.1) Suppose we want to compute an invariant $I_\beta(\mathcal{T})$. If the invariant is induced by X, it is assumed to be known by lemma 2.2. If β is purely exceptional, the invariant is known by lemma 2.4. In all other cases, use the algorithm 2.5 to compute the invariant recursively. We have to show that the equations to be used in fact do contain the desired invariants linearly with non-zero coefficient, and that the recursion stops after a finite number of calculations.

To do this, we will define a partial ordering on pairs $(\beta, \mathcal{T})$ where $\beta \in A_1(\tilde{X})$ is an effective homology class and $\mathcal{T}$ is a collection of cohomology classes. Choose an ordering of the effective homology classes in $A_1(X)$ such that, for $\alpha_1, \alpha_2 \neq 0$ being two such classes, we have $\alpha_1 < \alpha_1 + \alpha_2$ (this is possible since the effective classes in $A_1(X)$ form a semigroup with indecomposable zero). For a collection of classes $\mathcal{T} = T_{j_1} \otimes \ldots \otimes T_{j_n}$, we assume as in the description of the algorithm that the classes are ordered such that the non-exceptional classes are exactly $T_{j_1}, \ldots, T_{j_m}$, where codim $T_{j_1} \geq \cdots \geq$ codim T_{j_m}, and that codim $T_{j_1} + \cdots +$ codim $T_{j_m} \geq r+1$ (by possibly adding non-exceptional divisor classes). Then we define

$$\boldsymbol{v(\mathcal{T})} = \min \{k \,;\, \text{codim } T_{j_1} + \cdots + \text{codim } T_{j_k} \geq r+1\},$$

i.e. "the minimal number of non-exceptional classes in $\mathcal{T}$ whose codimensions sum up to at least $r+1$". With this, we now define the partial ordering on pairs $(\beta, \mathcal{T})$ as follows: say that $(\beta_1, \mathcal{T}_1) < (\beta_2, \mathcal{T}_2)$ if and only if one of the following holds:

- $d(\beta_1) < d(\beta_2)$,
- $d(\beta_1) = d(\beta_2)$ and $v(\mathcal{T}_1) < v(\mathcal{T}_2)$,
- $d(\beta_1) = d(\beta_2)$, $v(\mathcal{T}_1) = v(\mathcal{T}_2)$, and $e(\beta_1) < e(\beta_2)$.

Obviously, this defines a partial ordering satisfying the "descending chain condition", i.e. there do not exist infinite chains $(\beta_1, \mathcal{T}_1) > (\beta_2, \mathcal{T}_2) > (\beta_3, \mathcal{T}_3) > \ldots$. This means that, to prove that the recursion stops after finitely many calculations, it suffices to show that the equations in the algorithm compute the desired invariant $I_\beta(\mathcal{T})$ entirely in terms of invariants that are either known by the lemmas 2.2 and 2.4 or smaller with respect to the above partial ordering. We will do this now for the three cases (A), (B), and (C).

(A) The equation reads

$$
\begin{aligned}
0 &= I_\beta(\mathcal{T}' \otimes T_{j_1} \otimes T_{j_2} \otimes E_i \cdot E_i^{k-1}) && (1)\\
&+ I_\beta(\mathcal{T}' \otimes E_i \otimes E_i^{k-1} \otimes T_{j_1} \cdot T_{j_2}) && (2)\\
&+ (\text{no further } I_\beta(\,\cdot\,)\, I_0(\,\cdot\,)\text{-terms since } E_i \cdot T_{j_1} = E_i^{k-1} \cdot T_{j_2} = 0)\\
&+ (\text{some } I_{\beta - dE_i'}(\,\cdot\,)\, I_{dE_i'}(\,\cdot\,)\text{-terms}) && (3)\\
&+ (\text{some } I_{\beta_1}(\,\cdot\,)\, I_{\beta_2}(\,\cdot\,)\text{-terms with } d(\beta_1), d(\beta_2) \neq 0). && (4)
\end{aligned}
$$

The term (1) is the desired invariant. If the term in (2) is non-zero, it has the same $d(\beta)$ and smaller $v(\mathcal{T})$, since the two non-exceptional classes T_{j_1}, T_{j_2} of maximal codimensions codim T_{j_1}, codim T_{j_2} are replaced by one class of codimension codim T_{j_1} + codim T_{j_2}. Hence, the term (2) is smaller with respect to our partial ordering. The terms in (3) have the same d, the same or smaller v (note that all non-exceptional classes from the original invariant must be in the left invariant $I_{\beta - dE_i'}(\,\cdot\,)$), and smaller e. Finally, the terms in (4) have smaller d. Hence, all terms in (2), (3) and (4) are smaller with respect to our partial ordering.

(B) The equation reads

$$
\begin{aligned}
0 &= I_\beta(\mathcal{T}' \otimes E_i \otimes T_{j_2} \otimes \mu \cdot \nu) && (1)\\
&+ (\text{no further } I_\beta(\,\cdot\,)\, I_0(\,\cdot\,)\text{-terms since } E_i \cdot T_{j_2} = E_i \cdot \mu = T_{j_2} \cdot \nu = 0)\\
&+ (\text{no } I_{\beta - dE_i'}(\,\cdot\,)\, I_{dE_i'}(\,\cdot\,)\text{-terms since } I_{dE_i'}(\,\cdot\,) \text{ would have to contain at least}\\
&\qquad \text{one of the non-exceptional classes } T_{j_2}, \mu, \nu)\\
&+ (\text{some } I_{\beta_1}(\,\cdot\,)\, I_{\beta_2}(\,\cdot\,)\text{-terms with } d(\beta_1), d(\beta_2) \neq 0). && (2)
\end{aligned}
$$

Here, obviously, (1) is the desired invariant and the terms in (2) have smaller d and are therefore smaller with respect to the partial ordering.

(C) The equation reads

$$
\begin{aligned}
0 &= I_{\beta + E_i'}(\mathcal{T}' \otimes T_{j_1} \otimes T_{j_2} \otimes \underbrace{E_i \cdot E_i^{r-1}}_{(-1)^{r-1} pt}) && (1)\\
&+ I_{\beta + E_i'}(\mathcal{T}' \otimes E_i \otimes E_i^{r-1} \otimes T_{j_1} \cdot T_{j_2}) && (2)\\
&+ (\text{no further } I_\beta(\,\cdot\,)\, I_0(\,\cdot\,)\text{-terms})\\
&+ I_\beta(\mathcal{T}' \otimes T_{j_1} \otimes T_{j_2} \otimes E_i)\, \underbrace{I_{E_i'}(E_i \otimes E_i^{r-1} \otimes E_i^{r-1})}_{=-1}\, (-1)^{r-1} && (3)\\
&+ (\text{no further } I_{\beta - dE_i'}(\,\cdot\,)\, I_{dE_i'}(\,\cdot\,)\text{-terms since there are not enough exceptional}\\
&\qquad \text{classes to put into } I_{dE_i'}(\,\cdot\,))\\
&+ (\text{some } I_{\beta_1}(\,\cdot\,)\, I_{\beta_2}(\,\cdot\,)\text{-terms with } d(\beta_1), d(\beta_2) \neq 0). && (4)
\end{aligned}
$$

Here, (3) is the desired invariant. (4) has smaller d, and (2) has the same d and smaller v, as in case (A)-(2). The term (1) has the same d, but is not necessarily smaller with respect to the partial ordering. We distinguish two cases:

(i) If $\mathcal{T}' \otimes T_{j_1} \otimes T_{j_2}$ contains a non-divisorial (non-exceptional) class, then the invariant (1) will be computed in the next step using rule (B), which expresses it entirely in terms of invariants with smaller d.

(ii) If $\mathcal{T}' \otimes T_{j_1} \otimes T_{j_2}$ contains only divisor classes, the invariant (1) will be computed in the next step using (C). This time, (2) vanishes (for $T_{j_1} \cdot T_{j_2} = 0$ since $T_{j_1} = pt$), (4) has smaller d, and (1) will be computed by (B) as in (i) in terms of invariants with smaller d.

Hence, combining (C) with possibly one other application of (B) and/or (C), the desired invariant will again be computed in terms of invariants that are smaller with respect to the partial ordering.

This finishes the proof. □

Corollary 2.6 *There exists an explicit algorithm to compute all Gromov-Witten invariants on $\tilde{\mathbb{P}}^r(s)$ for all $r \geq 2$, $s \geq 1$.*

Proof Compute the invariants of $\mathbb{P}^r$ using the First Reconstruction Theorem [KM], and then use theorem 2.1. □

3 A vanishing theorem

We will now prove a vanishing theorem saying that a Gromov-Witten invariant $I_\beta(\mathcal{T})$ with $d(\beta) \neq 0$ and $e_i(\beta) \geq 0$ for some i vanishes under favourable conditions, mainly if $e_i(\beta) > 0$ and if there are "not too many" exceptional classes in $\mathcal{T}$. The proof of the proposition is quite involved, but as a reward it is also very sharp in the sense that numerical calculations on $\tilde{\mathbb{P}}^r(1)$ have shown that an invariant (with non-vanishing $d(\beta)$ and non-negative $e(\beta)$) is "unlikely to vanish" if the conditions of the proposition are not satisfied. We will then apply the proposition to prove corollary 3.2, which is a first hint that Gromov-Witten invariants on blow-ups will lead to enumeratively meaningful numbers.

To state the proposition, we need an auxiliary definition. For $T \in \tilde{\mathcal{B}}$ and $1 \leq i \leq s$ we define

$$\boldsymbol{w_i(T)} = \begin{cases} m-1 & \text{if } T = E_i^m \text{ for some } m, \\ 0 & \text{otherwise.} \end{cases}$$

If $\mathcal{T} = T_{j_1} \otimes \ldots \otimes T_{j_n}$ is a collection of classes, we set $w_i(\mathcal{T}) = w_i(T_{j_1}) + \cdots + w_i(T_{j_n})$.

Proposition 3.1 *Let* β *and* $\mathcal{T}$ *be such that for some* $1 \leq i_0 \leq s$ *the following three conditions hold:*

(i) $d(\beta) \neq 0$*,*

(ii) $w_{i_0}(\mathcal{T}) > 0$ *or* $e_{i_0}(\beta) > 0$*,*

(iii) $w_{i_0}(\mathcal{T}) < (e_{i_0}(\beta)+1)(r-1)$*.*

Then $I_\beta(\mathcal{T}) = 0$*.*

Proof The proof will be given inductively following the lines of the algorithm 2.5. For invariants induced by X or invariants with purely exceptional homology class, the proposition does not say anything, so all we have to do is to go through the three equations (A) to (C) and show that the statement of the proposition is correct for the invariant to be determined if it is correct for all the others.

For the proof of the proposition, we will refer to the classes T_i and T_j in the splitting axiom (see proposition 1.2 (iv))

$$0 = \sum g^{ij} \Big(I(\ldots \otimes T_i)\, I(\ldots \otimes T_j) \Big)$$

as the **additional classes** of a certain summand in the equation.

Assume that we are calculating an invariant $I_\beta(\mathcal{T})$ and that a term $I_{\beta_1}(\mathcal{T}_1)\, I_{\beta_2}(\mathcal{T}_2)$ occurs in the corresponding equation (A), (B), or (C) such that $(\beta, \mathcal{T})$ satisfies the conditions of the proposition, but neither $(\beta_1, \mathcal{T}_1)$ nor $(\beta_2, \mathcal{T}_2)$ does. We will show that this assumption leads to a contradiction.

We first distinguish the two cases $w_{i_0}(\mathcal{T}) > 0$ and $e_{i_0}(\beta) > 0$ according to $(\beta, \mathcal{T})$ satisfying (ii).

- $\underline{w_{i_0}(\mathcal{T}) > 0}$. This means that we have an exceptional non-divisorial class in the invariant and hence that we are in case (A) of the algorithm. Moreover, we can assume that we use case (A) of the algorithm with $i = i_0$. Since the term in (A)-(2) in the proof of theorem 2.1 satisfies the conditions of the proposition if the desired invariant (A)-(1) does, we only need to consider the terms (A)-(3) and (A)-(4).

  From (A)-(1) we know that

  $$w_i(\mathcal{T}) = w_i(\mathcal{T}') + w_i(E_i^k) = w_i(\mathcal{T}') + k - 1,$$

  whereas in all other terms $I_{\beta_1}(\mathcal{T}_1)\, I_{\beta_2}(\mathcal{T}_2)$ we have

  $$w_i(\mathcal{T}_1) + w_i(\mathcal{T}_2) = w_i(\mathcal{T}') + w_i(E_i^{k-1}) + \varepsilon(r-2) = w_i(\mathcal{T}') + k - 2 + \varepsilon(r-2), \tag{1}$$

where $\varepsilon = 1$ if the additional classes happen to be classes in the exceptional divisor E_i, and $\varepsilon = 0$ otherwise. Combining both equations, we get

$$w_i(\mathcal{T}_1) + w_i(\mathcal{T}_2) = w_i(\mathcal{T}) - 1 + \varepsilon(r-2). \qquad (*)$$

Now we again distinguish two cases.

(a) $(\beta_1, \mathcal{T}_1)$ and $(\beta_2, \mathcal{T}_2)$ satisfy (ii). If $(\beta_1, \mathcal{T}_1)$ does not satisfy (i), then β_1 is a purely exceptional class, so all classes in $\mathcal{T}_1$ must be exceptional, i.e.

$$\begin{aligned} w_i(\mathcal{T}_1) = \text{vdim } \bar{M}_{0,0}(\tilde{X}, \beta_1) &= e_i(\beta_1)\,(r-1) + r - 3 \\ &= (e_i(\beta_1)+1)(r-1) - 2. \end{aligned}$$

So we have the two possibilities

$$\begin{aligned} (\beta_1, \mathcal{T}_1) \text{ does not satisfy (i)} &\Rightarrow w_i(\mathcal{T}_1) \geq (e_i(\beta_1)+1)(r-1)-2, \\ (\beta_1, \mathcal{T}_1) \text{ does not satisfy (iii)} &\Rightarrow w_i(\mathcal{T}_1) \geq (e_i(\beta_1)+1)(r-1). \end{aligned}$$

The same is true for $(\beta_2, \mathcal{T}_2)$. However, since β is not purely exceptional, it is not possible that both $(\beta_1, \mathcal{T}_1)$ and $(\beta_2, \mathcal{T}_2)$ do not satisfy (i). We conclude that

$$\begin{aligned} w_i(\mathcal{T}_1) + w_i(\mathcal{T}_2) &\geq (e_i(\beta_1) + 1 + e_i(\beta_2) + 1)(r-1) - 2 \\ &= (e_i(\beta) + 2)(r-1) - 2 \\ &> w_i(\mathcal{T}) + r - 3 \qquad \text{since } (\beta, \mathcal{T}) \text{ satisfies (iii).} \end{aligned}$$

This is a contradiction to (1).

(b) $(\beta_1, \mathcal{T}_1)$ does not satisfy (ii), i.e. $w_i(\mathcal{T}_1) = e_i(\beta_1) = 0$. Since $w_i(\mathcal{T}_1) = 0$, $\mathcal{T}_1$ does not contain exceptional classes E_i^k for $k > 1$. Since $e_i(\beta_1) = 0$, $\mathcal{T}_1$ also does not contain E_i (otherwise $I_{\beta_1}(\mathcal{T}_1) = 0$ by the divisor axiom). Hence $\mathcal{T}_1$ does not contain E_i^k for any k, and in particular we conclude that $\varepsilon = 0$ in (1):

$$\begin{aligned} w_i(\mathcal{T}_2) = w_i(\mathcal{T}) - 1 < w_i(\mathcal{T}) & \\ &< (e_i(\beta)+1)(r-1) \\ &= (e_i(\beta_2)+1)(r-1). \end{aligned}$$

Therefore $(\beta_2, \mathcal{T}_2)$ satisfies (iii). It also satisfies (ii), since otherwise we would have $e_i(\beta_1) = e_i(\beta_2) = 0$ and hence get zero by the divisor axiom from the class E_i in (A). Hence, $(\beta_2, \mathcal{T}_2)$ cannot satisfy (i), i.e. we must be looking at the invariants (A)-(3). However, the invariant $I_{d'E_i'}(\,\cdot\,)$ appearing there can never be non-zero if the additional classes are non-exceptional. We reach a contradiction.

- $e_{i_0}(\beta) > 0$ and $w_{i_0}(\mathcal{T}) = 0$. Then we can be in any of the cases (A) to (C) of the algorithm. Note that $e_{i_0}(\beta_1) + e_{i_0}(\beta_2)$ is equal to $e_{i_0}(\beta)$ or $e_{i_0}(\beta) + 1$ (the latter case appearing exactly if we are in case (C) and $i = i_0$). In any case, it follows that

$$e_{i_0}(\beta_1) + e_{i_0}(\beta_2) \geq e_{i_0}(\beta) \geq 1,$$

hence we can assume without loss of generality that $e_{i_0}(\beta_1) \geq 1$. In particular, $(\beta_1, \mathcal{T}_1)$ satisfies (ii). We are going to show that it also satisfies (i) and (iii), which is then a contradiction to our assumptions.

The case that $(\beta_1, \mathcal{T}_1)$ does not satisfy (i), i.e. that $d(\beta_1) = 0$, could only occur in (A)-(3) and for $\beta_1 = dE'_i$. Since

$$1 \leq e_{i_0}(\beta_1) = e_{i_0}(dE'_i) = d\,\delta_{i,i_0}$$

we must have $i = i_0$. But this means that we have a class $E_i^k = E_{i_0}^k$ in $\mathcal{T}$ which is a contradiction to $w_{i_0}(\mathcal{T}) = 0$. Hence $(\beta_1, \mathcal{T}_1)$ must satisfy (i).

As for (iii), we compute $w_{i_0}(\mathcal{T}_1)$. There are no exceptional classes $E_{i_0}^2, \ldots, E_{i_0}^{r-1}$ in $\mathcal{T}'$ since $w_{i_0}(\mathcal{T}) = 0$. Hence the only such classes in $\mathcal{T}_1$ can come from

  - the additional classes,
  - the four special classes used in the equation (A), (B), or (C).

Both can contribute at most $r - 2$ to $w_{i_0}(\mathcal{T}_1)$, hence

$$w_{i_0}(\mathcal{T}_1) \leq 2r - 4 < 2(r-1) \leq (e_{i_0}(\beta_1) + 1)(r-1).$$

Therefore $(\beta_1, \mathcal{T}_1)$ also satisfies (iii), arriving at the contradiction we were looking for.

□

As a corollary we can now prove a relation between the Gromov-Witten invariants of $\tilde{X}$ that one would expect from geometry. Namely, if we want to express the condition that curves of homology class β pass through a generic point in X, we expect to be able to do this in two different ways: either we add the class of a point to $\mathcal{T}$, or we blow up the point and count curves with homology class $\beta - E'$. The following corollary states that these two methods will always give the same result, no matter whether the invariants are actually enumeratively meaningful or not.

Corollary 3.2 *Let $(\beta, \mathcal{T})$ be such that, for some $1 \leq i \leq s$, we have $e_i(\beta) = w_i(\mathcal{T}) = 0$ and $d(\beta) \neq 0$. Then*

$$I_{\beta - E'_i}(\mathcal{T}) = I_\beta(\mathcal{T} \otimes pt).$$

Proof Consider the equation $\mathcal{E}_\beta(\mathcal{T}\,;\,\lambda,\lambda \mid E_i, E_i^{r-1})$ for an arbitrary divisor $\lambda \in \mathcal{B}$ with $\lambda\cdot\beta \neq 0$:

$$\begin{aligned} 0 &= I_\beta(\mathcal{T}\otimes\lambda\otimes\lambda\otimes E_i\cdot E_i^{r-1}) && (1)\\ &\quad + (\text{no further } I_\beta(\,\cdot\,)\,I_0(\,\cdot\,)\text{-terms})\\ &\quad + I_{\beta-E_i'}(\mathcal{T}\otimes\lambda\otimes\lambda\otimes E_i)\,\underbrace{I_{E_i'}(E_i\otimes E_i^{r-1}\otimes E_i^{r-1})}_{=-1}\,(-1)^{r-1} && (2)\\ &\quad + (\text{no further } I_{\beta-dE_i'}(\,\cdot\,)\,I_{dE_i'}(\,\cdot\,)\text{-terms since there are not enough exceptional}\\ &\qquad\quad \text{classes to put into } I_{dE_i'}(\,\cdot\,))\\ &\quad + (\text{some } I_{\beta_1}(\,\cdot\,)\,I_{\beta_2}(\,\cdot\,)\text{-terms with } d(\beta_1), d(\beta_2)\neq 0). && (3) \end{aligned}$$

Using proposition 3.1, we will show for any term $I_{\beta_1}(\mathcal{T}_1)I_{\beta_2}(\mathcal{T}_2)$ in (3) that it vanishes. Since $e_i(\beta_1)+e_i(\beta_2) = e_i(\beta) = 0$, we have without loss of generality one of the following cases:

- $e_i(\beta_1) = e_i(\beta_2) = 0.$ Then $I_{\beta_1}(\mathcal{T}_1)\,I_{\beta_2}(\mathcal{T}_2) = 0$ by the divisor axiom because of the class E_i in the equation.
- $e_i(\beta_1) > 0.$ Then we show that $(\beta_1,\mathcal{T}_1)$ satisfies conditions (i) to (iii) of the proposition and hence vanishes. (i) and (ii) are obvious. As for (iii), the only classes contributing to $w_i(\mathcal{T}_1)$ can come from
  - the additional classes,
  - the special class E_i^{r-1} used in the equation.

  Both can contribute at most $r-2$ to $w_i(\mathcal{T}_1)$, hence

  $$w_i(\mathcal{T}_1) \leq 2r-4 < 2(r-1) \leq (e_i(\beta_1)+1)(r-1).$$

  Therefore $(\beta_1,\mathcal{T}_1)$ also satisfies (iii).

Now that we know that all terms in (3) vanish, the above equation becomes

$$I_\beta(\mathcal{T}\otimes\lambda\otimes\lambda\otimes E_i\cdot E_i^{r-1}) = I_{\beta-E_i'}(\mathcal{T}\otimes\lambda\otimes\lambda\otimes E_i)\,(-1)^{r-1}.$$

Since $E_i\cdot E_i^{r-1} = (-1)^{r-1}pt$ and $E_i\cdot(\beta - E_i') = 1$, the corollary follows. □

4 Enumerative significance — general remarks

After having computed all Gromov-Witten invariants on blow-ups of projective space (see corollary 2.6), we now come to the question of enumerative significance of the invariants. For most of the time, we will be concerned with invariants $I_\beta^{\tilde{X}}(\mathcal{T})$ where $\mathcal{T}$ is non-exceptional, leading to numbers of curves on X intersecting the blown-up points with prescribed multiplicities. Only in section 7 we will consider some invariants containing exceptional classes in $\mathcal{T}$, leading to numbers of curves on X with certain tangency conditions.

For the rest of the chapter, we will only work with $\tilde{X} = \tilde{\mathbb{P}}^r(s)$. We start by giving a precise definition of an enumeratively significant invariant.

Definition 4.1 *Let $\beta \in A_1(\tilde{X})$ a homology class with $d(\beta) \neq 0$ and $e_i(\beta) \leq 0$, and let $\mathcal{T} = \gamma_1 \otimes \ldots \otimes \gamma_n$ be a collection of non-exceptional effective classes $\gamma_i \in A^{\geq 1}(X)$ such that $\sum_i \operatorname{codim} \gamma_i = \operatorname{vdim} \bar{M}_{0,n}(\tilde{X}, \beta)$.*

*Then we call the Gromov-Witten invariant $I_\beta^{\tilde{X}}(\mathcal{T})$ **enumerative** if, for generic subschemes $V_i \subset \tilde{X}$ with $[V_i] = \gamma_i$, it is equal to the number of irreducible stable maps $(C, x_1, \ldots, x_n, f)$ with f being generically injective, $f_*[C] = \beta$, and $f(x_i) \in V_i$ for all i (where each such stable map is counted with multiplicity one).*

Note that irreducible stable maps $(C, x_1, \ldots, x_n, f)$ on $\tilde{X}$ of homology class β with f generically injective correspond bijectively to irreducible curves in $\tilde{X}$ of homology class β, and hence via strict transform to irreducible curves in X of homology class $d(\beta)$ intersecting the blown-up points P_i with global multiplicities $-e_i(\beta)$. Hence it is clear that we can also give the following interpretation of enumerative invariants:

Lemma 4.2 *If $I_\beta(\mathcal{T})$ is enumerative, then for generic subschemes $V_i \subset \tilde{X}$ with $[V_i] = \gamma_i$, it is equal to the number of irreducible rational curves $C \subset X$ of homology class $d(\beta)$ intersecting all V_i, and in addition passing through each P_i with global multiplicity $-e_i(\beta)$. Every such curve is counted with multiplicity $\sharp(C \cap V_1) \cdot \ldots \cdot \sharp(C \cap V_n)$.*

In general, one would then expect these curves to have $-e_i$ smooth local branches at every point P_i.

We will now give an overview of the results about enumerative significance of Gromov-Witten invariants on $\tilde{\mathbb{P}}^r(s)$. Assume that $d(\beta) \neq 0$, $e_i(\beta) \leq 0$, and that $\mathcal{T}$ is a collection of non-exceptional effective classes.

(i) If $s = 1$ then $I_\beta(\mathcal{T})$ is enumerative. This will be shown in theorem 5.3.

(ii) If $r = 2$ then $I_\beta(\mathcal{T})$ is enumerative if $e_i(\beta) \in \{-1, -2\}$ for some i or $\mathcal{T}$ contains at least one point class. This has been proven by L. Göttsche and R. Pandharipande in [GP].

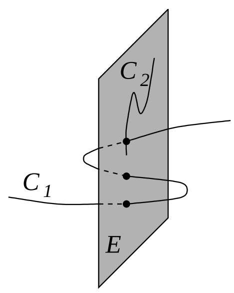
(iii) If $r = 3$, $s \leq 4$, and $\mathcal{T}$ contains only point classes, then $I_\beta(\mathcal{T})$ is enumerative if and only if β is not equal to $dH' - dE'_i - dE'_j$ for some $d \geq 2$ and $i \neq j$ with $1 \leq i, j \leq s$. We will prove this in theorem 6.4.

(iv) If $r = 3$ and $\mathcal{T}$ contains not only point classes, then $I_\beta(\mathcal{T})$ is in general not enumerative.

(v) If $r \geq 4$ and $s \geq 2$ then $I_\beta(\mathcal{T})$ is "almost never" enumerative.

We start our study of enumerative significance by showing the origin of potential problems with enumerative significance, thereby giving counterexamples to enumerative significance in the cases (iv) and (v) above.

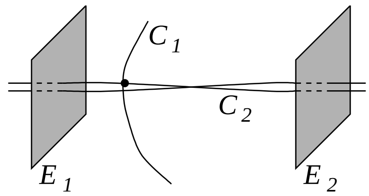
The most obvious problem is that a stable map $(C, x_1, \ldots, x_n, f)$ may be reducible, with some of the components mapped to the exceptional divisor. The part of the moduli space corresponding to such curves will in general have too big dimension. For example, consider the case $\tilde{X} = \tilde{\mathbb{P}}^3(1)$, $\beta = 4H'$. Stable maps in $M_{0,0}(\tilde{X}, \beta)$ will not intersect the exceptional divisor at all, hence $M_{0,0}(\tilde{X}, \beta)$ has the expected dimension. However, consider reducible curves $C = C_1 \cup C_2$ where f is of homology class $4H' - 3E'$ on C_1 and of homology class $3E'$ on C_2. These can be depicted as follows:



The space of such curves C_1 is (at least) of dimension vdim $\bar{M}_{0,0}(\tilde{X}, 4H' - 3E') = 4 \cdot 4 - 3 \cdot 2 = 10$, the space of curves C_2 of homology class $3E'$ through a given point (namely one of the points of intersection of C_1 with E) is of dimension $3 \cdot 3 - 1 - 1 = 7$ (note that $E \cong \mathbb{P}^2$). Hence the part of the moduli space $\bar{M}_{0,0}(\tilde{X}, \beta)$ corresponding to those curves has dimension (at least) 17, but we have vdim $\bar{M}_{0,0}(\tilde{X}, \beta) = 4 \cdot 4 = 16$. Note that this is in agreement with the fact that these curves certainly cannot be deformed into smooth quartics not intersecting the exceptional divisor, hence they are not contained in the closure of $M_{0,0}(\tilde{X}, \beta)$ in $\bar{M}_{0,0}(\tilde{X}, \beta)$.

However, this will cause no problems when computing Gromov-Witten invariants, since, intuitively speaking, the curve C_2 cannot satisfy any incidence conditions with generic non-exceptional varieties. So if we try to impose vdim $\bar{M}_{0,0}(\tilde{X}, \beta) = 16$ non-exceptional conditions on these curves, we will get zero, since the curve C_1 can satisfy at most 10 of the conditions and C_2 can satisfy none at all. For a mathematically more precise statement of this fact, see proposition 5.2 (i) which is the important step in the proof of enumerative significance in the case of only one blow-up.

When we consider more than one blow-up, things get more complicated, since then for example multiple coverings of the lines joining the blown-up points will cause problems. As an example, consider $\tilde{X} = \tilde{\mathbb{P}}^r(2)$, $\beta = (d+q)H' - qE_1' - qE_2'$ for some $r \geq 2$, $d \geq 1$, $q \geq 2$, and look at reducible stable maps as above with C_1 of homology class dH' and C_2 of homology class $qH' - qE_1' - qE_2'$, being a q-fold covering of the strict transform of the line between P_1 and P_2:



We have just learned that C_2 for itself will make no problems, since no generic (non-divisorial) non-exceptional incidence conditions can be satisfied on this component. However, it may well happen that the dimension of the moduli space of curves C_1 meeting the line through P_1 and P_2 (i.e. vdim $\bar{M}_{0,0}(\tilde{X}, dH') - (r-2)$) is *bigger* than that of both components together:

$$\begin{aligned} \text{vdim } \bar{M}_{0,0}(\tilde{X}, dH') - (r-2) &= (r+1)d + r - 3 - (r-2), \\ \text{vdim } \bar{M}_{0,0}(\tilde{X}, \beta) &= (r+1)d + (1-q)(r-3), \\ \Rightarrow \text{vdim } \bar{M}_{0,0}(\tilde{X}, dH') - (r-2) - \text{vdim } \bar{M}_{0,0}(\tilde{X}, \beta) &= \underline{(q-1)(r-3) - 1}. \end{aligned}$$

If this last number is non-negative, we will obviously get non-wanted contributions to our Gromov-Witten invariants from these reducible curves, since all vdim $\bar{M}_{0,0}(X, \beta)$ conditions that we impose on the curve can be satisfied on C_1. This will always happen if $r \geq 4$, showing that in this case there is no chance of getting enumerative invariants. The reader who wants to convince himself of this fact numerically can find some obviously non-enumerative invariants of this kind in example 8.4. For $r = 3$, we will see that multiple coverings of lines joining blown-up points only make problems if they form the only component of an irreducible curve, see theorem 6.4 and example 8.3. In fact, in the case where $\beta = dH' - dE_1' - dE_2'$, such that we "count" d-fold coverings of lines, we get other important invariants, see example 8.5.

Since the case of $\tilde{\mathbb{P}}^4(s)$ for $s \geq 2$ will not lead to enumerative invariants and the case of $\tilde{\mathbb{P}}^2(s)$ has been studied almost exhaustively in [GP], it only remains to look at blow-ups of $\mathbb{P}^3$. We will look at the case $\tilde{X} = \tilde{\mathbb{P}}^3(4)$ in detail in section 6 (which then includes, of course, also the cases $\tilde{X} = \tilde{\mathbb{P}}^3(s)$ with $s < 4$). Here, in analogy to the situation discussed above, one gets problems with too big dimensions for reducible curves as above, where C_2 is now a curve contained in a plane spanned by three of the blown-up points. These problems arise in particular because in this case it is no longer true that C_2 can satisfy no incidence conditions. To be more precise, C_2 can satisfy incidence conditions with generic curves, but *not* with generic points in $\tilde{\mathbb{P}}^3(4)$. This is the reason why we have to make the assumption that all cohomology classes in the invariant are point classes (see theorem 6.4). If we do not assume this, we can again easily get non-enumerative invariants, e.g. $I^{\tilde{\mathbb{P}}^3(4)}_{4H'-2E_1'-2E_2'-2E_3'}((H^2)^{\otimes 4}) = -1$, to mention the easiest one.

In the remainder of this section, we will prove some statements about irreducible curves in blow-ups that will be needed for both cases $\tilde{\mathbb{P}}^r(1)$ and $\tilde{\mathbb{P}}^3(4)$. We start by computing $h^1(\mathbb{P}^1, f^*T_{\tilde{X}})$ in the next two lemmas.

Lemma 4.3 *Let $p : \tilde{X} \to X$ be the blow-up of a smooth variety at some points $P_1, \dots, P_s$ and let $E = E_1 \cup \dots \cup E_s$ be the exceptional divisor. Let C be a smooth curve and $f : C \to \tilde{X}$ a map such that $f(C) \not\subset E$. Then there is an injective morphism of sheaves on $\tilde{X}$*

$$f^*p^*T_X(-f^*E) \to f^*T_{\tilde{X}}$$

which is an isomorphism away from $f^{-1}(E)$.

Proof Since $E = \{P_1, \dots, P_s\} \times_X \tilde{X}$, we have $i^*\Omega_{\tilde{X}/X} = \Omega_{E/\{P_1,\dots,P_s\}} = \Omega_E$ where $i : E \to \tilde{X}$ is the inclusion. As $\Omega_{\tilde{X}/X}$ has support on E, this can be rewritten as $i_*\Omega_E = \Omega_{\tilde{X}/X}$. Hence, there is an exact sequence of sheaves on $\tilde{X}$

$$0 \to p^*\Omega_X \to \Omega_{\tilde{X}} \to i_*\Omega_E \to 0.$$

Dualizing, we get

$$0 \to T_{\tilde{X}} \to p^*T_X \to \mathcal{E}xt^1(i_*\Omega_E, \mathcal{O}_X) \to 0.$$

By duality (see [H] theorem III 6.7), we have

$$\mathcal{E}xt^1(i_*\Omega_E, \mathcal{O}_X) = i_*\mathcal{E}xt^1(\Omega_E, N_{E/\tilde{X}}) = i_*T_E(-1)$$

where $\mathcal{O}(-1) := \mathcal{O}_{E_1}(-1) \otimes \dots \otimes \mathcal{O}_{E_s}(-1)$. Therefore we get a morphism $p^*T_X \to i_*T_E(-1)$ which we can restrict to E to get a morphism $p^*T_X|_E \to i_*T_E(-1)$ fitting into a commutative diagram

$$\begin{array}{ccccccccc}
0 & \longrightarrow & p^*T_X(-E) & \longrightarrow & p^*T_X & \longrightarrow & p^*T_X|_E & \longrightarrow & 0 \\
 & & & & \| & & \downarrow & & \\
0 & \longrightarrow & T_{\tilde{X}} & \longrightarrow & p^*T_X & \longrightarrow & i_*T_E(-1) & \longrightarrow & 0.
\end{array}$$

From this we can deduce the existence of an injective map $p^*T_X(-E) \to T_{\tilde{X}}$ which is clearly an isomorphism away from E. Applying the functor f^* we get the desired morphism $f^*p^*T_X(-f^*E) \to f^*T_{\tilde{X}}$. Since the image of f is not contained in E, this morphism is also injective and an isomorphism away from $f^{-1}(E)$. $\square$

Lemma 4.4 *Let $C = \mathbb{P}^1$, $\tilde{X} = \mathbb{P}^r(s)$, $f : C \to \tilde{X}$ a morphism, $\beta = f_*[C] \in A_1(\tilde{X})$, and $\varepsilon \in \{0, 1\}$.*

(i) If $f(C) \not\subset E$ or f is a constant map then $h^1(C, f^*T_{\tilde{X}}(-\varepsilon)) = 0$ whenever $d(\beta) + e(\beta) \geq 0$. (Here, $f^*T_{\tilde{X}}(-\varepsilon)$ is to be interpreted as $f^*T_{\tilde{X}} \otimes \mathcal{O}_C(-\varepsilon)$.) In particular, this always holds for $s = 1$ (since then $d(\beta) + e(\beta) = \deg\ f^*(H - E)$ and $f^*(H - E)$ is an effective divisor on C).

(ii) If $f(C) \subset E$ and the map $f : C \to E \cong \mathbb{P}^{r-1}$ has degree $e > 0$ then

$$h^1(C, f^* T_{\tilde{X}}(-\varepsilon)) = e + \varepsilon - 1.$$

Proof

(i) If f is a constant map then the assertion is trivial, so assume that $f(C) \not\subset E$ and set $d = \deg\ f^* H$, $e = -\deg\ f^* E$. By lemma 4.3 we have an exact sequence

$$0 \to f^* p^* T_X(e) \to f^* T_{\tilde{X}} \to Q \to 0$$

with some sheaf Q on C with zero-dimensional support. Hence to prove the lemma it suffices to show that $h^1(C, f^* p^* T_X(e-\varepsilon)) = 0$. But this follows from the Euler sequence on $\mathbb{P}^r$ pulled back to C and twisted by $\mathcal{O}_C(e-\varepsilon)$:

$$0 \to \mathcal{O}_C(e-\varepsilon) \to (r+1)\,\mathcal{O}_C(d+e-\varepsilon) \to f^* p^* T_X(e-\varepsilon) \to 0$$

since $d + e - \varepsilon \geq -1$ by assumption.

(ii) We consider the normal sequence

$$0 \to T_E \to i^* T_{\tilde{X}} \to N_{E/\tilde{X}} \to 0.$$

As $N_{E/\tilde{X}} = \mathcal{O}_E(-1)$, pulling back to C and twisting by $\mathcal{O}_C(-\varepsilon)$ yields

$$0 \to f^* T_E(-\varepsilon) \to f^* T_{\tilde{X}}(-\varepsilon) \to \mathcal{O}_C(-e-\varepsilon) \to 0. \tag{1}$$

In complete analogy to (i), it follows by the Euler sequence of $E \cong \mathbb{P}^{r-1}$

$$0 \to \mathcal{O}_C(-\varepsilon) \to r\,\mathcal{O}_C(e-\varepsilon) \to f^* T_E(-\varepsilon) \to 0$$

that $h^1(C, f^* T_E(-\varepsilon)) = 0$. Hence we deduce from (1) that

$$h^1(f^* T_{\tilde{X}}(-\varepsilon)) = h^1(C, \mathcal{O}_C(-e-\varepsilon)) = e + \varepsilon - 1.$$

□

We now come to the Bertini lemma 4.7 which is our main tool to prove the transversality of the intersection products in the Gromov-Witten invariants.

Lemma 4.5 *Let M be a scheme of finite type and $f : M \to \mathbb{P}^r$ a morphism. Then, for a generic hyperplane $H \subset \mathbb{P}^r$, we have:*

(i) $f^{-1}(H)$ is (empty or) of pure codimension 1 in M.

(ii) If M is smooth then the divisor $f^{-1}(H)$ is a smooth subscheme of M counted with multiplicity one.

Proof See e.g. [J] corollary 6.11. □

Lemma 4.6 *Let M be a scheme of finite type, X a smooth, connected, projective scheme, and $f : M \to X$ a morphism. Let L be a base point free linear system on X. Then, for generic $D \in L$, we have:*

(i) $f^{-1}(D)$ is (empty or) purely 1–codimensional.

(ii) If M is smooth then the divisor $f^{-1}(D)$ is a smooth subscheme of M counted with multiplicity one.

Proof The base point free linear system L on X gives rise to a morphism $s : X \to \mathbb{P}^m$ where $m = \dim L$. Composing with f yields a morphism $M \to \mathbb{P}^m$, and the divisors $D \in L$ correspond to the inverse images under s of the hyperplanes in $\mathbb{P}^m$. Hence, the statement follows from lemma 4.5, applied to the map $M \to \mathbb{P}^m$. □

Lemma 4.7 *Let M be a Deligne-Mumford stack of finite type, X a smooth, connected, projective scheme and $f_i : M \to X$ morphisms for $i = 1, \ldots, n$. Let $\gamma_i \in A^{c_i}(X)$ be cycles of codimensions $c_i \geq 1$ on X that can be written as intersection products of divisors on X*

$$\gamma_i = [D'_{i,1}] \cdot \cdots \cdot [D'_{i,c_i}] \qquad (i = 1, \ldots, n)$$

such that the complete linear systems $|D'_{i,j}|$ are base point free (this always applies, for example, for effective cycles in the case $X = \mathbb{P}^r$). Let $c = c_1 + \cdots + c_n$. Then, for generic $D_{i,j} \in |D'_{i,j}|$, we have:

(i) $V_i := D_{i,1} \cap \cdots \cap D_{i,c_i}$ is smooth of pure codimension c_i in X, and the intersection is transverse. In particular, $[V_i] = \gamma_i$.

(ii) $V := f_1^{-1}(V_1) \cap \cdots \cap f_n^{-1}(V_n)$ is of pure codimension c in M. In particular, if $\dim M < c$ then $V = \emptyset$.

(iii) If $\dim M = c$ and M contains a dense, open, smooth substack U such that each geometric point of U has no non-trivial automorphisms then V consists of exactly $(f_1^*\gamma_1 \cdot \ldots \cdot f_n^*\gamma_n)[X]$ points of M which lie in U and are counted with multiplicity one.

Proof

(i) follows immediately by recursive application of lemma 4.5 to the scheme X.

(ii) If M is a scheme, then the statement follows by recursive application of lemma 4.6. If M is a Deligne-Mumford stack, then it has an étale cover $S \to M$ by a scheme S, so (ii) holds for the composed maps $S \to M \to X$. But since the map $S \to M$ is étale, the statement is also true for the maps $M \to X$.

(iii) A Deligne-Mumford stack U whose generic geometric point has no non-trivial automorphisms always has a dense open substack U' which is a scheme (see e.g. [V]. To be more precise, U is a functor and hence an algebraic space ([DM] ex. 4.9), but an algebraic space always contains a dense open subset U' which is a scheme ([Kn] p. 25)). Since U' is dense in M and therefore $M\backslash U'$ has smaller dimension, applying (ii) to the restrictions $f_i|_{M\backslash U'} : M\backslash U' \to X$ gives that V is contained in the smooth scheme U', hence it suffices to consider the restrictions $f_i|_{U'} : U' \to X$. But since U' is a smooth scheme, we can apply lemma 4.6 (ii) recursively and get the desired result.

□

As we needed for lemma 4.7 (iii) that the generic element of M has no non-trivial automorphisms, we now give a criterion under which circumstances this is satisfied for our moduli spaces of stable maps.

Lemma 4.8 *Let $\tilde{X} = \mathbb{P}^r(s)$ and $\beta \in A_1(\tilde{X})$ with $d(\beta) > 0$ and $d(\beta) + e(\beta) \geq 0$. Assume that β is not of the form $d\,H' - d\,E_i'$ for $1 \leq i \leq s$ and $d \geq 2$. Then, if $M_{0,n}(\tilde{X}, \beta)$ is not empty, it is a smooth stack of the expected dimension, and if $\mathcal{C} = (C, x_1, \ldots, x_n, f)$ is a generic element of $M_{0,n}(\tilde{X}, \beta)$ then $\mathcal{C}$ has no automorphisms and f is generically injective.*

Proof Set $d = d(\beta)$ and $e = e(\beta)$. We can assume that $e \leq 0$ since otherwise $M_{0,n}(\tilde{X}, \beta)$ is empty.

It follows from lemma 4.4 (i) that $M_{0,n}(\tilde{X}, \beta)$ is a smooth stack of the expected dimension. Note that an irreducible stable map can only have automorphisms if it is a multiple covering map onto its image. Therefore it suffices if we compute, for all $N \geq 2$, the dimension of the subspace $Z_N \subset M_{0,n}(\tilde{X}, \beta)$ consisting of N-fold coverings and show that it is smaller than the dimension of $M_{0,n}(\tilde{X}, \beta)$.

So assume that $N \geq 2$ and that $Z_N \neq \emptyset$, so that $\beta = N\beta'$ for some $\beta' \in A_1(\tilde{X})$. We set $d' = d(\beta')$ and $e' = e(\beta')$. Since $d' + e' \geq 0$, we can apply lemma 4.4 (i) to see that the space of stable maps of homology class β' is of the expected dimension $(r+1)d' + (r-1)e' + r + n - 3$. The dimension of Z_N is exactly bigger by $2N - 2$ because of the moduli of the covering. Hence we have

$$\begin{aligned} \dim Z_N &= (r+1)d' + (r-1)e' + r + n - 3 + 2N - 2 \\ &= (r+1)d + (r-1)e + r + n - 3 + ((r+1)d' + (r-1)e')(1-N) + 2N - 2 \\ &= \dim M_{0,n}(\tilde{X}, \beta) + ((r+1)d' + (r-1)e' - 2)(1-N). \end{aligned}$$

Therefore, to prove the lemma, it suffices to show that $(r+1)d' + (r-1)e' > 2$. We distinguish two cases:

- If $e' = 0$, then

$$(r+1)d' + (r-1)e' = (r+1)d' \geq (2+1)\cdot 1 = 3 > 2.$$

- If $e' \leq -1$, then

$$(r+1)d' + (r-1)e' = (r+1)(d'+e') - 2e' \geq -2e' \geq 2,$$

  but if we had equality, this would mean $d' + e' = 0$ and $e' = -1$, hence $\beta' = H' - E'_i$ for some i and therefore $\beta = NH' - NE'_i$, which is the case we excluded in the lemma.

This finishes the proof. □

5 Enumerative significance — the case $\tilde{\mathbb{P}}^r(1)$

In this section we will prove that all invariants $I_\beta(\mathcal{T})$ on $\tilde{X} = \tilde{\mathbb{P}}^r(1)$ are enumerative. We start with the computation of $h^1(C, f^*T_{\tilde{X}})$ for arbitrary stable maps. To state the result, we need the following definition: for any prestable map $(C, x_1, \ldots, x_n, f)$ to $\tilde{X}$ we define $\eta(C, f)$ to be "the sum of the exceptional degrees of all irreducible components of C which are mapped into E", i.e.

$$\eta(C,f) := \sum_{C'} \{\ e \mid C' \text{ is an irreducible component of } C \text{ such that } f_*[C'] = eE'\ \}.$$

Obviously, $\eta(C, f)$ only depends on the topology τ of the prestable map in the sense of section 1, so we will write $\eta(\tau) = \eta(C, f)$.

Lemma 5.1 *Let C be a prestable curve, $\tilde{X} = \tilde{\mathbb{P}}^r(1)$, and $f : C \to \tilde{X}$ a morphism. Then $h^1(C, f^*T_{\tilde{X}}) \leq \eta(C, f)$, with strict inequality holding if $\eta(C, f) > 0$.*

Proof The proof is by induction on the number of irreducible components of C. If C itself is irreducible, the statement follows immediately from lemma 4.4 for $\varepsilon = 0$.

Now let C be reducible, so assume $C = C_0 \cup C'$ where $C' \cong \mathbb{P}^1$, $C_0 \cap C' = \{Q\}$, and where C_0 is a prestable curve for which the induction hypothesis holds. If $\eta(C, f) > 0$, we can arrange this such that $\eta(C_0, f_0) > 0$.

Consider the exact sequences

$$0 \to f^*T_{\tilde{X}} \to f_0^*T_{\tilde{X}} \oplus f'^*T_{\tilde{X}} \xrightarrow{\varphi} f_Q^*T_{\tilde{X}} \to 0$$
$$0 \to f'^*T_{\tilde{X}}(-Q) \to f'^*T_{\tilde{X}} \xrightarrow{\psi} f_Q^*T_{\tilde{X}} \to 0$$

where f_0, f', and f_Q denote the restrictions of f to C_0, C', and Q, respectively.

From these sequences we deduce that

$$\dim \operatorname{coker} H^0(\varphi) = h^1(C, f^*T_{\tilde{X}}) - h^1(C_0, f_0^*T_{\tilde{X}}) - h^1(C', f'^*T_{\tilde{X}})$$
$$\dim \operatorname{coker} H^0(\psi) = h^1(C', f'^*T_{\tilde{X}}(-Q)) - h^1(C', f'^*T_{\tilde{X}}).$$

Since we certainly have $\dim \operatorname{coker} H^0(\varphi) \leq \dim \operatorname{coker} H^0(\psi)$, we can combine these equations into the single inequality

$$h^1(C, f^*T_{\tilde{X}}) \leq h^1(C_0, f_0^*T_{\tilde{X}}) + h^1(C', f'^*T_{\tilde{X}}(-Q)).$$

Now, by the induction hypothesis on f_0, we have $h^1(C_0, f_0^*T_{\tilde{X}}) \leq \eta(C_0, f_0)$ with strict inequality holding if $\eta(C_0, f_0) > 0$. On the other hand, we get $h^1(C', f'^*T_{\tilde{X}}(-Q)) \leq \eta(C', f')$ by lemma 4.4 for $\varepsilon = 1$. As $\eta(C, f) = \eta(C_0, f_0) + \eta(C', f')$, the proposition follows by induction. □

We now come to the central proposition already alluded to in section 4: given a part $M(\tilde{X}, \tau)$ of the moduli space $\bar{M}_{0,n}(\tilde{X}, \beta)$ corresponding to the topology τ (see section 1), we consider the map

$$\phi : M(\tilde{X}, \tau) \hookrightarrow \bar{M}_{0,n}(\tilde{X}, \beta) \to \bar{M}_{0,n}(X, d(\beta))$$

given by mapping $(C, x_1, \dots, x_n, f)$ to $(C, x_1, \dots, x_n, p \circ f)$ and stabilizing if necessary (ϕ exists by the functoriality of the moduli spaces of stable maps, see [BM] remark after theorem 3.14). We show that, although $M(\tilde{X}, \tau)$ may have too big dimension, the image $\phi(M(\tilde{X}, \tau))$ has not. Part (ii) of the proposition, which is of similar type, will be needed later in section 7.

Proposition 5.2 *Let $\tilde{X} = \tilde{\mathbb{P}}^r(1)$ and $\beta \in A_1(\tilde{X})$ with $d(\beta) > 0$. Let $\phi : \bar{M}_{0,n}(\tilde{X}, \beta) \to \bar{M}_{0,n}(X, d(\beta))$ be the morphism as above, and let τ be a topology of stable maps of homology class β (so that $M(\tilde{X}, \tau) \subset \bar{M}_{0,n}(\tilde{X}, \beta)$). Then we have*

(i) $\dim \phi(M(\tilde{X}, \tau)) \leq \operatorname{vdim} \bar{M}_{0,n}(\tilde{X}, \beta)$. *Moreover, strict inequality holds if and only if τ is a topology corresponding to reducible curves.*

(ii) At least one of the following holds:

- *(a)* $\dim \phi(M(\tilde{X}, \tau)) \leq \operatorname{vdim} \bar{M}_{0,n}(\tilde{X}, \beta) - r$,
- *(b)* $\dim M(\tilde{X}, \tau) \leq \operatorname{vdim} \bar{M}_{0,n}(\tilde{X}, \beta) - 2$,
- *(c)* $\dim M(\tilde{X}, \tau) \leq \operatorname{vdim} \bar{M}_{0,n}(\tilde{X}, \beta) - 1$ *and* $\eta(\tau) = 0$,
- *(d)* $\dim M(\tilde{X}, \tau) = \operatorname{vdim} \bar{M}_{0,n}(\tilde{X}, \beta)$ *and τ is the topology corresponding to irreducible curves,*
- *(e)* $\dim M(\tilde{X}, \tau) = \operatorname{vdim} \bar{M}_{0,n}(\tilde{X}, \beta) - 1$ *and τ is a topology corresponding to reducible curves having exactly two irreducible components, one with homology class $\beta - E'$ and the other with homology class E'.*

Proof We start by defining some numerical invariants of the topology τ that will be needed in the proof.

- Let $\boldsymbol{S}$ be the number of nodes of a curve with topology τ. We divide this number into $S = S_{EE} + S_{XX} + S_{XE}$, where $\boldsymbol{S_{EE}}$ (resp. $\boldsymbol{S_{XX}}$, $\boldsymbol{S_{XE}}$) denotes the number of nodes joining two exceptional components of C (resp. two non-exceptional components, or one exceptional with one non-exceptional component). Here and in the following we call an irreducible component of C exceptional if it is mapped by f into the exceptional divisor and it is not contracted by f, and non-exceptional otherwise.
- Let $\boldsymbol{P}$ be the (minimal) number of additional marked points which are necessary to stabilize C. We divide the number P into $P = P_E + P_X$, where $\boldsymbol{P_E}$ (resp. $\boldsymbol{P_X}$) is the number of marked points that have to be added on exceptional components (resp. non-exceptional components) of C to stabilize C.

Now we give an estimate for the dimension of $M(\tilde{X}, \tau)$. The tangent space $T_{M(\tilde{X},\tau),\mathcal{C}}$ at a point $\mathcal{C} = (C, x_1, \dots, x_n, f) \in M(\tilde{X}, \tau)$ is given by the hypercohomology group (see [K] section 1.3.2)

$$T_{M(\tilde{X},\tau),\mathcal{C}} = \mathbb{H}^1(T'_C \to f^* T_{\tilde{X}})$$

where $T'_C = T_C(-x_1 - \dots - x_n)$ and where we put the sheaves T'_C and $f^* T_{\tilde{X}}$ in degrees 0 and 1, respectively. This means that there is an exact sequence

$$0 \to H^0(C, T'_C) \to H^0(C, f^* T_{\tilde{X}}) \to T_{M(\tilde{X},\tau),\mathcal{C}} \to H^1(C, T'_C) \tag{1}$$

(note that the first map is injective because $\mathcal{C}$ is a stable map). By lemma 5.1, we have

$$\dim H^0(C, f^* T_{\tilde{X}}) \le \chi(C, f^* T_{\tilde{X}}) + \eta(C, f). \tag{2}$$

Moreover, by Riemann-Roch we get $\chi(C, T'_C) = S + 3 - n$. It follows that

$$\begin{aligned} \dim T_{M(\tilde{X},\tau),\mathcal{C}} &\le \chi(C, f^* T_{\tilde{X}}) + \eta(C, f) + n - S - 3 \\ &= \operatorname{vdim} \bar{M}_{0,n}(\tilde{X}, \beta) + \eta(C, f) - S, \end{aligned}$$

and therefore

$$\dim M(\tilde{X}, \tau) \le \operatorname{vdim} \bar{M}_{0,n}(\tilde{X}, \beta) + \eta(\tau) - S.$$

If $\eta(\tau) - S < 0$, then statement (i) is obviously satisfied. Moreover, if $\eta(\tau) = 0$ then we also have (ii)-(c), and if $\eta(\tau) > 0$ then we have strict inequality also in (2) and therefore (ii)-(b). Therefore we can assume from now on that $\underline{\eta(\tau) - S \ge 0}$. If $\eta(\tau) = 0$, then we must also have $S = 0$, which means that the curve is irreducible. But then (i) and (ii)-(d) are satisfied. So we can also assume in the sequel that $\underline{\eta(\tau) > 0}$. It follows then from lemma 5.1 that we have strict inequality in (2), hence

$$\dim\, T_{M(\tilde{X},\tau),\mathcal{C}} \le \text{vdim}\, \bar{M}_{0,n}(\tilde{X},\beta) + \eta(C,f) - S - 1. \tag{3}$$

We now give an estimate of the dimension of the image $\phi(M(\tilde{X},\tau))$. As we always work over the ground field $\mathbb{C}$, we can do this on the level of tangent spaces, i.e. we have

$$\dim\, \phi(M(\tilde{X},\tau)) \le \max{}_{\mathcal{C}\in M(\tilde{X},\tau)} \dim\,(d\phi)(T_{M(\tilde{X},\tau),\mathcal{C}}).$$

Hence our goal is to find as many vectors in ker $d\phi$ as possible. We do this by finding elements in the kernel of the composite map (see (1))

$$H^0(C, f^*T_{\tilde{X}})/H^0(C,T'_C) \hookrightarrow T_{M(\tilde{X},\tau),\mathcal{C}} \to T_{\bar{M}_{0,n}(X,d(\beta)),\phi(\mathcal{C})}.$$

Let C_0 be a maximal connected subscheme of C consisting only of exceptional components of C. Let f_0 be the restriction of f to C_0 and let $Q_1,\dots,Q_a$ be the nodes of C which join C_0 with the rest of C (they are of type S_{XE}). Now every section of $f_0^*T_E(-Q_1-\cdots-Q_a)$ can be extended by zero to a section of $f^*T_{\tilde{X}}$ which is mapped to zero by $d\phi$ since these deformations of the map take place entirely within the exceptional divisor. As $E \cong \mathbb{P}^{r-1}$ is a convex variety, we have

$$h^0(C_0, f_0^*T_E) = \chi(C_0, f_0^*T_E) = r-1+r\eta(C_0,f_0)$$

and therefore we can estimate the dimension of the space of deformations that we have just found:

$$h^0(C_0, f_0^*T_E(-Q_1-\cdots-Q_a)) \ge r-1+r\eta(C_0,f_0)-(r-1)\,a.$$

(The right hand side of this inequality may well be negative, but nevertheless the statement is correct also in this case, of course.)

We will now add up these numbers for all possible C_0, say there are $\boldsymbol{B}$ of them. The sum of the $\eta(C_0,f_0)$ will then give $\eta(C,f)$, and the sum of the a will give S_{XE}. Note that there is a P_E-dimensional space of infinitesimal automorphisms of C, i.e. a subspace of $H^0(C,T'_C)$, included in the deformations that we have just found, and that these are exactly the trivial elements in the kernel of $d\phi$. Therefore we have

$$\begin{aligned}
\dim\ker d\phi &\ge B(r-1)+r\eta(C,f)-(r-1)S_{XE}-P_E \\
&= (r-2)(\underbrace{B}_{\ge 1}+\underbrace{\eta(C,f)-S_{XE}}_{\ge 0})+B+2\eta(C,f)-S_{XE}-P_E \\
&\qquad (B\ge 1 \text{ since } \eta(C,f)>0 \\
&\qquad\ \text{and } \eta(C,f)-S_{XE}\ge 0 \text{ since } \eta(C,f)-S\ge 0) \\
&\ge (r-2)+B+2\eta(C,f)-S_{XE}-P_E.
\end{aligned}$$

Combining this with (3), we get the estimate

$$\begin{aligned}\dim\,\phi(M(\tilde{X},\tau)) &\leq \dim\, T_{M(\tilde{X},\tau),\mathcal{C}} - \dim\,\ker\, d\phi \\ &\leq \operatorname{vdim}\, \bar{M}_{0,n}(\tilde{X},\beta) - r + 1 - (S_{XX} + S_{EE} + B + \eta(\tau) - P_E).\end{aligned}$$

To prove the proposition, it remains to look at the term in brackets. First we will show that

$$P_E \leq S_{XX} + S_{EE} + B + \eta(\tau). \tag{4}$$

Look at P_E, i.e. the exceptional components of C where marked points have to be added to stabilize C. We have to distinguish three cases:

(A) Components on which two points have to be added, and whose (only) node is of type S_{EE}: those give a contribution of 2 to P_E, but they also give at least 1 to $\eta(\tau)$ and to S_{EE} (and every node of type S_{EE} belongs to at most one such component).

(B) Components on which two points have to be added, and whose (only) node is of type S_{XE}: those give a contribution of 2 to P_E, but they also give at least 1 to $\eta(\tau)$ and to B (since such a component alone is one of the C_0 considered above).

(C) Components on which only one point has to be added: those give a contribution of 1 to P_E, but they also give at least 1 to $\eta(\tau)$.

This shows (4), finishing the proof of (i). As for (ii), (a) is satisfied if we have strict inequality in (4), so we assume from now on that this is not the case and determine necessary conditions for equality by looking at the proof of (4) above. First of all, we see that every maximal connected subscheme of C consisting only of exceptional components contributes 1 to B, but this gets accounted for only in case (B) above, so if we want to have equality, every such maximal connected subscheme must actually be an irreducible component of type (B), which in addition gives a contribution of *exactly* 2 to P_E and *exactly* 1 to $\eta(\tau)$. So all exceptional components of the curve must actually be lines with no marked points, connected at exactly one point to a non-exceptional component of the curve. Moreover, for equality we must also have $S_{XX} = 0$, since these nodes have not been considered above at all.

Hence, in summary, we must have one non-exceptional irreducible component C_0 of homology class $\beta - qE'$, and q exceptional components of homology class E' with no marked points, each connected at exactly one point to C_0. But it is easy to compute the dimension of $\phi(M(X,\tau))$ for these topologies: the map ϕ simply forgets the q exceptional components, so

$$\begin{aligned}\dim\,\phi(M(\tilde{X},\tau)) &= \dim\, M_{0,n}(\tilde{X},\beta - qE') \\ &= \operatorname{vdim}\, \bar{M}_{0,n}(\tilde{X},\beta - qE') \qquad \text{by (i)} \\ &= \operatorname{vdim}\, \bar{M}_{0,n}(\tilde{X},\beta) - q(r-1).\end{aligned}$$

Hence we see that (ii)-(a) is satisfied for $q > 1$ and (ii)-(e) for $q = 1$.

This completes the proof. □

We now combine our results to prove the enumerative significance of the Gromov-Witten invariants of $\tilde{\mathbb{P}}^r(1)$. Some examples of these numbers can be found in 8.1 and 8.2.

Theorem 5.3 *Let $\tilde{X} = \tilde{\mathbb{P}}^r(1)$, $\beta = dH' + eE' \in A_1(\tilde{X})$ an effective homology class with $d > 0$ and $e \leq 0$, and $\mathcal{T} = \gamma_1 \otimes \ldots \otimes \gamma_n$ a collection of non-exceptional effective classes such that $\sum_i \operatorname{codim} \gamma_i = \operatorname{vdim} \bar{M}_{0,n}(\tilde{X}, \beta)$. Then $I_\beta(\mathcal{T})$ is enumerative.*

Proof The proof goes along the same lines as that of lemma 2.2. For irreducible stable maps $(C, x_1, \ldots, x_n, f)$ we have $h^1(C, f^* T_{\tilde{X}}) = 0$ by lemma 4.4 (i). Therefore, if $Z \subset \bar{M}_{0,n}(\tilde{X}, \beta)$ denotes the closure of $M_{0,n}(\tilde{X}, \beta)$, then lemma 1.1 tells us that

$$[\bar{M}_{0,n}(\tilde{X}, \beta)]^{virt} = [Z] + \alpha$$

where α is a cycle of dimension vdim $\bar{M}_{0,n}(\tilde{X}, \beta)$ supported on $\bar{M}_{0,n}(\tilde{X}, \beta) \backslash M_{0,n}(\tilde{X}, \beta)$. But if $\phi : \bar{M}_{0,n}(\tilde{X}, dH' + eE') \to \bar{M}_{0,n}(X, dH')$ denotes the morphism induced by the map $p : \tilde{X} \to X$, we must have $\phi_* \alpha = 0$ by proposition 5.2 (i). So, considering the commutative diagram

$$\begin{array}{ccc} \bar{M}_{0,n}(\tilde{X}, \beta) & \xrightarrow{\phi} & \bar{M}_{0,n}(X, dH') \\ \downarrow{\scriptstyle ev_i} & & \downarrow{\scriptstyle ev_i} \\ \tilde{X} & \xrightarrow{p} & X \end{array}$$

for $1 \leq i \leq n$, it follows by the projection formula that

$$\begin{aligned} I_\beta^{\tilde{X}}(\mathcal{T}) &= (\prod_i ev_i^* p^* \gamma_i) \cdot [\bar{M}_{0,n}(\tilde{X}, \beta)]^{virt} \\ &= (\prod_i ev_i^* \gamma_i) \cdot \phi_* [\bar{M}_{0,n}(\tilde{X}, \beta)]^{virt} \\ &= (\prod_i ev_i^* \gamma_i) \cdot \phi_* [Z]. \\ &= (\prod_i ev_i^* p^* \gamma_i) \cdot [Z]. \end{aligned}$$

Hence we are evaluating an intersection product on the stack Z.

Unless $d + e = 0$ and $d \geq 2$, the theorem now follows from the Bertini lemma 4.7 (iii) in combination with lemma 4.8 saying that the generic element of Z has no automorphisms and corresponds to a generically injective stable map. However, if $d + e = 0$ and $d \geq 2$, then the image of every stable map in $M_{0,n}(\tilde{X}, dH' - dE')$ is a line through the blown-up point. These curves can obviously only satisfy as many incidence conditions as the curves in $M_{0,n}(\tilde{X}, H' - E')$. But vdim $\bar{M}_{0,n}(\tilde{X}, dH' - dE') >$ vdim $\bar{M}_{0,n}(\tilde{X}, H' - E')$, so the Gromov-Witten invariant will be zero, which is also the enumeratively correct number. □

6 Enumerative significance — the case $\tilde{\mathbb{P}}^3(4)$

In this section, we discuss the enumerative significance of the Gromov-Witten invariants on $\tilde{X} = \tilde{\mathbb{P}}^3(4)$. First we fix some notation. As the four points to blow up on $X = \mathbb{P}^3$ we choose $P_1 = (1:0:0:0)$, $P_2 = (0:1:0:0)$, $P_3 = (0:0:1:0)$, and $P_4 = (0:0:0:1)$. For $1 \le i < j \le 4$, we denote by $L_{ij} \subset \tilde{X}$ the strict transform of the line $\overline{P_iP_j}$. The L_{ij} are disjoint from each other, and we set $\mathcal{L} = \bigcup_{i<j} L_{ij}$. For $1 \le i \le 4$, we let H_i be the strict transform of the hyperplane in X spanned by the three points P_j with $j \neq i$, and we set $\mathcal{H} = \bigcup_i H_i$. As usual, E_i denotes the exceptional divisor over P_i. We set $\mathcal{E} = \bigcup_i E_i$.

Let $\beta \in A_1(\tilde{X})$ be an effective homology class with $d(\beta) > 0$. The first thing to do is to look at *irreducible* curves of homology class β and to see whether their moduli space $M_{0,0}(\tilde{X}, \beta)$ is smooth and of the expected dimension, which in this case is

$$\text{vdim } \bar{M}_{0,0}(\tilde{X}, \beta) = 4d(\beta) + 2e(\beta).$$

In the case of one blow-up in section 5, this followed easily from lemma 4.4 (i) since there we always have $d(\beta) + e(\beta) \ge 0$. However, for multiple blow-ups, this is not necessarily the case. Our way to solve this problem is to use a certain Cremona map to transform curves with $d(\beta) + e(\beta) \le 0$ into others with $d(\beta) + e(\beta) \ge 0$, so that lemma 4.4 can be applied again. Before we can describe this map, we need a definition.

Definition 6.1 *Let $(C, f) \in M_{0,0}(\tilde{\mathbb{P}}^3(4), \beta)$ be an irreducible stable map with $f(C) \not\subset \mathcal{L}$. Then we set $\lambda_{ij}(C, f)$ to be the "multiplicity of f along L_{ij}", defined as follows: if $\varphi_1 : \tilde{Y} \to \tilde{\mathbb{P}}^3(4)$ is the blow-up of $\tilde{\mathbb{P}}^3(4)$ along $\mathcal{L}$ with exceptional divisors F_{ij} over L_{ij}, then there is a well-defined map $\varphi_1^{-1} \circ f : C \to \tilde{Y}$, and we define*

$$\boldsymbol{\lambda_{ij}(C, f)} := F_{ij} \cdot (\varphi_1^{-1} \circ f)_*[C] \ge 0.$$

Finally, we define $\vec{\boldsymbol{\lambda}}\boldsymbol{(C, f)}$ to be the vector consisting of all $\lambda_{ij}(C, f)$, and set

$$\boldsymbol{\lambda(C, f)} = \sum_{i<j} \lambda_{ij}(C, f).$$

We can now describe the Cremona map announced above.

Lemma 6.2 *There exists a birational map $\varphi : \tilde{\mathbb{P}}^3(4) \dashrightarrow \tilde{\mathbb{P}}^3(4)$ which is an isomorphism outside $\mathcal{L}$ with the following property:*

If $(C, f) \in M_{0,0}(\tilde{\mathbb{P}}^3(4), \beta)$ is an irreducible stable map such that $f(C) \not\subset \mathcal{L}$, so that the transformed stable map $(C, \varphi \circ f) \in M_{0,0}(\tilde{\mathbb{P}}^3(4), \beta')$ exists, then the homology class β' of the transformed stable map satisfies

$$d(\beta') = 3d(\beta) + 2e(\beta) - \lambda(C, f),$$
$$e(\beta') = -4d(\beta) - 3e(\beta) + 2\lambda(C, f).$$

Hence, in particular, we have

- $4d(\beta') + 2e(\beta') = 4d(\beta) + 2e(\beta)$,
- *if* $d(\beta) + e(\beta) \leq 0$, *then* $d(\beta') + e(\beta') \geq 0$.

Proof The birational map $\varphi : \tilde{\mathbb{P}}^3(4) \dashrightarrow \tilde{\mathbb{P}}^3(4)$ we want to consider is most easily described in the language of toric geometry (see e.g. [F2]). Let Δ' in $\mathbb{R}^3$ be the complete simplicial fan with one-dimensional cones $\{\langle v_i \rangle \mid 1 \leq i \leq 4\}$, where

$$v_1 = (1,0,0),\ v_2 = (0,1,0),\ v_3 = (0,0,1),\ v_4 = (-1,-1,-1),$$

corresponding to the toric variety $X_{\Delta'} = \mathbb{P}^3$. Let Δ be the blow-up of Δ' at the four torus-invariant points as described in [F2] section 2.4, so that the toric variety X_Δ associated to Δ is $\tilde{\mathbb{P}}^3(4)$. The fan Δ can be described explicitly as follows: it is the complete fan with one-dimensional cones

$$\{\pm\langle v_i \rangle \mid 1 \leq i \leq 4\}$$

and two-dimensional cones

$$\{\langle v_i, -v_j \rangle \mid 1 \leq i,j \leq 4; i \neq j\} \cup \{\langle v_i, v_j \rangle\ ;\ 1 \leq i < j \leq 4\}.$$

The Picard group of X_Δ is generated by the divisors corresponding to the one-dimensional cones, we will denote the divisor corresponding to the cone $\langle v_i \rangle$ by H_i and the divisor corresponding to the cone $-\langle v_i \rangle$ by E_i. This coincides with the definition of H_i and E_i given above, and these divisors satisfy the three relations

$$\begin{aligned} H := {} & H_1 + E_2 + E_3 + E_4 \\ = {} & H_2 + E_1 + E_3 + E_4 \\ = {} & H_3 + E_1 + E_2 + E_4 \\ = {} & H_4 + E_1 + E_2 + E_3 \end{aligned} \tag{1}$$

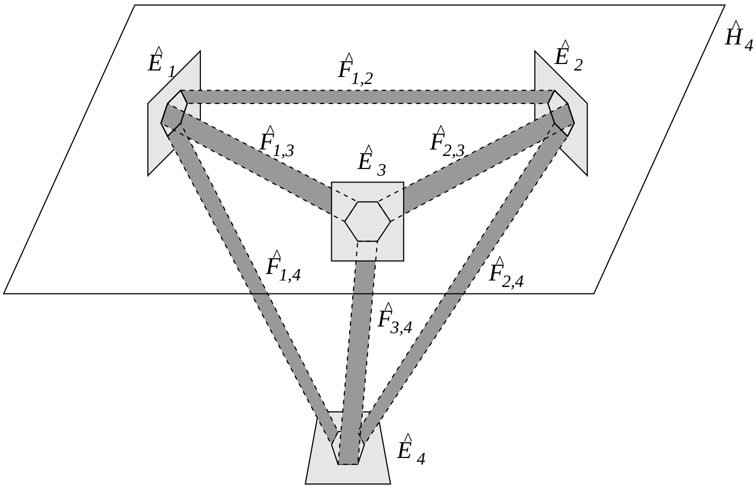
where H denotes the pullback of the hyperplane class under the map $p : \tilde{\mathbb{P}}^3(4) \to \mathbb{P}^3$.

Now denote by $-\Delta$ the fan obtained by mirroring Δ at the origin in $\mathbb{R}^3$. Then, of course, we also have $X_{-\Delta} \cong \tilde{\mathbb{P}}^3(4)$. The map φ we want to consider is now the obvious rational map $\varphi : X_\Delta \dashrightarrow X_{-\Delta}$ which is the identity on the torus $(\mathbb{C}^*)^3$ contained in both X_Δ and $X_{-\Delta}$. Note that the one-dimensional cones of Δ and $-\Delta$ are the same, so that φ is an isomorphism away from a subvariety of $\tilde{\mathbb{P}}^3(4)$ of codimension 2.

In more geometric terms, we can describe φ as the so-called "flip" of the 6 lines $\mathcal{L}$, i.e. one blows up these lines (that have normal bundle $\mathcal{O}(-1) \oplus \mathcal{O}(-1)$ in $\tilde{\mathbb{P}}^3(4)$) to get a variety $\tilde{Y}$ with the 6 exceptional divisors $\hat{F}_{ij} \cong \mathbb{P}^1 \times \mathbb{P}^1$ corresponding to L_{ij}, and then blows down the F_{ij} again with the roles of base and fibre reversed in $\mathbb{P}^1 \times \mathbb{P}^1$. One can write these two steps as in the following diagram:

$$\begin{array}{ccc} & \tilde{Y} & \\ {}^{\varphi_1}\swarrow & & \searrow^{\varphi_2} \\ X_\Delta \cong \tilde{\mathbb{P}}^3(4) & \overset{\varphi}{\dashrightarrow} & X_{-\Delta} \cong \tilde{\mathbb{P}}^3(4). \end{array}$$

The variety $\tilde{Y}$ can be depicted as follows:



Here, we denoted the strict transforms of H_i and E_i under φ_1 by $\hat{H}_i$ and $\hat{E}_i$, respectively. These are all isomorphic to $\tilde{\mathbb{P}}^2(3)$. The divisors $\hat{H}_1$, $\hat{H}_2$, and $\hat{H}_3$ have not been drawn to keep the picture simple.

We now look more closely at the divisors in $\tilde{Y}$. Obviously, we have

$$\begin{aligned}\varphi_1^* H_1 &= \hat{H}_1 + \hat{F}_{23} + \hat{F}_{24} + \hat{F}_{34},\\ \varphi_1^* E_1 &= \hat{E}_1,\end{aligned}$$

and similarly for H_i and E_i with $i = 2,3,4$. The Picard group of $\tilde{Y}$ is the free abelian group generated by the $\hat{H}_i$, $\hat{E}_i$, and $\hat{F}_{ij}$, modulo the three relations induced by (1)

$$\begin{aligned}\hat{H} := \varphi_1^* H &= \hat{H}_1 + \hat{E}_2 + \hat{E}_3 + \hat{E}_4 + \hat{F}_{23} + \hat{F}_{24} + \hat{F}_{34}\\ &= \hat{H}_2 + \hat{E}_1 + \hat{E}_3 + \hat{E}_4 + \hat{F}_{13} + \hat{F}_{14} + \hat{F}_{34}\\ &= \hat{H}_3 + \hat{E}_1 + \hat{E}_2 + \hat{E}_4 + \hat{F}_{12} + \hat{F}_{14} + \hat{F}_{24}\\ &= \hat{H}_4 + \hat{E}_1 + \hat{E}_2 + \hat{E}_3 + \hat{F}_{12} + \hat{F}_{13} + \hat{F}_{23}.\end{aligned} \tag{2}$$

If we now have a stable map in (C, f) in $\tilde{Y}$, we also get stable maps (C_i, f_i) in $\tilde{\mathbb{P}}^3(4)$ by composing f with φ_i for $i = 1,2$. We will now compute the homology classes of these two stable maps.

The homology class of (C_1, f_1) is $\beta = dH' + \sum_i e_i E_i'$ where

$$\begin{aligned}d &= H \cdot \varphi_{1*} f_*[C]\\ &= \hat{H} \cdot f_*[C]\\ &= (\hat{H}_1 + \hat{E}_2 + \hat{E}_3 + \hat{E}_4 + \hat{F}_{23} + \hat{F}_{24} + \hat{F}_{34}) \cdot f_*[C],\\ e_i &= -E_i \cdot \varphi_{1*} f_*[C]\\ &= -\hat{E}_i \cdot f_*[C].\end{aligned}$$

The homology class of (C_2, f_2) is obtained by reversing the roles of $\hat{H}_i$ and $\hat{E}_i$ and substituting $\hat{F}_{12} \leftrightarrow \hat{F}_{34}$, $\hat{F}_{13} \leftrightarrow \hat{F}_{24}$, and $\hat{F}_{14} \leftrightarrow \hat{F}_{23}$, so it is $\beta' = d' H' + \sum_i e'_i E'_i$ where

$$\begin{aligned} d' &= (\hat{E}_1 + \hat{H}_2 + \hat{H}_3 + \hat{H}_4 + \hat{F}_{14} + \hat{F}_{13} + \hat{F}_{12}) \cdot f_*[C] \\ &= (3\hat{H}_1 - 2\hat{E}_1 + \hat{E}_2 + \hat{E}_3 + \hat{E}_4 - \hat{F}_{12} - \hat{F}_{13} - \hat{F}_{14} + 2\hat{F}_{23} + 2\hat{F}_{24} + 2\hat{F}_{34}) \cdot f_*[C] \\ &\qquad \text{(by substituting } \hat{H}_2, \hat{H}_3, \text{ and } \hat{H}_4 \text{ from (2))} \\ &= 3d + 2(e_1 + e_2 + e_3 + e_4) - \underbrace{(\sum_{i<j} F_{ij}) \cdot f_*[C]}_{=\lambda(C_1,f_1)=\lambda(C_2,f_2)=:\lambda}, \\ e'_1 &= -\hat{H}_1 \cdot f_*[C] \\ &= -d - e_2 - e_3 - e_4 + (\hat{F}_{23} + \hat{F}_{24} + \hat{F}_{34}) \cdot f_*[C], \end{aligned}$$

and similarly for e_2, e_3, and e_4. Defining $e = \sum_i e_i$ and $e' = \sum_i e'_i$, we arrive at the equations

$$\begin{aligned} d' &= 3d + 2e - \lambda, \\ e' &= -4d - 3e + 2\lambda. \end{aligned}$$

In particular, we see that $4d' + 2e' = 4d + 2e$ and that, if $d + e \leq 0$, then

$$d' + e' = -d - e + \lambda \geq \lambda \geq 0.$$

□

We now use this map to prove some properties of irreducible stable maps in $\tilde{X} = \tilde{\mathbb{P}}^3(4)$. As already mentioned in section 4, apart from the case where $M_{0,n}(\tilde{X}, \beta)$ is smooth of the expected dimension (case (iii) below), we have to consider the cases where the curves are multiple coverings of one of the L_{ij} (case (i)) and where they are contained in one of the H_i (such that they cannot satisfy any incidence conditions with generic points in $\tilde{X}$, see case (ii)). One of the most important statements of the next lemma is the final conclusion that, although the dimension of the moduli space may be too big, the curves can never satisfy more incidence conditions (with points) as one would expect from the virtual dimension of the moduli space.

Lemma 6.3 *Let $\beta \in A_1(\tilde{X})$ be a homology class such that $M_{0,0}(\tilde{X}, \beta) \neq \emptyset$. Set*

$$n := \frac{1}{2} \text{vdim } \bar{M}_{0,0}(\tilde{X}, \beta) = 2d(\beta) + e(\beta).$$

Then at least one of the following statements holds:

(i) $n = 0$ and $\beta = d H' - d E'_i - d E'_j$ for some $d > 0$, $1 \leq i < j \leq 4$. All curves in $M_{0,0}(\tilde{X}, \beta)$ are contained in L_{ij}.

(ii) $n > 0$*, and for generic points* $Q_1,\dots,Q_n \in \tilde{X}$*, we have*

$$ev_1^{-1}(Q_1)\cap\cdots\cap ev_n^{-1}(Q_n) = \emptyset$$

in $M_{0,n}(\tilde{X},\beta)$*.*

(iii) $n > 0$*,* $\dim M_{0,0}(\tilde{X},\beta) = \operatorname{vdim} \bar{M}_{0,0}(\tilde{X},\beta)$*, and for a generic element* $\mathcal{C} = (C, f) \in M_{0,0}(\tilde{X},\beta)$*,* f *is generically injective,* $\mathcal{C}$ *has no automorphisms, and* $f(C)$ *intersects neither* $\mathcal{L}$ *(which is a disjoint union of 6 smooth rational curves) nor* $\mathcal{H}\cap\mathcal{E}$ *(which is a union of 12 smooth rational curves).*

In particular, it is impossible that $n < 0$*, and in any case we have*

$$ev_1^{-1}(Q_1)\cap\cdots\cap ev_{n'}^{-1}(Q_{n'}) = \emptyset$$

in $M_{0,n'}(\tilde{X},\beta)$ *for generic points* $Q_1,\dots,Q_{n'} \in \tilde{X}$ *if* $n' > n$*.*

Proof Let $(C, f) \in M_{0,0}(\tilde{X},\beta)$ be a stable map, $d = d(\beta)$, $e_i = e_i(\beta)$, $e = \sum_i e_i$, and assume that $\beta \neq 0$ (since otherwise $M_{0,0}(\tilde{X},\beta) = \emptyset$).

If $d = 0$, then $n = e(\beta) > 0$ and $f(C)$ is contained in an exceptional divisor. Then it is clear that for a generic point in $\tilde{X}$, no curve in $M_{0,0}(\tilde{X},\beta)$ meets this point. Therefore, (ii) is satisfied.

Now assume $d > 0$, then we must have $e_i \le 0$ for all i. The curve $f(C)$ cannot be contained at the same time in three of the H_i, since their intersection is empty. This means that there are at least two of the H_i, say H_1 and H_2, in which $f(C)$ is not contained. It follows that

$$d+e_2+e_3+e_4 = \deg\ f^*H_1 \ge 0 \quad \text{and} \quad d+e_1+e_3+e_4 = \deg\ f^*H_2 \ge 0.$$

Since $e_4 \le 0$ and $e_3 \le 0$, this also means that $d+e_2+e_3 \ge 0$ and $d+e_1+e_4 \ge 0$, and therefore $n = 2d+e \ge 0$: the virtual dimension of the moduli space cannot be negative. Moreover, if $n = 0$ then we must have equality everywhere, which means

$$e_1 = -d,\ e_2 = -d,\ e_3 = 0,\ e_4 = 0.$$

Hence we are in case (i), and it is clear that all these curves are d-fold coverings of L_{12}.

It remains to consider the case when $n > 0$. We distinguish four cases.

Case 1: $\beta = dH' - dE_i'$ for $d > 1$ and some $1 \le i \le 4$. Then the curves in $M_{0,0}(\tilde{X},\beta)$ must obviously be d-fold coverings of a line through the exceptional divisor E_i. Those cannot pass through two generic points, however $n = 2d-d = d \ge 2$, hence (ii) is satisfied.

We assume therefore from now on that β is not of this form.

Case 2: $d+e \ge 0$. We show that (iii) is satisfied.

- dim $M_{0,0}(\tilde{X},\beta) = $ vdim $\bar{M}_{0,0}(\tilde{X},\beta)$: This follows because $h^1(C, f^*T_{\tilde{X}}) = 0$ by lemma 4.4 (i).

- the generic element of $M_{0,0}(\tilde{X},\beta)$ has no automorphisms and corresponds to a generically injective map: This follows from lemma 4.8.

- the generic element of $M_{0,0}(\tilde{X},\beta)$ does not intersect $\mathcal{L}$ and $\mathcal{H}\cap\mathcal{E}$: Let L be one of the 18 smooth rational curves in $\mathcal{L}\cup(\mathcal{H}\cap\mathcal{E})$, we will show that the generic element of $M_{0,0}(\tilde{X},\beta)$ does not intersect L. Assume that (C, f) is a stable map in $\tilde{X}$ such that there is a point $x \in C$ with $f(x) = Q \in L$. Consider $\mathcal{C} = (C,x, f)$ as an element of $M = M_{0,1}(\tilde{X},\beta)$. The tangent space to M at the point $\mathcal{C}$ is (see [K] section 1.3.2)

  $$T_{M,\mathcal{C}} = H^0(C, f^*T_{\tilde{X}})/H^0(C,T_C(-x)).$$

  If $Z \subset M$ denotes the substack of those stable maps with $f(x) \in L$, then the tangent space to Z at $\mathcal{C}$ is

  $$T_{Z,\mathcal{C}} = \{s \in T_{M,\mathcal{C}}\ ;\ s(x) \in f^*T_{L,Q}\}.$$

  However, by lemma 4.4 (i) for $\varepsilon = 1$ we see that

  $$h^0(C, f^*T_{\tilde{X}}(-x)) = h^0(C, f^*T_{\tilde{X}}) - 3,$$

  i.e. that the map $H^0(C, f^*T_{\tilde{X}}) \to f^*T_{\tilde{X},Q}$, $s \mapsto s(x)$ is surjective. Therefore the tangent space to Z at $\mathcal{C}$ has smaller dimension than that to M. Since M is smooth at $\mathcal{C}$, it follows that Z has smaller dimension than M at $\mathcal{C}$, proving the statement that the generic element of $M_{0,0}(\tilde{X},\beta)$ does not intersect L.

Case 3: $d+e < 0$ and $e_i = 0$ for some i. Without loss of generality assume that $e_4 = 0$. Since then $0 > d+e = \deg\ f^*(H-E_1-E_2-E_3) = \deg\ f^*H_4$, we conclude that $f(C)$ must be contained in H_4. Hence (ii) is satisfied.

Case 4: $d+e < 0$ and all $e_i \neq 0$. We show that (iii) is satisfied using the Cremona map of lemma 6.2. We use in the following proof the notations of this lemma. Certainly no curve in $M_{0,0}(\tilde{X},\beta)$ is contained in $\mathcal{L}$. So if we decompose $M_{0,0}(\tilde{X},\beta)$ into parts $M_{\vec{\lambda}}$ according to the value of $\vec{\lambda}(C)$ then φ gives injective morphisms from $M_{\vec{\lambda}}$ to $M_{0,0}(\tilde{X},\beta_{\vec{\lambda}})$ with $\beta_{\vec{\lambda}}$ calculated in the proof of lemma 6.2. In particular we have $d(\beta_{\vec{\lambda}})+e(\beta_{\vec{\lambda}}) \geq 0$, so that we can apply the results of case 2 to $M_{0,0}(\tilde{X},\beta_{\vec{\lambda}})$. We therefore have

$$\begin{aligned} \dim M_{\vec{\lambda}} &\leq \dim M_{0,0}(\tilde{X},\beta_{\vec{\lambda}}) \\ &= \operatorname{vdim} \bar{M}_{0,0}(\tilde{X},\beta_{\vec{\lambda}}) \quad \text{by case 2} \\ &= 4d(\beta_{\vec{\lambda}})+2e(\beta_{\vec{\lambda}}) \\ &= 4d(\beta)+2e(\beta) \quad \text{by lemma 6.2} \\ &= \operatorname{vdim} \bar{M}_{0,0}(\tilde{X},\beta). \end{aligned} \tag{1}$$

If $\vec{\lambda} \neq 0$, i.e. if all curves in $M_{\vec{\lambda}}$ intersect $\mathcal{L}$, then the transformed curves in $M_{0,0}(\tilde{X}, \beta_{\vec{\lambda}})$ also have to intersect $\mathcal{L}$. But the generic curve in $M_{0,0}(\tilde{X}, \beta_{\vec{\lambda}})$ does not intersect $\mathcal{L}$ by the results of case 2, so it follows that we must have strict inequality in (1). Since the dimension of $\bar{M}_{0,0}(\tilde{X}, \beta)$ cannot be smaller than its virtual dimension, this means that $M_{\vec{\lambda}}$ is nowhere dense in $M_{0,0}(\tilde{X}, \beta)$ for $\vec{\lambda} \neq \vec{0}$. In other words, $M_{\vec{0}}$ is dense in $M_{0,0}(\tilde{X}, \beta)$, so it obviously suffices to prove (iii) for $M_{\vec{0}}$.

But this is now easy: it follows from the above calculation that the dimension of $M_{\vec{0}}$ is equal to the virtual dimension of $\bar{M}_{0,0}(\tilde{X}, \beta)$. The other statements of (iii) about the generic curves in the moduli space are obviously preserved by the Cremona map φ, so they follow from the fact that the space $M_{0,0}(\tilde{X}, \beta_{\vec{0}})$ has these properties.

This completes the proof that we always have one of the cases (i) to (iii). The statement that $n \geq 0$ has already been proven, and the fact that

$$ev_1^{-1}(Q_1) \cap \cdots \cap ev_{n'}^{-1}(Q_{n'}) = \emptyset$$

in $M_{0,n'}(\tilde{X}, \beta)$ for generic points $Q_1, \ldots, Q_{n'} \in \tilde{X}$ if $n' > n$ follows easily in all cases: for (i) because the image of all curves in the moduli space is contained in an L_{ij}, for (ii) it is trivial, and for (iii) it follows from the Bertini lemma 4.7 (ii). □

To prove enumerative significance for the Gromov-Witten invariants on $\tilde{\mathbb{P}}^3(4)$, we now finally have to consider reducible stable maps. Some numerical examples can be found in 8.3.

Theorem 6.4 *Let $\tilde{X} = \tilde{\mathbb{P}}^3(4)$ and $\beta \in A_1(\tilde{X})$ an effective homology class which is not of the form $dH' - dE'_i - dE'_j$ for some $d \geq 2$ and $i \neq j$. Let $\mathcal{T} = pt^{\otimes n}$, where $n = 2d(\beta) + e(\beta)$. Then $I_\beta(\mathcal{T})$ is enumerative.*

Proof Let $Q_1, \ldots, Q_n$ be generic points in $\tilde{X}$. First we want to show that all points in the intersection

$$I := ev_1^{-1}(Q_1) \cap \cdots \cap ev_n^{-1}(Q_n) \tag{1}$$

on $\bar{M}_{0,n}(\tilde{X}, \beta)$ correspond to irreducible stable maps. To do this, we decompose the moduli space $\bar{M}_{0,n}(\tilde{X}, \beta)$ into the spaces $M_\tau := M(\tilde{X}, \tau)$ according to the topology of the curves and show that $I \cap M_\tau$ is empty for each τ corresponding to reducible curves.

So assume that τ is a topology corresponding to stable maps (C, f) whose irreducible components *that are not contracted by* f are $C_1, \ldots, C_a$. For $1 \leq i \leq a$, let $\beta_i \neq 0$ be the homology class of f on C_i and let n_i be the number of markings on the component C_i.

By a **maximal contracted subscheme** we will mean a maximal connected subscheme of C consisting only of components of C that are contracted by f. A maximal contracted subscheme will be called **marked** if it contains at least one of the marked points. For each $1 \leq i \leq a$, we define ρ_i to be the number of marked maximal contracted subschemes of C that have non-empty intersection with C_i.

We can assume that each maximal contracted subscheme hast at most one marked point, since otherwise the intersection (1) will certainly be empty. This means that each maximal contracted subscheme must have at least two points of intersection with the other components of the curve, since otherwise the prestable map $(C, x_1, \dots, x_n, f)$ would not be stable. We conclude that each marked point that lies in a contracted component (there are $(n - \sum_i n_i)$ of them) must be counted in at least two of the ρ_i:

$$\sum_i \rho_i \geq 2(n - \sum_i n_i). \tag{2}$$

Now there is a morphism

$$\Phi : M_\tau \to M_{0,n_1+\rho_1}(\tilde{X}, \beta_1) \times \cdots \times M_{0,n_a+\rho_a}(\tilde{X}, \beta_a) \tag{3}$$

mapping a stable map $\mathcal{C}$ to its non-contracted components, where on each such component we take as marked points the n_i marked points of $\mathcal{C}$ lying on this component together with the intersection points of the component with the maximal contracted subschemes. We denote by $\Phi_i : M_\tau \to M_{0,n_i+\rho_i}(\tilde{X}, \beta_i)$ the composition of Φ with the projections onto the factors of the right hand side of (3).

We now consider again the intersection I in (1) and show that $\Phi(I \cap M_\tau)$ is empty for all topologies τ but the trivial one, hence showing that $I \cap M_\tau$ is empty. Note that in $\Phi_i(I \cap M_\tau)$ the image point of each of the $n_i + \rho_i$ marked points is fixed to be a certain Q_j. But we have seen in lemma 6.3 that, if $\Phi_i(I \cap M_\tau) \subset M_{0,n_i+\rho_i}(\tilde{X}, \beta_i)$ is non-empty, this requires $n_i + \rho_i$ to be at most $2d(\beta_i) + e(\beta_i)$. Therefore we get

$$\begin{aligned} n \leq 2n - \sum_i n_i &\overset{(2)}{\leq} \sum (n_i + \rho_i) \leq \sum_i (2d(\beta_i) + e(\beta_i)) \\ &= 2d(\beta) + e(\beta) = \frac{1}{2} \operatorname{vdim} \bar{M}_{0,0}(\tilde{X}, \beta) = n. \end{aligned}$$

Hence we must have equality everywhere, which means first of all that $\sum_i n_i = n$ and therefore $\rho_i = 0$ for all i. Moreover, it follows that the number n_i of marked points with prescribed image in $\Phi_i(I \cap M_\tau)$ is equal to $2d(\beta_i) + e(\beta_i)$ for all i, showing that there can be no component of C of type (ii) according to the classification of lemma 6.3 (to be precise, that for all i, C is mapped under Φ_i to a moduli space which is not of type (ii)). If there are only components of type (i), then we have the case that $\beta = dH - dE'_i - dE'_j$ for some $d > 2$ and $i \neq j$ (note that there cannot be two components of type (i) with different (i, j) since the L_{ij} do not intersect). As we excluded this case in the theorem, we conclude that there must be at least one component of C of type (iii). We are going to show that there is in fact only one component which must then necessarily be of type (iii).

We first exclude components of type (i). Note that on each component C_i of type (iii) we impose n_i generic point conditions. Since dim $M_{0,n_i}(\tilde{X}, \beta_i) = 3n_i$, this means by the Bertini lemma 4.7 (ii) that $\Phi_i(I \cap M_\tau) \subset M_{0,n_i}(\tilde{X},\beta_i)$ is zero-dimensional (if not empty). Moreover, if we let $Z_i \subset M_{0,n_i}(\tilde{X},\beta_i)$ be the substack of curves intersecting $\mathcal{L} \cup (\mathcal{H} \cap \mathcal{E})$, then dim $Z_i < 3n_i$ by lemma 6.3, and hence again by Bertini, $\Phi_i(I \cap M_\tau)$ will not intersect Z_i, i.e. the curves in $\Phi_i(I \cap M_\tau)$ do not intersect $\mathcal{L} \cup (\mathcal{H} \cap \mathcal{E})$. This is true for any component of type (iii). Hence, if there were also a component of type (i) which is contained in an L_{ij}, the curve would not be connected, which is impossible. Therefore we can only have components of type (iii).

Assume now that we have at least two components of type (iii). We will again show that these components do not intersect, leading to a contradiction. We define

$$V_1 := \bigcup_{(C,x_1,\dots,x_{n_1},f) \in \Phi_1(I \cap M_\tau)} f(C) \subset \tilde{X},$$

$$V_2 := \bigcup_{i=2}^{a} \bigcup_{(C,x_1,\dots,x_{n_i},f) \in \Phi_i(I \cap M_\tau)} f(C) \subset \tilde{X}.$$

We already remarked that $\Phi_i(I \cap M_\tau)$ is zero-dimensional for all i and corresponds to curves none of which intersects $\mathcal{L} \cup (\mathcal{H} \cap \mathcal{E})$, hence V_1 and V_2 are one-dimensional subvarieties of $\tilde{X} \backslash (\mathcal{L} \cup (\mathcal{H} \cap \mathcal{E}))$. We now define

$$\mathcal{M} := \{\text{diag}\,(v_0, v_1, v_2, v_3) \mid v_i \in \mathbb{C}^*\}/\mathbb{C}^* \subset \text{PGL}\,(3)$$

to be the space of all invertible projective diagonal matrices. Obviously the elements of $\mathcal{M}$ can be considered as automorphisms of $\tilde{\mathbb{P}}^3(4)$ with our choice of the blown-up points. We now consider the map

$$\Psi : V_1 \times \mathcal{M} \to \tilde{X} \backslash (\mathcal{L} \cup (\mathcal{H} \cap \mathcal{E}))$$
$$(Q, \mu) \mapsto \mu(Q)$$

and determine the dimension of its fibres. Fix a point $Q' \in \tilde{X} \backslash (\mathcal{L} \cup (\mathcal{H} \cap \mathcal{E}))$.

- If $Q' \notin \mathcal{H} \cup \mathcal{E}$, then for any $Q \in \tilde{X} \backslash (\mathcal{L} \cup (\mathcal{H} \cap \mathcal{E}))$ there is at most one $\mu \in \mathcal{M}$ such that $\mu(Q) = Q'$ (in fact, there is exactly one such μ if $Q \notin \mathcal{H} \cup \mathcal{E}$ and no such μ otherwise). Therefore the fibre $\Psi^{-1}(Q')$ is one-dimensional (in fact, isomorphic to $V_1 \backslash (\mathcal{H} \cup \mathcal{E})$).

- If $Q' \in H_i$ for some i, then any $Q \in \tilde{X} \backslash (\mathcal{L} \cup (\mathcal{H} \cap \mathcal{E}))$ that can be transformed into Q' by an element of $\mathcal{M}$ must also lie in H_i. In this case, we then have a $\mathbb{C}^*$-family of elements of $\mathcal{M}$ mapping Q to Q'. Since V_1 meets H_i only in finitely many points (otherwise we would be in case (ii) of lemma 6.3), the fibre $\Psi^{-1}(Q')$ is again (at most) one-dimensional.

- If $Q' \in E_i$ for some i, we again get at most one-dimensional fibres by exactly the same reasoning as for the H_i.

We have thus shown that all fibres of Ψ are at most one-dimensional. Hence $\Psi^{-1}(V_2)$ can be at most two-dimensional. But this means that there must be a $\mu \in \mathcal{M}$ such that $V_1 \times \{\mu\} \cap \Psi^{-1}(V_2) = \emptyset$, or in other words such that $\mu(V_1) \cap V_2 = \emptyset$. So if we now transform the prescribed images $Q_i \in \tilde{X}$ of those marked points lying on the component C_1 by μ, this will transform V_1 to $\mu(V_1)$, with the result that the component C_1 does not intersect the others. This would lead to curves that are not connected, which is a contradiction.

So we finally see that only the trivial topology τ corresponding to irreducible curves can contribute to I, and moreover that these irreducible curves are of type (iii) according to lemma 6.3. Hence if we let $Z \subset \bar{M}_{0,n}(\tilde{X}, \beta)$ be the closure of the substack corresponding to irreducible curves and R be the union of the other irreducible components, then by lemma 1.1 we can write

$$[\bar{M}_{0,n}(\tilde{X}, \beta)]^{virt} = [Z] + \text{some cycle supported on } R.$$

But as we have just shown, the intersection I to be considered is disjoint from R, so we can drop this additional cycle and evaluate the intersection on Z. Then it follows from the Bertini lemma 4.7 (iii) that the invariant $I_\beta(\mathcal{T})$ is enumerative, since the generic element of Z has no automorphisms, as shown in lemma 6.3. □

7 Tangency conditions via blow-ups

In this section we will show how to count curves in $X = \mathbb{P}^r$ of given homology class β that intersect a fixed point $P \in X$ with tangent direction in a specified linear subspace of $T_{X,P}$. One would expect that this can be done on the blow-up $\tilde{X}$ of X at P, since the condition that a curve in X has tangent direction in a specified linear subspace of $T_{X,P}$ of codimension k (where $1 \le k \le r-1$) translates into the statement that the strict transform of the curve intersects the exceptional divisor E in a specified k-codimensional projective subspace of $E \cong \mathbb{P}^{r-1}$. As such a k-codimensional projective subspace of E has class $-(-E)^{k+1}$, we would expect that the answer to our problem is

$$I^{\tilde{X}}_{\beta - E'}(\mathcal{T} \otimes -(-E)^{k+1})$$

where $\mathcal{T}$ denotes as usual the other incidence conditions that the curves should satisfy.

We will show in theorem 7.1 that this is in fact the case as long as $k \neq r-1$. However, if $k = r-1$, so that we want to have a fixed tangent direction at P, things get more complicated. This can be seen as follows: consider the invariant $I^X_\beta(\mathcal{T} \otimes pt^{\otimes 2})$ on X, about which we know that it counts the number of curves on X through the classes in $\mathcal{T}$ and through two generic points P and P' in X. We now want to see what happens if P' and P approach each other and finally coincide. Basically, if P' approaches P, there are two possibilities: either the two points x and x' on the curve that are mapped to P and P' also approach each other (left picture), or they do not (right picture):

In the limit $P' \to P$, the curves on the left become curves through P tangent to the limit of the lines $\overline{PP'}$, and those on the right simply become curves intersecting P with global multiplicity two. But the latter we have already counted in theorem 5.3. So we expect in this case

$$\begin{aligned} I^X_\beta(\mathcal{T} \otimes pt^{\otimes 2}) &= (\text{curves through } \mathcal{T} \text{ and through } P \text{ with specified tangent}) \\ &\quad + 2 I^{\tilde{X}}_{\beta - 2E'}(\mathcal{T}) \end{aligned}$$

where the factor two arises because in the right picture, the points x and x' on the curve can be interchanged in the limit where $P = P'$ and $x \neq x'$. This should motivate the results of the following theorem. Some numerical examples can be found in 8.6.

Theorem 7.1 *Let $X = \mathbb{P}^r$ and let $0 \neq \beta \in A_1(X)$ be an effective homology class. Let $P \in X$ be a point, $k \in \{1, \dots, r-1\}$ and W a generic projective subspace of $\mathbb{P}(T_{X,P})$ of codimension k. Let $\mathcal{T} = \gamma_1 \otimes \dots \otimes \gamma_n$ be a collection of effective classes in X such that $\sum_i \operatorname{codim} \gamma_i = \operatorname{vdim} \bar{M}_{0,n}(X, \beta) - r + 1 - k$.*

Then, for generic subschemes $V_i \subset X$ with $[V_i] = \gamma_i$, the number of irreducible stable maps $(C, x_1, \dots, x_{n+1}, f)$ satisfying the conditions

- *f generically injective,*
- *$f_*[C] = \beta$,*
- *$f(x_i) \in V_i$ for all i,*
- *$f(x_{n+1}) = P$,*
- *the tangent direction of f at x_{n+1} lies in W (i.e. if $\tilde{f}: C \to \tilde{X}$ is the strict transform, then $\tilde{f}(x_{n+1}) \in W \subset \mathbb{P}(T_{X,P}) \cong E$),*

is equal to

$$\begin{aligned} &I^{\tilde{X}}_{\beta - E'}(\mathcal{T} \otimes -(-E)^{k+1}) && \text{if } k < r-1, \\ &I^X_\beta(\mathcal{T} \otimes pt^{\otimes 2}) - 2 I^{\tilde{X}}_{\beta - 2E'}(\mathcal{T}) && \text{if } k = r-1, \end{aligned}$$

where each such curve is counted with multiplicity one.

Proof Consider the Gromov-Witten invariant $I^{\tilde{X}}_{\beta-E'}(\mathcal{T}\otimes -(-E)^{k+1})$. We will show that this invariant counts what we want, apart from a correction term in the case $k = r-1$.

As usual, we decompose the moduli space $\bar{M}_{0,n+1}(\tilde{X},\beta-E')$ according to the topology of the curves

$$\bar{M}_{0,n+1}(\tilde{X},\beta-E') = \bigcup_\tau M(\tilde{X},\tau)$$

and determine which parts $M(\tilde{X},\tau)$ give rise to contributions to the intersection

$$ev_1^{-1}(V_1)\cap\cdots\cap ev_n^{-1}(V_n)\cap ev_{n+1}^{-1}(W) \tag{1}$$

on $\bar{M}_{0,n+1}(\tilde{X},\beta-E')$ (note that $[W] = -(-E)^{k+1}$ on $\tilde{X}$).

We use proposition 5.2 (ii) and distinguish the five cases of this proposition. Assume that $M(\tilde{X},\tau)$ satisfies (a). Set $I := ev_1^{-1}(V_1)\cap\cdots\cap ev_n^{-1}(V_n)$ on $\bar{M}_{0,n+1}(X,\beta)$. By the Bertini lemma 4.7 (ii), this intersection is of codimension

$$\begin{aligned}\sum_i \text{codim } V_i &= \text{vdim } \bar{M}_{0,n}(\tilde{X},\beta) - r + 1 - k \\ &= \text{vdim } \bar{M}_{0,n+1}(\tilde{X},\beta-E') - k - 1 \\ &\geq \dim \phi(M(\tilde{X},\tau)) + r - k - 1 \qquad \text{(by (a))} \\ &\geq \dim \phi(M(\tilde{X},\tau)), \qquad \text{(since } k \leq r-1)\end{aligned}$$

where $\phi : M(\tilde{X},\tau) \hookrightarrow \bar{M}_{0,n+1}(\tilde{X},\beta-E') \to \bar{M}_{0,n+1}(X,\beta)$ is the morphism given by the functoriality of the moduli spaces of stable maps. Hence, by Bertini again, $\phi^{-1}(I)$ will be a finite set of points. But since the point x_{n+1} of the curves in $\phi^{-1}(I)$ is not restricted at all, it is actually impossible that $\phi^{-1}(I)$ is finite unless it is empty. So we see that we get no contribution to the intersection (1) from $M(\tilde{X},\tau)$.

Before we look at the cases (b) to (e) of proposition 5.2 (ii), we set $Z = ev_{n+1}^{-1}(E) \subset \bar{M}_{0,n+1}(\tilde{X},\beta-E')$ and decompose Z analogously to $\bar{M}_{0,n+1}(\tilde{X},\beta-E')$ as $Z = \bigcup_\tau Z(\tau)$. Then we obviously have

$$\dim Z(\tau) = \begin{cases} \dim M(\tilde{X},\tau) - 1 & \text{if } x_{n+1} \text{ is on a non-exceptional component of the curve,} \\ \dim M(\tilde{X},\tau) & \text{if } x_{n+1} \text{ is on an exceptional component of the curve.}\end{cases} \tag{2}$$

There are evaluation maps $ev_i : Z(\tau) \to \tilde{X}$ for $1 \leq i \leq n$ and $\widetilde{ev}_{n+1} : Z(\tau) \to E \cong \mathbb{P}^{r-1}$, and the intersection (1) now becomes the intersection

$$ev_1^{-1}(V_1)\cap\cdots\cap ev_n^{-1}(V_n)\cap \widetilde{ev}_{n+1}^{-1}(W), \tag{3}$$

on $Z(\tau)$, where $V_i \subset \tilde{X}$ and $W \subset \mathbb{P}^{r-1}$ are chosen generically.

We now continue to look at the cases (b) to (e) of proposition 5.2 (ii). If $M(\tilde{X},\tau)$ satisfies (b), then the intersection (3) will be empty by Bertini, since

$$\begin{aligned}\sum_i \text{codim } \gamma_i + \text{codim } W &= \text{vdim } \bar{M}_{0,n}(X,\beta) - r + 1 \\ &= \text{vdim } \bar{M}_{0,n+1}(\tilde{X},\beta - E') - 1 \\ &\geq \dim M(\tilde{X},\tau) + 1 \qquad \text{(by (b))} \\ &\geq \dim Z(\tau) + 1. \qquad \text{(by (2))}\end{aligned}$$

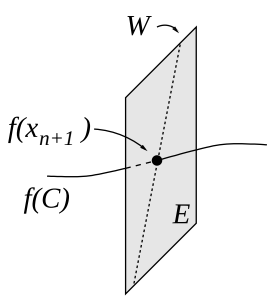
Similarly, this follows for (c): because of $\eta(\tau) = 0$ we have no exceptional component, hence we must have the first possibility in (2), i.e.

$$\begin{aligned}\sum_i \text{codim } \gamma_i + \text{codim } W &= \text{vdim } \bar{M}_{0,n+1}(\tilde{X},\beta - E') - 1 \\ &\geq \dim M(\tilde{X},\tau) \qquad \text{(by (c))} \\ &\geq \dim Z(\tau) + 1. \qquad \text{(by (2))}\end{aligned}$$

Hence we are only left with the cases (d) and (e). In case (d) we must have the first possibility in (2) since the curve is irreducible, hence

$$\begin{aligned}\sum_i \text{codim } \gamma_i + \text{codim } W &= \text{vdim } \bar{M}_{0,n+1}(\tilde{X},\beta - E') - 1 \\ &= \dim M(\tilde{X},\tau) - 1 \qquad \text{(by (d))} \\ &= \dim Z(\tau). \qquad \text{(by (2))}\end{aligned}$$

The intersection (3) is transverse and finite by Bertini. Moreover, the dimension of $M(\tilde{X},\tau)$ coincides with vdim $\bar{M}_{0,n+1}(\tilde{X},\beta - E')$, and there are no obstructions on $\bar{M}(\tilde{X},\tau)$ by lemma 4.4 (i). Hence, using lemma 1.1 in the same way as we did in the proof of theorem 5.3, we see that we get a contribution to the Gromov-Witten invariant $I^{\tilde{X}}_{\beta-E'}(\mathcal{T} \otimes -(-E)^{k+1})$ from exactly the curves we wanted. One can depict these curves as follows:



Note that, by corollary 3.2, in the case $k = r - 1$ we have

$$I^{\tilde{X}}_{\beta-E'}(\mathcal{T} \otimes -(-E)^r) = I^{\tilde{X}}_{\beta-E'}(\mathcal{T} \otimes pt) = I^X_\beta(\mathcal{T} \otimes pt^{\otimes 2}).$$

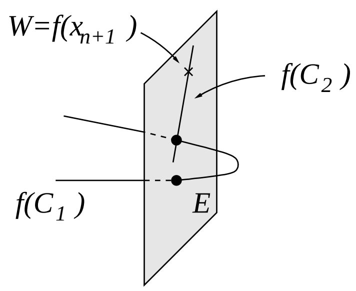
It remains to look at case (e). There we have

$$\begin{aligned}\sum_i \text{codim } \gamma_i + \text{codim } W &= \text{vdim } \bar{M}_{0,n+1}(\tilde{X}, \beta - E') - 1 \\ &= \dim M(\tilde{X}, \tau) \qquad \text{(by (e))} \\ &\geq \dim Z(\tau). \qquad \text{(by (2))}\end{aligned}$$

Note that again there are no obstructions on $\bar{M}(\tilde{X}, \tau)$ by lemma 5.1.

Hence, to get a non-zero contribution from (e) to the intersection (3), we must have equality in the last line, which fixes the component where x_{n+1} lies. We thus have reducible curves with exactly two components, one component C_1 with marked points $x_1, \ldots, x_n$ and homology class $\beta - 2E'$, and the other component C_2 with marked point x_{n+1} and homology class E'. Moreover, the intersection (3) must be transverse and finite by Bertini. But this is only possible if $k = r - 1$, since the only conditions on the exceptional line C_2 are that it has to intersect C_1 and that x_{n+1} maps to W, and this cannot fix C_2 uniquely unless W is a point, i.e. $k = r - 1$. This finishes the proof of the theorem in the case $k < r - 1$.

In the case $k = r - 1$, we have just shown that the curves in the intersection (3) look as follows:



Here, one has to show that the generic curve of homology class $\beta - 2E'$ intersects the exceptional divisor twice, and not only once with multiplicity two. But this is easy to see: irreducible curves of homology class $\beta - 2E'$ intersecting the exceptional divisor once with multiplicity two correspond via strict transform to curves of homology class β in $\mathbb{P}^r$ having a cusp at P. For maps $f : \mathbb{P}^1 \to X = \mathbb{P}^r$ it is however easy to see that the requirement that a specified point $x \in \mathbb{P}^1$ is mapped to P and that $df(x) = 0$ imposes $2r$ independent conditions, so the space of irreducible stable maps of homology class β with a cusp at P has dimension

$$\dim M_{0,1}(X, \beta) - 2r = \dim M_{0,0}(\tilde{X}, \beta - 2E') - 1,$$

so the generic curve in $\tilde{X}$ of homology class $\beta - 2E'$ does indeed intersect the exceptional divisor twice and looks as in the picture above.

Therefore, to get the correct enumerative answer, we have to subtract the contribution from this case (e). But this is easily done, since we now know that this contribution is twice the number of curves of homology class $\beta - 2E'$ satisfying the conditions $\mathcal{T}$ (the factor two arises since the component C_2 can be attached to both points of intersection of the component $f(C_1)$ with E). By theorem 5.3, we know that this number is $I^{\tilde{X}}_{\beta-2E'}(\mathcal{T})$. This finishes the proof also in the case $k = r-1$. □

One can of course ask whether the analogue of theorem 7.1 is true also for several tangency conditions at different points. As imaginable from our work in this chapter, the answer in general is no, and the problems arising here are essentially the same as those discussed in the previous sections when considering multiple blow-ups.

However, as (most) invariants on $\tilde{\mathbb{P}}^2(s)$ are enumerative by [GP], one can expect an analogue of theorem 7.1 in this case. Indeed, numerical calculations show that this seems to be true: if one calculates with these methods what should be the number of rational curves in $\mathbb{P}^2$ tangent to c general lines at c fixed points, and intersecting additional a general points, one obtains exactly the numbers $N(a,0,c)$ of Ernström and Kennedy [EK] that have been computed by completely different methods and shown to be enumeratively correct.

8 Numerical examples

Example 8.1 *Gromov-Witten invariants on* $\tilde{\mathbb{P}}^2(1)$

According to theorem 5.3, the Gromov-Witten invariants $I^{\tilde{\mathbb{P}}^2(1)}_{dH'+eE'}(pt^{\otimes(3d+e-1)})$ for $d > 0$ are equal to the numbers of degree d plane rational curves meeting $3d+e-1$ generic points in the plane, and in addition passing through a fixed point in $\mathbb{P}^2$ with global multiplicity $-e$. All these curves are counted with multiplicity one. Some of the invariants are listed in the following table.

	$d=1$	$d=2$	$d=3$	$d=4$	$d=5$	$d=6$	$d=7$
$e=0$	1	1	12	620	87304	26312976	14616808192
$e=-1$	1	1	12	620	87304	26312976	14616808192
$e=-2$	0	0	1	96	18132	6506400	4059366000
$e=-3$	–	0	0	1	640	401172	347987200
$e=-4$	–	0	0	0	1	3840	7492040
$e=-5$	–	0	0	0	0	1	21504
$e=-6$	–	–	0	0	0	0	1

The equality of the first two lines follows from the geometric meaning of the invariants (see theorem 5.3) as well as from corollary 3.2. In [GP], L. Göttsche and R. Pandharipande also compute the numbers given here, together with those for blow-ups of $\mathbb{P}^2$ in any number of points, and they prove the enumerative significance of all these numbers if the prescribed multiplicity in at least one of the blown-up points is one or two. The numbers for $e=-2$ have been computed earlier by different methods in [P]. The fact that $I^{\tilde{\mathbb{P}}^2(1)}_{dH'-(d-1)E'}(pt^{\otimes 2d})=1$ can also be understood geometrically: a curve C of degree d in $\mathbb{P}^2$ passing with multiplicity $d-1$ through a point P has genus

$$\frac{1}{2}(d-1)(d-2)-\frac{1}{2}(d-1)(d-2)=0,$$

i.e. it is always a rational curve. Hence the space of degree d rational curves with a $(d-1)$-fold point in P is simply a linear system of the expected dimension, showing that the corresponding Gromov-Witten invariant must be 1.

Example 8.2 *Gromov-Witten invariants on* $\tilde{\mathbb{P}}^3(1)$

As in the previous example, the Gromov-Witten invariants $I^{\tilde{\mathbb{P}}^3(1)}_{dH'+eE'}(pt^{\otimes(2d+e)})$ for $d>0$ are equal to the numbers of degree d rational curves in $\mathbb{P}^3$ meeting $2d+e$ generic points, and in addition passing through a fixed point in $\mathbb{P}^3$ with global multiplicity $-e$.

	$d=1$	$d=2$	$d=3$	$d=4$	$d=5$	$d=6$	$d=7$	$d=8$
$e=0$	1	0	1	4	105	2576	122129	7397760
$e=-1$	1	0	1	4	105	2576	122129	7397760
$e=-2$	0	0	0	0	12	384	23892	1666128
$e=-3$	–	0	0	0	0	0	620	72528
$e=-4$	–	0	0	0	0	0	0	0

Example 8.3 *Gromov-Witten invariants on* $\tilde{\mathbb{P}}^3(2)$

By theorem 6.4, the numbers $I^{\tilde{\mathbb{P}}^3(2)}_{dH'+e_1E'_1+e_2E'_2}(pt^{\otimes(2d+e_1+e_2)})$ for $d>0$ are enumerative unless $d>2$, $e_1=-d$, $e_2=-d$ (for those cases, see proposition 8.5). This means that they are equal to the numbers of degree d rational curves in $\mathbb{P}^3$ meeting $2d+e_1+e_2$ generic points in $\mathbb{P}^3$, and in addition passing through two fixed points with global multiplicities $-e_1$ and $-e_2$, respectively.

(e_1,e_2)	$d=2$	$d=3$	$d=4$	$d=5$	$d=6$	$d=7$	$d=8$	$d=9$
$(-2,-2)$	1/8	0	0	1	48	4374	360416	39100431
$(-3,-2)$	–	0	0	0	0	96	14040	2346168
$(-3,-3)$	–	1/27	0	0	0	1	384	119134
$(-4,-2)$	–	0	0	0	0	0	0	18132
$(-4,-3)$	–	–	0	0	0	0	0	640
$(-4,-4)$	–	–	1/64	0	0	0	0	1

The numbers with one of the $e_i=-1$ can be obtained from corollary 3.2 and example 8.2.

Example 8.4 *Gromov-Witten invariants on* $\tilde{\mathbb{P}}^4(2)$

The invariants $I^{\tilde{\mathbb{P}}^4(2)}_{dH'+e_1E'_1+e_2E'_2}(\,\cdot\,)$ for $d > 0$ are enumerative if only one of the blown-up points is involved (i.e. if one of the e_i is zero) or if one of the e_i is equal to -1 (by corollary 3.2). It has already been mentioned that in almost all other cases, the invariants are not enumerative. As examples, we list in the following table some invariants $I^{\tilde{\mathbb{P}}^4(2)}_{dH'+e_1E'_1+e_2E'_2}(\mathcal{T})$ where $\mathcal{T} = pt^{\otimes a} \otimes (H^2)^{\otimes b}$ with $a \geq 0$, $0 \leq b \leq 2$ being the unique numbers such that $5d+3e_1+3e_2+1 = 3a+b$.

(e_1,e_2)	$d=2$	$d=3$	$d=4$	$d=5$	$d=6$	$d=7$	$d=8$
$(-1,-1)$	1	0	1	161	270	831	1351863
$(-2,-1)$	0	0	0	9	16	105	233040
$(-2,-2)$	–	$1/4$	0	$5/4$	$9/4$	$29/2$	$154683/4$
$(-3,-1)$	–	0	0	0	0	0	2625
$(-3,-2)$	–	0	0	0	$3/4$	1	$2533/2$
$(-3,-3)$	–	–	$1/27$	$13/108$	$-1/12$	$-1/54$	$32471/108$
$(-4,-1)$	–	0	0	0	0	0	0
$(-4,-2)$	–	–	0	0	0	0	16

Example 8.5 *Non-enumerative invariants on* $\tilde{\mathbb{P}}^3(4)$

We have seen in theorem 6.4 that the only non-enumerative invariants on $\tilde{\mathbb{P}}^3(4)$ involving only point classes are those of the form $I_{dH'-dE'_1-dE'_2}(1)$ for $d \geq 2$ (where the 1 is to be understood as an element of $A^*(\tilde{X})^{\otimes 0}$, i.e. there are no cohomology classes in the invariant). We will now explicitly compute these invariants and discuss their meaning.

Let $\tilde{X} = \tilde{\mathbb{P}}^3(2)$. Let L be the strict transform of the line joining the two blown-up points, its normal bundle in $\tilde{X}$ is $\mathcal{O}(-1)\oplus\mathcal{O}(-1)$. If we let $\beta = dH' - dE'_1 - dE'_2$ for some $d \geq 2$, then stable maps of homology class β correspond to degree d coverings of L. In fact, the moduli space $\bar{M}_{0,0}(\tilde{X},\beta)$ of these coverings is equal to $\bar{M}_{0,0}(\mathbb{P}^1,d)$ and has dimension $2d-2$. Applying [BF] proposition 7.3 we see that the Gromov-Witten invariant $I^{\tilde{\mathbb{P}}^3(2)}_{dH'-dE'_1-dE'_2}(1)$ is equal to the integral

$$\int_{\bar{M}_{0,0}(\mathbb{P}^1,d)} c_{2d-2}\left(R^1\pi_* f^*(\mathcal{O}(-1)\oplus\mathcal{O}(-1))\right)$$

where $\pi : \bar{M}_{0,1}(\mathbb{P}^1,d) \to \bar{M}_{0,0}(\mathbb{P}^1,d)$ is the universal curve and $f : \bar{M}_{0,1}(\mathbb{P}^1,d) \to \mathbb{P}^1$ the evaluation map. One can see that this does not depend on $\tilde{X}$ any more, but just on the normal bundle of L.

Before we do the actual computation — the integral will turn out to be d^{-3} — one should note that this number has some history. Its most important application is the case of a quintic threefold Q, where rigid rational curves (of any degree) also have normal bundle $\mathcal{O}(-1)\oplus\mathcal{O}(-1)$. All methods to compute the numbers of rational curves of a given degree on Q will determine the degree of the zero-cycle

$[\bar{M}_{0,0}(Q,\beta)]^{virt} \in A_0(\bar{M}_{0,0}(Q,\beta))$, but this number counts not only the number of rational curves of class β, but also d-fold covering maps of all rational curves of class β/d. Knowing that these multiple coverings are counted with multiplicity d^{-3}, one can then subtract them from the degree of the zero-cycle $[\bar{M}_{0,0}(Q,\beta)]^{virt}$ to get the actual number of rational curves of degree β on Q.

When the numbers of rational curves on the quintic threefold had been computed first by physicists [COGP], they just guessed the multiplicity d^{-3} because it was the only one that turned their predictions of the number of rational curves into non-negative integers. Later, Yu. Manin [M] and independently P. Aspinwall and D. Morrison [AM] (using an a priori different definition of the multiplicity) derived this multiplicity rigorously, however their methods are very complicated. We can now give a remarkably simple way to compute it as a byproduct of our work on Gromov-Witten invariants of blow-ups.

To compute the invariant, we use the equation $\mathcal{E}_{\beta+E_1'}(1\,;\,H,H\mid E_1,E_1^2)$. The only possibilities how the homology class $\beta+E_1' = d\,H' - (d-1)\,E_1' - d\,E_2'$ can split up into two effective classes are

$$\beta_1 = d_1 H' - d_1 E_1' - d_1 E_2', \qquad \beta_2 = d_2 H' - (d_2-1)\,E_1' - d_2 E_2'$$

for $d_1+d_2 = d$ and $d_1,d_2 \geq 0$. First we look at the invariants with homology class β_2 and claim that they all vanish for $d_2 \geq 2$. The virtual dimension of $\bar{M}_{0,0}(\tilde{X},\beta_2)$ is 2, so we have to impose two conditions on the curves we are counting. It is easy to see that all stable maps with homology class β_2 are reducible, such that one component maps to a line in the exceptional divisor $E_1 \cong \mathbb{P}^2$, and all the others into L. This means that no such curve can intersect the strict transform of a general line in $\tilde{\mathbb{P}}^3(2)$ or of a general line through P_2, and hence $I_{\beta_2}(\mathcal{T})$ vanishes whenever $\mathcal{T}$ contains one of the classes H^2, E_2^2, and pt. But also no such curve can intersect *two* strict transforms of general lines in $\tilde{\mathbb{P}}^3(2)$ through P_1, so we also have $I_{\beta_2}((H^2-E_1^2)^{\otimes 2}) = 0$. Hence, by the multilinearity of the Gromov-Witten invariants it follows that all invariants with homology class β_2 vanish for $d_2 \geq 2$.

The equation $\mathcal{E}_{\beta+E_1'}(1\,;\,H,H\mid E_1,E_1^2)$ reduces therefore to the simple statement

$$\begin{aligned} 0 = {} & I_{dH'-dE_1'-dE_2'}(H\otimes H\otimes E_1)\, \underbrace{I_{E_1'}(E_1\otimes E_1^2\otimes E_1^2)}_{=-1} \\ & - I_{(d-1)H'-(d-1)E_1'-(d-1)E_2'}(H\otimes E_1\otimes E_1)\, I_{H'-E_2'}(H\otimes E_1^2\otimes E_1^2). \end{aligned}$$

The invariant $I_{H'-E_2'}(H\otimes E_1^2\otimes E_1^2)$ is easily computed to be -1, e.g. using the algorithm 2.5. Hence, by the divisor axiom we get

$$d^3 I_{dH'-dE_1'-dE_2'}(1) = (d-1)^3 I_{(d-1)H'-(d-1)E_1'-(d-1)E_2'}(1).$$

Together with $I_{H'-E_1'-E_2'}(1) = 1$ (which follows for example from corollary 3.2), we see that

$$I_{dH'-dE_1'-dE_2'}(1) = d^{-3}.$$

It should be noted that our additional considerations above to prove the vanishing of Gromov-Witten invariants of homology class $d_2 H' - (d_2 - 1) E_1' - d_2 E_2'$ for $d_2 > 0$ would not have been necessary to compute the desired invariants, they just made the calculation easier. According to theorem 2.1, we could of course also use the algorithm 2.5 without further thinking, and everything would take care of itself.

Example 8.6 *Curves with tangency conditions*

The following table shows some of the numbers

$$N_{r,k,d,\mathcal{T}} = \begin{cases} I^{\tilde{\mathbb{P}}^r(1)}_{dH'-E'}(\mathcal{T} \otimes -(-E)^{k+1}) & \text{if } k < r-1 \\ I^{\mathbb{P}^r}_{dH'}(\mathcal{T} \otimes pt^{\otimes 2}) - 2 I^{\tilde{\mathbb{P}}^r(1)}_{dH'-2E'}(\mathcal{T}) & \text{if } k = r-1 \end{cases}$$

which are according to theorem 7.1 equal to the numbers of curves in $\mathbb{P}^r$ of degree d through generic subspaces of $\mathbb{P}^r$ according to $\mathcal{T}$, and intersecting a fixed point $P \in \mathbb{P}^r$ with tangent direction contained in a given linear subspace of $T_{\mathbb{P}^r,P}$ of codimension k.

(r,k)	$\mathcal{T}$	$d=2$	$d=3$	$d=4$	$d=5$	$d=6$	$d=7$
$(2,1)$	$pt^{\otimes(3d-3)}$	1	10	428	51040	13300176	6498076192
$(3,1)$	$pt^{\otimes(2d-2)} \otimes H^2$	1	3	28	485	14376	639695
$(3,2)$	$pt^{\otimes(2d-2)}$	0	1	4	81	1808	74345

The numbers in the first row have already been computed by L. Ernström and G. Kennedy [EK] by different methods.

9 Blow-ups of subvarieties

In the last section of this chapter we will discuss two examples of blow-ups of $\mathbb{P}^r$ along higher-dimensional subvarieties, leading to well-known classical results about multisecants of space curves and abelian surfaces in $\mathbb{P}^4$, respectively.

Example 9.1 *Blow-ups of curves in* $\mathbb{P}^3$

Let $X = \mathbb{P}^3$ and $Y \subset X$ be a smooth curve of degree d and genus g. Let $\tilde{X}$ be the blow-up of X along Y. We are going to compute the Gromov-Witten invariants

$$q := I^{\tilde{X}}_{H'-4E'}(1) \quad \text{and} \quad t := I^{\tilde{X}}_{H'-3E'}(H^2)$$

where E' is the class of a fibre over a point in Y. Irreducible curves of homology class $H' + eE'$ for $e < 0$ obviously correspond to lines in Y intersecting the curve Y with multiplicity $-e$, i.e. to $(-e)$-secants of Y. Hence, we expect t to be the number of 3-secants of Y intersecting a fixed line and q to be the number of 4-secants of Y. It is however not at all clear that this interpretation is valid, and indeed in some cases it is not, since there are e.g. space curves with infinitely many 4-secants. We will be able to see this already from the result since the numbers t and q can well be negative.

Nevertheless, t and q can be regarded to be the "virtual" number of 3-secants through a line and 4-secants, respectively. These (virtual) numbers have already been computed classically — the computation goes back to Cayley (1863). Some more recent work on this topic has been done by Le Barz [L]. We will see that the numbers we obtain by Gromov-Witten theory are the same, although it is not clear that, in the case where there are infinitely many such multisecants, the classical and the Gromov-Witten definition of the "virtual number" agree.

Of course, the algorithms we developed so far do not tell us how to compute the numbers, so we will sketch here a possible way to calculate them.

Step 1: Intersection ring. (This can be computed easily using the methods of [F1].) The ring structure of $A^*(\tilde{X})$ is determined by $A^1(\tilde{X}) = \langle H,E\rangle$ and $A^2(\tilde{X}) = \langle H^2,F\rangle$ (where E is the exceptional divisor and F is the Poincaré dual of the homology class E' introduced above) and the following non-zero intersection products involving at least one exceptional class:

$$\begin{aligned} E\cdot E &= (4d+2g-2)F - d\,H^2, \\ E\cdot H &= d\,F, \\ E\cdot F &= -pt. \end{aligned}$$

Step 2: Invariants with homology class $\beta = eE'$, $e>0$. Since these curves have to be contained in the exceptional divisor, the invariants $I_{eE'}(\mathcal{T})$ are certainly zero if $\mathcal{T}$ contains a non-exceptional class. By the divisor axiom, the only independent classes to compute are therefore $I_{eE'}(F^{\otimes e})$. The curves that are counted there must be e-fold coverings of a fibre over a point in Y, so this invariant is zero for $e\geq 2$ since we then require the curve to lie in two different fibres. Finally, the geometric statement that $I_{E'}(H^2-F)=1$ (we count curves that are a fibre over a point in Y, and the condition H^2-F fixes the point) means that $I_{E'}(F)=-1$.

Step 3: Invariants with homology class $\beta = H'$. For geometric reasons, the invariant $I_{H'}(\mathcal{T})$ is zero if $\mathcal{T}$ contains an exceptional class and coincides with the corresponding one on $\mathbb{P}^3$ otherwise, i.e.

$$I_{H'}((H^2)^{\otimes 4}) = 2, \quad I_{H'}((H^2)^{\otimes 2}\otimes pt) = 1, \quad I_{H'}(pt^{\otimes 2}) = 1.$$

Step 4: Invariants with homology class $\beta = H' + eE'$, $e<0$. The main equation that we use is $\mathcal{E}_{H'+(e+1)E'}(\mathcal{T}\,;\,H,H\mid E,E)$ for $e<0$. Assume that $\mathcal{T}$ contains no divisor classes. Let α be the number of classes F in $\mathcal{T}$ and assume further that $\alpha + e \neq 0$. Then the equation reads after some ordering of the terms

$$I_{H'+eE'}(\mathcal{T}) = \frac{1}{\alpha+e}\Big((2g-2+(6+2e)d)I_{H'+(e+1)E'}(\mathcal{T}\otimes F) + ((e+1)^2-d)I_{H'+(e+1)E'}(\mathcal{T}\otimes H^2)\Big).$$

We now list the results in the order they can be computed recursively (and state the equations used to compute the invariant in the cases where $\alpha + e = 0$ such that the above equation is not applicable).

$$\begin{aligned}
I_{H'-E'}((H^2)^{\otimes 3}) &= 2d, \\
I_{H'-E'}(H^2 \otimes pt) &= d, \\
I_{H'-E'}(\mathcal{T}\otimes F^{\otimes 2}) &= 0 \qquad \text{for any } \mathcal{T}, \\
I_{H'-E'}(F\otimes H^2\otimes H^2) &= 1 \qquad \text{using } \mathcal{E}_{H'}(H^2\otimes H^2\,;\, H,H \mid E,F), \\
I_{H'-E'}(F\otimes pt) &= 1 \qquad \text{using } \mathcal{E}_{H'}(pt\,;\, H,H \mid E,F), \\
I_{H'-2E'}(H^2\otimes H^2) &= d(d-2)+1-g, \\
I_{H'-2E'}(pt) &= \frac{d(d-3)}{2}+1-g, \\
I_{H'-2E'}(F\otimes H^2) &= d-1, \\
I_{H'-2E'}(F\otimes F) &= 1 \qquad \text{using } \mathcal{E}_{H'-E'}(F\,;\, H,H \mid E,F), \\
I_{H'-3E'}(H^2) &= \boxed{t = \frac{(d-1)(d-2)(d-3)}{3} - g(d-2),} \\
I_{H'-3E'}(F) &= \frac{(d-1)(d-4)}{2}+1-g, \\
I_{H'-4E'}(1) &= \boxed{q = \frac{1}{12}(d-2)(d-3)^2(d-4) - \frac{g}{2}(d^2-7d+13-g).}
\end{aligned}$$

The numbers t and q coincide with the classical ones stated in [L].

Example 9.2 *Blow-up of an abelian surface in* $\mathbb{P}^4$

In analogy to example 9.1 we will now blow up an abelian surface Y of degree 10 in $X = \mathbb{P}^4$. The invariant $I_{H'-6E'}(1)$, where E' again denotes the fibre over a point in Y, is expected to be the number of 6-secants of the abelian variety, which is known to be 25. One can show that this is indeed the case. Since the calculation is very similar to the one in 9.1, we will sketch only very briefly the steps to obtain the result.

Step 1: Intersection ring. Assume that Y is generic such that $A^1(Y)$ is one-dimensional. Let $\alpha \in A^1(Y)$ be a hyperplane section of Y. Define $\gamma = j_* g^* \alpha$, where $j : E \to \tilde{X}$ is the inclusion and $g : E \to Y$ the projection. Let F be the Poincaré dual of E' introduced above. Then $A^*(\tilde{X})$ is determined by

$$A^1(\tilde{X}) = \langle H, E\rangle,\ A^2(\tilde{X}) = \langle H^2, \gamma\rangle,\ A^3(\tilde{X}) = \langle H^3, F\rangle$$

and the following non-zero intersection products involving at least one of the exceptional classes:

$$\begin{aligned}
E\cdot E &= 5\gamma - 10H^2,\\
E\cdot H &= \gamma,\\
E\cdot\gamma &= 50F - 10H^3,\\
E\cdot H^2 &= 10F,\\
E\cdot F &= -pt,\\
\gamma\cdot\gamma &= -10pt,\\
\gamma\cdot H &= 10F.
\end{aligned}$$

Step 2: Initial data for the recursion. The invariants with homology class H' again coincide with those on $\mathbb{P}^4$ or are zero if they contain an exceptional cohomology class. Invariants with homology class eE' are zero for $e \geq 2$, and the relevant invariants for $e = 1$ are $I_{E'}(F) = -1$ and $I_{E'}(\gamma\otimes\gamma) = 10$.

Step 3: Recursion relations. To determine an invariant $I_{H'+eE'}(\mathcal{T})$ for $e < 0$, use the following equations:

- If $\mathcal{T}$ contains a class F, use equation $\mathcal{E}_{H'+(e+1)E'}(\mathcal{T}' \,;\, H,H \mid E,F)$, where $\mathcal{T}'$ is defined by $\mathcal{T} = \mathcal{T}'\otimes F$.
- If $\mathcal{T}$ contains a class γ, use equation $\mathcal{E}_{H'+(e+1)E'}(\mathcal{T}' \,;\, H,H \mid \gamma,E)$, where $\mathcal{T}'$ is defined by $\mathcal{T} = \mathcal{T}'\otimes\gamma$.
- If $\mathcal{T}$ contains no exceptional class, use $\mathcal{E}_{H'+(e+1)E'}(\mathcal{T} \,;\, H,H \mid E,E)$.

Using these equations, one can determine the invariants recursively for decreasing values of e and finally obtain $I_{H'-6E'}(1) = 25$.

It should be remarked that this calculation can be done for any surface in $\mathbb{P}^4$. The computations can then still be done in the same way, however they get of course much more complicated since they will involve the numerical invariants of the surface.

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