THE CAPORASO-HARRIS FORMULA AND PLANE RELATIVE GROMOV-WITTEN INVARIANTS IN TROPICAL GEOMETRY

ANDREAS GATHMANN AND HANNAH MARKWIG

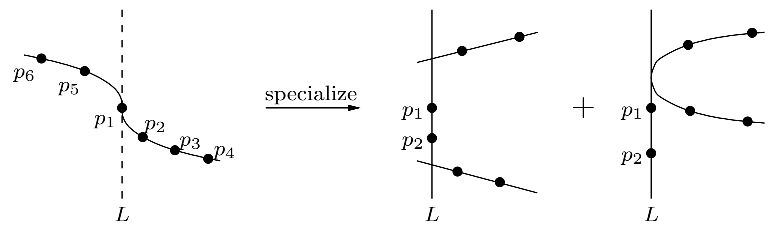
Abstract. Some years ago Caporaso and Harris have found a nice way to compute the numbers $N(d,g)$ of complex plane curves of degree d and genus g through $3d+g-1$ general points with the help of relative Gromov-Witten invariants. Recently, Mikhalkin has found a way to reinterpret the numbers $N(d,g)$ in terms of tropical geometry and to compute them by counting certain lattice paths in integral polytopes. We relate these two results by defining an analogue of the relative Gromov-Witten invariants and rederiving the Caporaso-Harris formula in terms of both tropical geometry and lattice paths.

1. Introduction

Let $N(d,g)$ be the number of complex curves of degree d and genus g in the projective plane $\mathbb{P}^2$ through $3d+g-1$ general fixed points. These numbers have first been computed by Caporaso and Harris [CH98]. Their strategy is to define "relative Gromov-Witten invariants" that count plane curves of given degree and genus having specified local contact orders to a fixed line L and passing in addition through the appropriate number of general points. By specializing one point after the other to lie on L they derive recursive relations among these relative Gromov-Witten invariants that finally suffice to compute all the numbers $N(d,g)$.

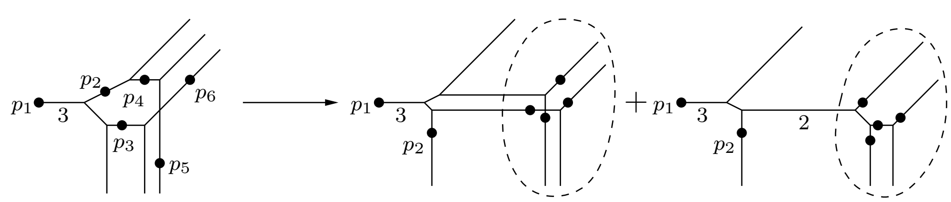
As an example of what happens in this specialization process we consider plane rational cubics having a point of contact order 3 to L at a fixed point $p_1 \in L$ and passing in addition through 5 general points $p_2,\dots,p_6 \in \mathbb{P}^2$. To compute the number of such curves we move p_2 to L. What happens to the cubics under this specialization? As they intersect L already with multiplicity 3 at p_1 they cannot pass through another point on L unless they become reducible and have L as a component. There are two ways how this can happen: they can degenerate into a union of three lines $L \cup L_1 \cup L_2$ where L_1 and L_2 each pass through two of the points $p_3,\dots,p_6$, or they can degenerate into $L \cup C$, where C is a conic tangent to L and passing through $p_3,\dots,p_6$:

2000 Mathematics Subject Classification: Primary 14N35, 52B20, Secondary 14N10, 51M20.
The second author has been funded by the DFG grant Ga 636/2.



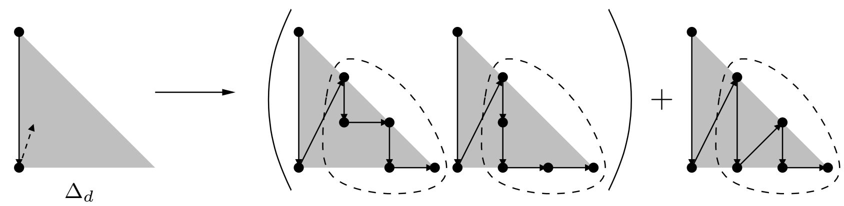
The initial number of rational cubics with a point of contact order 3 to L at a fixed point and passing through 5 more general points is therefore a sum of two numbers (counted with suitable multiplicities) related to only lines and conics. This is the general idea of Caporaso and Harris how specialization finally reduces the degree of the curves and allows a recursive solution to compute the numbers $N(d, g)$.

Recently, Mikhalkin found a different way to compute the numbers $N(d, g)$ using tropical geometry [Mik03]. He proved the so-called "Correspondence Theorem" that asserts that $N(d, g)$ can be reinterpreted as the number of *tropical* plane curves of degree d and genus g through $3d + g - 1$ points in general position. The goal of this paper is to establish a connection between the complex and the tropical point of view. We show that relative Gromov-Witten invariants correspond to tropical curves with unbounded edges of higher weight to the left. For example, our cubics above having a point of contact order 3 to L in a fixed point correspond to tropical curves having a fixed unbounded end of weight 3 to the left (and passing in addition through 5 more general points $p_2, \dots, p_6$). The specialization process above is simply accomplished by moving p_2 to the very far left:



We see that the resulting tropical curves "split" into two parts: a left part (through p_2) and a right part (through the remaining points, circled in the picture above). We get the same "degenerations" as in the complex case: one where the right part consists of two lines through two of the points $p_3, \dots, p_6$ each, and one where it consists of a conic "tangent to a line" (i.e. with an unbounded edge of multiplicity 2 to the left).

To arrive at the actual numbers $N(d, g)$ Mikhalkin did not count these tropical curves directly however. Instead, he showed by purely combinatorial arguments that the number $N(d, g)$ is also equal to the number of certain "increasing" lattice paths of length $3d+g-1$ from $(0, d)$ to $(d, 0)$ in the integral triangle $\Delta_d = \{(x, y) \in \mathbb{Z}^2;\ x \geq 0, y \geq 0, x + y \leq d\}$, counted with suitable multiplicities. We will show that the idea of Caporaso and Harris can also be seen directly in this lattice path set-up: relative Gromov-Witten invariants simply correspond to lattice paths with fixed integral steps on the left edge of Δ_d. For example, the cubics above with triple contact to L correspond to lattice paths starting with the two points $(0, 3)$ and $(0, 0)$, i.e. with a step of length 3. The remaining steps are then arbitrary, as long as the number of steps in the path is correct (in this case 6 as we have 6 marked points):



If we look at the triangle Δ_{d-1} obtained from Δ_d by removing the left edge (circled in the picture above) we see again two possible types: one that corresponds to a union of two lines (the first two cases) and one that corresponds to conics tangent to L (the last case, with a step of length 2 at the left side of Δ_{d-1}).

The aim of this paper is to make the above ideas rigorous. We will define "relative Gromov-Witten invariants" both in terms of tropical curves and lattice paths, and prove the Caporaso-Harris formula in both settings. For simplicity we will work with not necessarily irreducible curves most of the time (except for section 4.3). We will also restrict to complex curves in the plane (i.e. tropical curves with Newton polyhedron Δ_d) to keep the notation simple. However, the same ideas can be applied for other toric surfaces as well (see e.g. remarks 3.10 and 3.12). We hope that our ideas can also be generalized to curves in higher-dimensional spaces. Work on this question is in progress.

This paper is organized as follows. In section 2 we will briefly review the known results on complex and tropical curves. We will then construct analogues of relative Gromov-Witten invariants and prove the Caporaso-Harris formula in the lattice path set-up in section 3. The same is then done for the tropical curves set-up in section 4.

We would like to thank Ilia Itenberg for pointing out a serious error in a previous version of this paper.

2. Complex and tropical curves

In this section we will briefly review the notations and results on complex and tropical curves that we need in our paper. Our main references are [CH98] and [Mik03].

2.1. **Complex curves and the Caporaso-Harris formula.**

We start by defining the "relative Gromov-Witten invariants" used by Caporaso and Harris to compute the numbers of complex plane curves of given degree and genus through the appropriate number of given points.

Definition 2.1
A *(finite) sequence* is a collection $\alpha = (\alpha_1, \alpha_2, \dots)$ of natural numbers almost all of which are zero. If $\alpha_k = 0$ for all $k > n$ we will also write this sequence as

$\alpha = (\alpha_1, \dots, \alpha_n)$. For two sequences α and β we define

$$\begin{aligned}
|\alpha| &:= \alpha_1 + \alpha_2 + \cdots ; \\
I\alpha &:= 1\alpha_1 + 2\alpha_2 + 3\alpha_3 + \cdots ; \\
I^\alpha &:= 1^{\alpha_1} \cdot 2^{\alpha_2} \cdot 3^{\alpha_3} \cdot \cdots ; \\
\alpha + \beta &:= (\alpha_1 + \beta_1, \alpha_2 + \beta_2, \dots); \\
\alpha \geq \beta &:\Leftrightarrow \alpha_n \geq \beta_n \text{ for all } n; \\
\tbinom{\alpha}{\beta} &:= \tbinom{\alpha_1}{\beta_1} \cdot \tbinom{\alpha_2}{\beta_2} \cdot \cdots .
\end{aligned}$$

We denote by e_k the sequence which has a 1 at the k-th place and zeros everywhere else.

Definition 2.2
Let $d \geq 0$ and g be integers, and let α and β be two sequences with $I\alpha + I\beta = d$. Pick a fixed line $L \subset \mathbb{P}^2$. Then we denote by $N_{\text{cplx}}^{\alpha,\beta}(d,g)$ the number of smooth but not necessarily irreducible curves (or more precisely: stable maps) of degree d and genus g to $\mathbb{P}^2$ that

- intersect L in α_i fixed general points of L with multiplicity i each for all $i \geq 1$;
- intersect L in β_i more arbitrary points of L with multiplicity i each for all $i \geq 1$; and
- pass in addition through $2d + g + |\beta| - 1$ more general points in $\mathbb{P}^2$.

In other words, we consider the numbers of complex plane curves of given degree and genus that have fixed contact orders to a given line. Note that this set of numbers includes the numbers $N_{\text{cplx}}(d,g) := N_{\text{cplx}}^{(0),(d)}(d,g)$ of complex plane curves of degree d and genus g through $3d + g - 1$ general points (without multiplicity conditions).

The main result of Caporaso and Harris is how these numbers can be computed recursively:

Definition 2.3
We say that a collection of numbers $N^{\alpha,\beta}(d,g)$ defined for all integers $d \geq 0$ and g and all sequences α, β with $I\alpha + I\beta = d$ *satisfies the Caporaso-Harris formula* if

$$\begin{aligned}
N^{\alpha,\beta}(d,g) = &\sum_{k:\beta_k>0} k \cdot N^{\alpha+e_k,\beta-e_k}(d,g) \\
&+ \sum I^{\beta'-\beta} \cdot \binom{\alpha}{\alpha'} \cdot \binom{\beta'}{\beta} \cdot N^{\alpha',\beta'}(d-1,g')
\end{aligned}$$

for all d, g, α, β as above with $d > 1$, where the second sum is taken over all α', β' and g' satisfying

$$\begin{aligned}
\alpha' &\leq \alpha; \\
\beta' &\geq \beta; \\
I\alpha' + I\beta' &= d - 1; \\
g - g' &= |\beta' - \beta| - 1; \\
d - 2 &\geq g - g'.
\end{aligned}$$

Theorem 2.4
The numbers $N^{\alpha,\beta}_{\mathrm{cplx}}(d,g)$ of definition 2.2 satisfy the Caporaso-Harris formula.

Proof:
See [CH98]. □

Note that a collection of numbers $N^{\alpha,\beta}(d,g)$ satisfying the Caporaso-Harris formula is determined uniquely by their values for $d=1$. In particular, theorem 2.4 allows us to compute all numbers $N_{\mathrm{cplx}}(d,g)$ from the starting information that there is exactly one line through two points in the plane.

2.2. **Tropical curves.** We will move on to the tropical set-up and recall the definition of tropical curves and their Newton polyhedra (following [Mik03] and [NS04]).

Let $\overline{\Gamma}$ be a weighted finite graph without divalent vertices. The weight of an edge E of $\overline{\Gamma}$ will be written as $\omega(E) \in \mathbb{N} \setminus \{0\}$. Denote the set of 1-valent vertices by $\overline{\Gamma}^0_\infty$. We remove the one-valent vertices from $\overline{\Gamma}$ and set $\Gamma := \overline{\Gamma} \setminus \overline{\Gamma}^0_\infty$. The graph Γ may then have non-compact edges which are called *unbounded edges* or *ends*. We write Γ^0 for the set of vertices and Γ^1 for the set of edges of Γ. We define the set of *flags* of Γ by $\mathbb{F}\Gamma := \{(V,E) \in \Gamma^0 \times \Gamma^1 \mid V \in \partial E\}$. Let $h : \Gamma \to \mathbb{R}^2$ be a continuous proper map such that the image $h(E)$ of any edge $E \in \Gamma^1$ is contained in an affine line with rational slope. Then we can define a map $u : \mathbb{F}\Gamma \to \mathbb{Z}^2$ that sends (V,E) to the primitive integer vector that starts at $h(V)$ and points in the direction of $h(E)$.

Definition 2.5
A *parametrized tropical (plane) curve* is a pair (Γ, h) as above such that

(a) for every edge $E \in \Gamma^1$ the restriction $h|_E$ is an embedding;
(b) for every vertex $V \in \Gamma^0$ the *balancing condition*

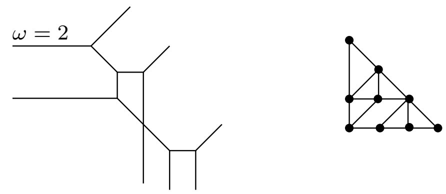
$$\sum_{E \in \Gamma^1 : V \in \partial E} \omega(E) \cdot u(V,E) = 0.$$

holds.

A *tropical curve* in $\mathbb{R}^2$ is the image $h(\Gamma)$ of a parametrized tropical curve.

Let V be an r-valent vertex of a tropical curve Γ and let $E_1, \ldots, E_r$ be the counterclockwise enumerated edges adjacent to V. Draw in the $\mathbb{Z}^2$-lattice an orthogonal line $l(E_i)$ of integer length $\omega(E_i)$ to $h(E_i)$, where $l(E_1)$ starts at any lattice point and $l(E_i)$ starts at the endpoint of $l(E_{i-1})$, and where by "integer length" we mean $\sharp(\mathbb{Z}^2 \cap l(E_i)) - 1$. The balancing condition tells us that we end up with a closed r-gon. If we do this for every vertex we end up with a polygon in $\mathbb{Z}^2$ that is divided into smaller polygons. The polygon is called the *Newton polygon* of the tropical curve, and the division the corresponding *Newton subdivision*. Note that the ends of the curve correspond to lines on the boundary of the Newton polygon.

Example 2.6
The following picture shows an example of a tropical curve and its Newton polygon and subdivision:



Most properties of tropical curves can be read off both from their image in $\mathbb{R}^2$ as well as in the dual picture from their Newton polygon and subdivision. Here are some examples:

Definition 2.7

(a) A parametrized tropical plane curve is called *simple* if Γ is 3-valent, the map h is injective on the set of vertices, for a vertex V and an edge E we have $h(V) \cap h(E) = \emptyset$, for two edges E_1 and E_2 we have $\#\{h(E_1) \cap h(E_2)\} \leq 1$, and for all $p \in \mathbb{R}^2$ we have $\sharp h^{-1}(p) \leq 2$. A tropical plane curve is called simple if it allows a simple parametrization. In the dual language, a curve is simple if and only if its subdivision contains only triangles and parallelograms.

(b) The *genus* of a parametrized tropical curve (Γ, h) is defined to be the number $1 - \dim H^0(\Gamma, \mathbb{Z}) + \dim H^1(\Gamma, \mathbb{Z}) = 1 - \#\Gamma^0 + \#\Gamma^1_0$. The genus of a simple tropical plane curve is the least genus of all parametrizations that the curve allows. In the dual of a simple tropical curve, the genus is the number of interior lattice points of the subdivision minus the number of parallelograms.

(c) A parametrized tropical curve is called *irreducible* if the graph Γ is connected. A simple tropical plane curve is called *irreducible* if it allows only irreducible parametrizations. An *irreducible component* of a simple tropical plane curve is the image of a connected component of the graph Γ of a parametrization with the maximum possible number of connected components.

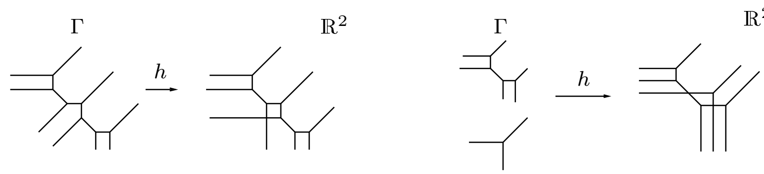
(d) The *degree* Δ of a tropical plane curve is the collection of directions of the unbounded edges together with the sum of the weights for each direction. In the dual language the degree is just the Newton polygon, which is therefore also denoted by Δ.

(e) The *(combinatorial) type* of a parametrized tropical curve is given by the weighted graph Γ together with the map $u : \mathbb{F}\Gamma \to \mathbb{Z}^2$. The (combinatorial) type of a tropical curve is the combinatorial type of any parametrization of least possible genus. In the dual setting, the type is just the Newton subdivision.

(f) Let V be a trivalent vertex of Γ and E_1, E_2, E_3 the edges adjacent to V. The *multiplicity* of V is the product of the area of the parallelogram spanned by $u(V, E_1)$ and $u(V, E_2)$ times the weights of the edges E_1 and E_2. The balancing condition tells us that this definition is independent of the order of E_1, E_2 and E_3. In the dual language, the multiplicity of 3-valent vertex is equal to 2 times the area of the dual triangle.

(g) The *multiplicity* $\text{mult}(C)$ of a simple tropical plane curve is the product of the multiplicities of all trivalent vertices of Γ as in (f). In the dual language, the multiplicity is the product over all double areas of triangles of the dual subdivision.

Example 2.8
The following picture shows an irreducible simple tropical curve (on the left) and a simple tropical curve with two components (on the right), together with their parametrizations of least possible genus. Both curves are of degree $(3 \cdot (0, -1), 3 \cdot (-1, 0), 3 \cdot (1, 1))$.



We are now ready to state the results of [Mik03] that are relevant for our purposes.

Definition 2.9
Let $d \geq 0$ and g be integers, and let Δ_d be the Newton polyhedron $\{(x, y) \in \mathbb{Z}^2;\ x \geq 0, y \geq 0, x + y \leq d\}$. Let $\mathcal{P} = \{p_1, \ldots, p_{3d+g-1}\} \subset \mathbb{R}^2$ be a set of points in (tropical) general position (see [Mik03] section 4.2 for a precise definition). Then by [Mik03] lemma 4.22 the number of tropical curves through all points of $\mathcal{P}$ is finite, and all such curves are simple. We denote by $N_{\text{trop}}(d, g)$ the number of tropical curves of degree Δ_d and genus g passing through all points of $\mathcal{P}$, counted with their multiplicities as in definition 2.7 (g). By [GM05] this definition does not depend on the choice of points.

Remark 2.10
Let $\mathcal{P} = (p_1, \ldots, p_n)$ be a configuration of points in general position, and let (Γ, h) be a (simple) tropical curve through $\mathcal{P}$. Then by [Mik03] lemma 4.20 each connected component of $\Gamma \setminus \{h^{-1}(p_1), \ldots, h^{-1}(p_n)\}$ is a tree and contains exactly one unbounded edge. That is, in a tropical curve through $\mathcal{P}$, we can neither find a path which connects two unbounded edges nor a path around a loop without meeting a marked point.

Theorem 2.11 ("Correspondence Theorem")
For all d and g we have $N_{\text{trop}}(d, g) = N_{\text{cplx}}(d, g)$.

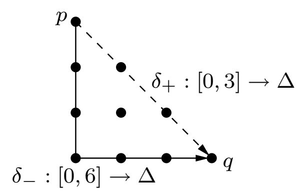
Proof:
See [Mik03] theorem 1. □

2.3. **Lattice paths.** In his paper [Mik03] Mikhalkin also gave an algorithm to compute the numbers $N_{\text{trop}}(d, g)$ combinatorially. He did not calculate them directly however, but rather by a trick that relates them to certain lattice paths that we will now introduce.

Definition 2.12
A path $\gamma : [0, n] \to \mathbb{R}^2$ is called a lattice path if $\gamma|_{[j-1,j]}, j = 1, \ldots, n$ is an affine-linear map and $\gamma(j) \in \mathbb{Z}^2$ for all $j = 0 \ldots, n$.

Let λ be a fixed linear map of the form $\lambda : \mathbb{R}^2 \to \mathbb{R}, \lambda(x, y) = x - \varepsilon y$, where ε is a small irrational number. A lattice path is called λ-increasing if $\lambda \circ \gamma$ is strictly increasing. Let $p := (0, d)$ and $q := (d, 0)$ be the points in $\Delta := \Delta_d$ where $\lambda|_\Delta$

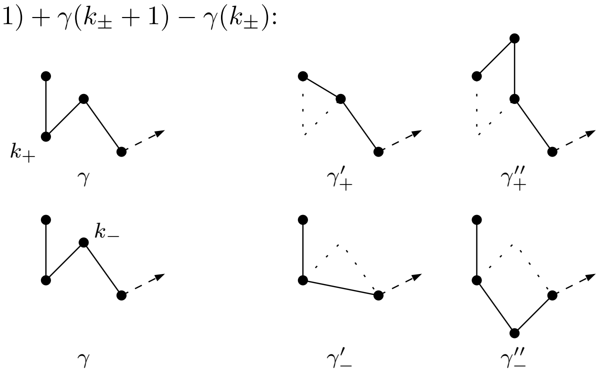
reaches its minimum (resp. maximum). The points p and q divide the boundary $\partial\Delta$ into two λ-increasing lattice paths $\delta_+ : [0, n_+] \to \partial\Delta$ (going clockwise around $\partial\Delta$) and $\delta_- : [0, n_-] \to \partial\Delta$ (going counterclockwise around $\partial\Delta$), where $n_\pm$ denotes the number of integer points in the $\pm$-part of the boundary. The following picture shows an example for $d = 3$:



We will now define the multiplicity of λ-increasing paths as in [Mik03]:

Definition 2.13
Let $\gamma : [0, n] \to \Delta$ be a λ-increasing path from p to q, that is, $\gamma(0) = p$ and $\gamma(n) = q$. The *multiplicities* $\mu_+(\gamma)$ and $\mu_-(\gamma)$ are defined recursively as follows:

(a) $\mu_\pm(\delta_\pm) := 1$.
(b) If $\gamma \neq \delta_\pm$ let $k_\pm \in [0, n]$ be the smallest number such that γ makes a left turn (respectively a right turn for μ_-) at $\gamma(k_\pm)$. (If no such $k_\pm$ exists we set $\mu_\pm(\gamma) := 0$). Define two other λ-increasing lattice paths as follows:
  - $\gamma'_\pm : [0, n-1] \to \Delta$ is the path that cuts the corner of $\gamma(k_\pm)$, i.e. $\gamma'_\pm(j) := \gamma(j)$ for $j < k_\pm$ and $\gamma'_\pm(j) := \gamma(j+1)$ for $j \geq k_\pm$.
  - $\gamma''_\pm : [0, n] \to \Delta$ is the path that completes the corner of $\gamma(k_\pm)$ to a parallelogram, i.e. $\gamma''_\pm(j) := \gamma(j)$ for all $j \neq k_\pm$ and $\gamma''_\pm(k_\pm) := \gamma(k_\pm - 1) + \gamma(k_\pm + 1) - \gamma(k_\pm)$:



  Let T be the triangle with vertices $\gamma(k_\pm - 1), \gamma(k_\pm), \gamma(k_\pm + 1)$. Then we set
  $$\mu_\pm(\gamma) := 2 \cdot \operatorname{Area} T \cdot \mu_\pm(\gamma'_\pm) + \mu_\pm(\gamma''_\pm).$$
  As both paths $\gamma'_\pm$ and $\gamma''_\pm$ include a smaller area with $\delta_\pm$, we can assume that their multiplicity is known. If $\gamma''_\pm$ does not map to Δ, $\mu_\pm(\gamma''_\pm)$ is defined to be zero.

Finally, the multiplicity $\mu(\gamma)$ is defined to be the product $\mu(\gamma) := \mu_+(\gamma)\mu_-(\gamma)$.

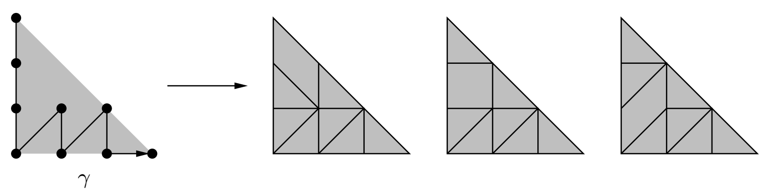
Note that the multiplicity of a path γ is positive only if the recursion above ends with the path $\delta_+ : [0, n_+] \to \Delta$ (respectively δ_-). In other words, if we end up with a "faster" path $\delta' : [0, n'] \to \Delta$ such that $\delta_+([0, n_+]) = \delta'([0, n'])$ but $n' < n_+$ then the multiplicity is zero.

Definition 2.14
Let $d \geq 0$ and g be integers. We denote by $N_{\text{path}}(d, g)$ the number of λ-increasing lattice paths $\gamma : [0, 3d + g - 1] \to \Delta$ with $\gamma(0) = p$ and $\gamma(3d + g - 1) = q$ counted with their multiplicities as in definition 2.13.

Theorem 2.15
For all d and g we have $N_{\text{path}}(d, g) = N_{\text{trop}}(d, g)$.

Proof:
See [Mik03] theorem 2. The idea is to choose a line H orthogonal to the kernel of λ and $n = 3d + g - 1$ points $p_1, \ldots, p_n$ on H such that the distance between p_i and p_{i+1} is much bigger than the distance of p_{i-1} and p_i for all i. These points are then in tropical general position. Consider a tropical curve of degree Δ_d and genus g through these points. If we take the edges on which the marked points lie and consider their dual edges in the Newton subdivision then these dual edges can be shown to form a λ-increasing path from p to q. Furthermore, the multiplicity of a path coincides with the number of possible Newton subdivisions times their multiplicity. (For a path γ there can be several possible Newton subdivisions.) The multiplicity μ_+ counts the possible Newton subdivisions times their multiplicity in the half-plane above H, whereas μ_- counts below H. Passing from γ to γ' and γ'' corresponds to moving the line up (for μ_+) respectively down. The path γ' leaves a 3-valent vertex of multiplicity $2\,\text{Area}(T)$ out, and γ'' counts the possibility that there might be a crossing of two edges, dual to a parallelogram.

As an example, the following picture shows a path γ and the possible Newton subdivisions for γ, all dual to tropical curves of multiplicity 1. The multiplicity of the path is 3.



□

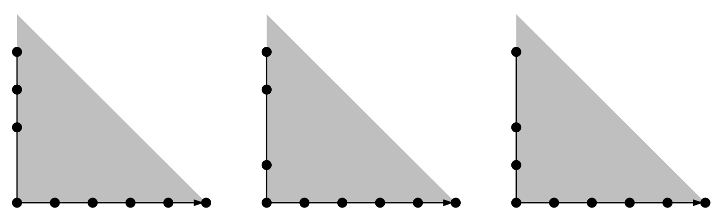
For integers $d \geq 0$ and g we have now defined three sets of numbers $N_{\text{cplx}}(d, g)$, $N_{\text{trop}}(d, g)$, and $N_{\text{path}}(d, g)$. Moreover, we know that these three sets of numbers coincide by theorems 2.11 and 2.15. However, only in the complex picture have we defined numbers $N^{\alpha,\beta}_{\text{cplx}}(d, g)$ corresponding to curves with specified contact orders that satisfied the Caporaso-Harris formula. It is the goal of this paper to establish the same thing for the tropical and lattice path set-ups.

3. The Caporaso-Harris formula in the lattice path set-up

3.1. **Generalized lattice paths.** We will now slightly generalize the definitions of section 2.3 in order to allow more lattice paths and arrive at lattice path analogues of the numbers $N^{\alpha,\beta}_{\text{cplx}}(d,g)$. Although this concept can be used for some other polygons (see 3.10 and 3.12) we will present it here only for the triangle Δ_d with vertices $(0,0),(0,d),(d,0)$. Choose two sequences α and β with $I\alpha + I\beta = d$. As above let $\lambda(x,y) = x - \varepsilon y$ where ε is a small irrational number.

Let $\gamma : [0,n] \to \Delta_d$ be a λ-increasing path with $\gamma(0) = (0, d - I\alpha) = (0, I\beta)$ and $\gamma(n) = q = (d,0)$. We are going to define a multiplicity for such a path γ. Again this multiplicity will be the product of a "positive" and a "negative" multiplicity that we define separately.

Definition 3.1
Let $\delta_\beta : [0, |\beta| + d] \to \Delta_d$ be a path such that the image $\delta_\beta([0,|\beta|+d])$ is equal to the piece of boundary of Δ_d between $(0, I\beta)$ and $q = (d,0)$, and such that there are β_i steps (i.e. images of a size one interval $[j, j+1]$) of integer length i at the side s (and hence at $\{y = 0\}$ only steps of integer length 1). We define the negative multiplicity $\mu_{\beta,-}(\delta_\beta)$ of all such paths to be 1. For example, the following picture shows all paths δ_β for $\beta = (2,1)$ and $d = 5$:



Using these starting paths the *negative multiplicity* $\mu_{\beta,-}(\gamma)$ of an arbitrary path as above is now defined recursively by the same procedure as in definition 2.13 (b).

Definition 3.2
To compute the *positive multiplicity* $\mu_{\alpha,+}(\gamma)$ we extend γ to a path $\gamma_\alpha : [0, |\alpha| + n] \to \Delta_d$ by adding α_i steps of integer length i at $\{x = 0\}$ from $\gamma_\alpha(0) = p$ to $\gamma_\alpha(|\alpha|) = (0, I\beta)$. Then we compute $\mu_+(\gamma_\alpha)$ as in definition 2.13 and set $\mu_{\alpha,+}(\gamma) := \frac{1}{I^\alpha} \cdot \mu_+(\gamma_\alpha)$.

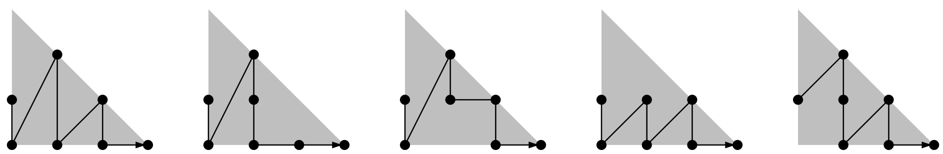
Remark 3.3
Note that definition 3.2 seems to depend on the order in which we add the α_i steps of lengths i to the path γ to obtain the path γ_α. It will follow however from the alternative description of the positive multiplicity in proposition 3.8 (b) that this is indeed not the case.

We can now define the analogue of relative Gromov-Witten invariants in the lattice path set-up.

Definition 3.4
Let $d \geq 0$ and g be integers, and let α and β be sequences with $I\alpha + I\beta = d$. We define $N^{\alpha,\beta}_{\text{path}}(d,g)$ to be the number of λ-increasing paths $\gamma : [0, 2d+g+|\beta|-1] \to \Delta_d$ that start at $(0, d - I\alpha) = (0, I\beta)$ and end at $(d,0)$, where each such path is counted with multiplicity $\mu_{\alpha,\beta}(\gamma) := \mu_{\alpha,+}(\gamma) \cdot \mu_{\beta,-}(\gamma)$.

Note that as expected (i.e. as in the complex case) we always have $N_{\text{path}}(d,g) = N_{\text{path}}^{(0),(d)}(d,g)$ by definition.

Example 3.5
The following picture shows that $N_{\text{path}}^{(0,1),(1)}(3,0) = 4+2+1+1+2 = 10$:



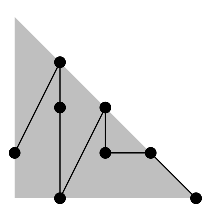
$\mu_{(0,1),+} = 2$	$\mu_{(0,1),+} = 2$	$\mu_{(0,1),+} = 1$	$\mu_{(0,1),+} = 1$	$\mu_{(0,1),+} = 1$
$\mu_{(1),-} = 2$	$\mu_{(1),-} = 1$	$\mu_{(1),-} = 1$	$\mu_{(1),-} = 1$	$\mu_{(1),-} = 2$
$\mu_{(0,1),(1)} = 4$	$\mu_{(0,1),(1)} = 2$	$\mu_{(0,1),(1)} = 1$	$\mu_{(0,1),(1)} = 1$	$\mu_{(0,1),(1)} = 2$

3.2. **The Caporaso-Harris formula.** In order to prove the Caporaso-Harris formula for the numbers $N_{\text{path}}^{\alpha,\beta}(d,g)$ of definition 3.4 we will first express the negative and positive multiplicities of a generalized lattice path in a different, non-recursive way. For this we need an easy preliminary lemma:

Lemma 3.6
Let α and β be two sequences with $I\alpha + I\beta = d$, and let γ be a generalized lattice path as in section 3.1. If γ has a step that "moves at least two columns to the right", i.e. that starts on a line $\{x = i\}$ and ends on a line $\{x = j\}$ for some i, j with $j - i \geq 2$ then $\mu_{\beta,-}(\gamma) = \mu_{\alpha,+}(\gamma) = \mu_{\alpha,\beta}(\gamma) = 0$.

Proof:
If γ is a path with a step that moves at least two columns to the right then the same is true for the paths $\gamma'_\pm$ and $\gamma''_\pm$ of definition 2.13. Hence the lemma follows by induction since the only end paths δ_β (see definition 3.1) and δ_+ (see definition 2.13) with non-zero multiplicity do not have such a step. □

Remark 3.7
We can therefore conclude that any generalized lattice path with non-zero multiplicity has only two types of steps: some that go down vertically and others that move exactly one column to the right (with a simultaneous move up or down):



For such a path we will fix the following notation for the rest of this section: for the vertical line $\{x = i\}$ we let $h(i)$ be the highest y-coordinate of a point of γ on this line, and we denote by α^i the sequence describing the lengths of the vertical steps of γ on this line. For example, for the path shown above we have $h(0) = 1$, $h(1) = 3$, $h(2) = 2$, $h(3) = 1$ and $\alpha^0 = (0), \alpha^1 = (1,1), \alpha^2 = (1), \alpha^3 = (0)$.

Proposition 3.8
Let α and β be two sequences with $I\alpha + I\beta = d$, and let γ be a generalized lattice path as above.

(a) *The negative multiplicity of γ is given by the formula*

$$\mu_{\beta,-}(\gamma) = \sum_{(\beta^0,\dots,\beta^d)} \left(\prod_{i=0}^{d-1} I^{\alpha^{i+1}+\beta^{i+1}-\beta^i} \cdot \binom{\alpha^{i+1}+\beta^{i+1}}{\beta^i} \right)$$

where the sum is taken over all $(d+1)$-tuples of sequences $(\beta^0,\dots,\beta^d)$ such that $\alpha^0+\beta^0=\beta$ and $I\alpha^i + I\beta^i = h(i)$ for all i.

(b) *The positive multiplicity of γ is given by the formula*

$$\mu_{\alpha,+}(\gamma) = \frac{1}{I^\alpha} \cdot \sum_{(\beta^0,\dots,\beta^d)} \left(\prod_{i=0}^{d-1} I^{\alpha^i+\beta^i-\beta^{i+1}} \cdot \binom{\alpha^i+\beta^i}{\beta^{i+1}} \right)$$

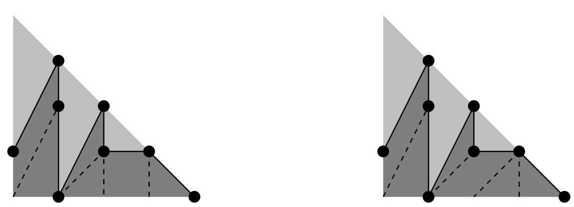
where the sum is taken over all $(d+1)$-tuples of sequences $(\beta^0,\dots,\beta^d)$ such that $\beta^0=\alpha$ and $d-i-I\beta^i = h(i)$ for all i.

Remark 3.9
Before we give the proof of this proposition let us briefly comment on the geometric meaning of these formulas.

The formula of proposition 3.8 (a) can be interpreted as the number of ways to subdivide the area of Δ_d below γ into parallelograms and triangles such that

- the subdivision contains all vertical lines $\{x=i\}$ below γ; and
- each triangle in the subdivision "is pointing to the left", i.e. the vertex opposite to its vertical edge lies to the left of this edge,

where each such subdivision is counted with a multiplicity equal to the product of the double areas of its triangles. Indeed, the sequences β^i describe the lengths of the vertical edges in the subdivisions below γ. The binomial coefficients $\binom{\alpha^{i+1}+\beta^{i+1}}{\beta^i}$ in the formula count the number of ways to arrange the parallelograms and triangles, and the factors $I^{\alpha^{i+1}+\beta^{i+1}-\beta^i}$ are simply the double areas of the triangles. As an example let us consider the path of remark 3.7 with $\beta=(1)$. In this case there is only one possibility to fill the area below γ with parallelograms and triangles in this way, namely as in the following picture on the left:



(corresponding to $\beta^0=\beta^2=\beta^3=(1), \beta^1=(0)$). As there is one triangle in this subdivision with double area 2 we see that $\mu_{\beta,-}=2$.

The surprising fact about this statement is that the original definition of the negative multiplicity (in definition 3.1 as well as in [Mik03]) was set up in a way so that it also counts certain subdivisions of the area Δ_d below γ — but *different ones*, namely those subdivisions that can occur as the Newton subdivisions of tropical curves passing through given points in a certain special position. In our concrete example above one can in fact see that the subdivision counted above *does not* correspond to an actual tropical curve through the given points, whereas the subdivision in the picture above on the right does. One can therefore interpret proposition 3.8 as the combinatorial statement that the number of "column-wise" subdivisions as described above agrees with the number of "tropical" subdivisions of the area of Δ_d below γ (when counted with the correct multiplicities).

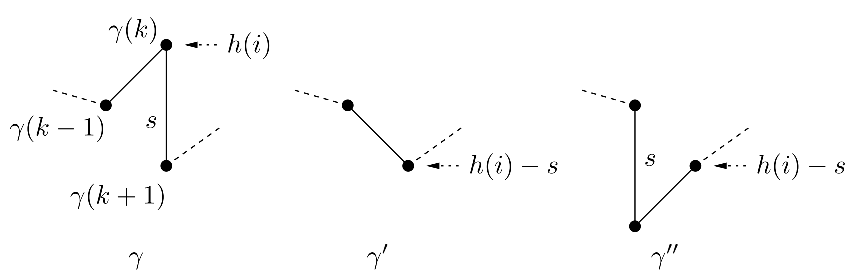
It is interesting to note that for the positive multiplicity there is no such difference: it can be checked that the "tropical subdivisions" are just equal to the "column-wise subdivisions" of the area of Δ_d above γ in this case. Given this fact the statement of proposition 3.8 (b) is then almost obvious.

In any case the nice thing about proposition 3.8 (resp. the "column-wise" subdivisions) is that in these subdivisions it is easy to split off the first column to obtain a similar subdivision of Δ_{d-1}. This will be the key ingredient in the proof of the Caporaso-Harris formula in the lattice path set-up in theorem 3.11.

Proof of proposition 3.8:
We prove only part (a) of the proposition since part (b) is entirely analogous. The proof will be by induction on the recursive definition of $\mu_{\beta,-}$ in definition 3.1. It is obvious that the end paths in this recursion (the paths that go from $(0, I\beta)$ to $(d, 0)$ along the border of Δ_d) satisfy the stated formula: all these paths have $\beta^0 = (0)$, so the condition $\alpha^0 + \beta^0 = \beta$ requires $\alpha^0 = \beta$, i.e. that the path is one of the paths δ_β as in definition 3.1.

Let us now assume that $\gamma : [0, n] \to \Delta_d$ is an arbitrary generalized lattice path. By induction we know that the paths γ'_- and γ''_- of definition 2.13 satisfy the formula of the proposition. Recall that if $k \in [1, n-1]$ is the first vertex at which γ makes a right turn then γ' and γ'' are defined by cutting this vertex $\gamma(k)$ (resp. flipping it to a parallelogram). By lemma 3.6 we know that $\gamma(k-1)$ (resp. $\gamma(k+1)$) can be at most one column to the left (resp. right) of $\gamma(k)$. But note that $\gamma(k-1)$ cannot be in the same column as $\gamma(k)$ as otherwise the λ-increasing path γ could not make a right turn at $\gamma(k)$. Hence $\gamma(k-1)$ is precisely one column left of $\gamma(k)$, and we are left with two possibilities:

- $\gamma(k+1)$ is in the same column i as $\gamma(k)$:



  Then the path γ' has the same values of $h(j)$ and α^j (see remark 3.7) as γ, except for $h(i)$ being replaced by $h(i) - s$ and α^i by $\alpha^i - e_s$, where s is the length of the vertical step from $\gamma(k)$ to $\gamma(k+1)$. So by induction we have

$$\begin{aligned}\mu_{\beta,-}(\gamma') &= \sum_{(\beta^0,\dots,\beta^d)} \left(\prod_{j=0}^{d-1} I^{\alpha^{j+1}+\beta^{j+1}-\beta^j-\delta_{i,j+1}e_s} \cdot \binom{\alpha^{j+1}+\beta^{j+1}-\delta_{i,j+1}e_s}{\beta^j} \right) \\ &= \frac{1}{s} \cdot \sum_{(\beta^0,\dots,\beta^d)} \left(\prod_{j=0}^{d-1} I^{\alpha^{j+1}+\beta^{j+1}-\beta^j} \cdot \binom{\alpha^{j+1}+\beta^{j+1}-\delta_{i,j+1}e_s}{\beta^j} \right)\end{aligned}$$

where the sum is taken over the same β as in the proposition. The path γ'' has the same values of $h(j)$ and α^j as γ except for $h(i)$ being replaced by $h(i)-s$, α^i by $\alpha^i - e_s$, and α^{i-1} by $\alpha^{i-1}+e_s$. So by induction it follows that

$$\mu_{\beta,-}(\gamma'') = \sum_{(\beta^0,\dots,\beta^d)} \left(\prod_{j=0}^{d-1} I^{\alpha^{j+1}+\beta^{j+1}-\beta^j} \cdot \binom{\alpha^{j+1}+\beta^{j+1}+\delta_{i-1,j+1}e_s-\delta_{i,j+1}e_s}{\beta^j} \right)$$

where the conditions on the summation variables β^i are $\alpha^0+\delta_{i-1,0}e_s+\beta^0=\beta$ and $I(\alpha^j-\delta_{i,j}e_s+\delta_{i-1,j}e_s)+I\beta^j = h(j)-s\delta_{i,j}$. We can make these conditions the same as in the proposition by replacing the summation variables β^{i-1} by $\beta^{i-1}-e_s$, arriving at the formula

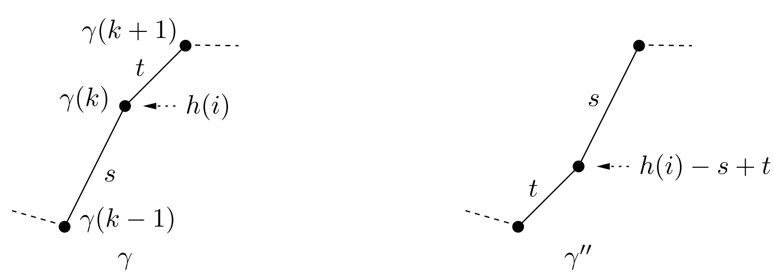
$$\mu_{\beta,-}(\gamma'') = \sum_{(\beta^0,\dots,\beta^d)} \left(\prod_{j=0}^{d-1} I^{\alpha^{j+1}+\beta^{j+1}-\beta^j} \cdot \binom{\alpha^{j+1}+\beta^{j+1}-\delta_{i,j+1}e_s}{\beta^j-\delta_{i,j+1}e_s} \right).$$

Plugging these expressions into the defining formula

$$\mu_{\beta,-}(\gamma) = s\cdot\mu_{\beta,-}(\gamma')+\mu_{\beta,-}(\gamma'')$$

we now arrive at the desired formula of the proposition (where we simply use the standard binomial identity $\binom{n-1}{k}+\binom{n-1}{k-1}=\binom{n}{k}$).

- $\gamma(k+1)$ is one column right of $\gamma(k)$: The idea here is the same as in the previous case. The actual computation is simpler however as the path γ' does not give a contribution by lemma 3.6.



The path γ'' has the same values of $h(j)$ and α^j as γ, except for $h(i)$ being replaced by $h(i)-s+t$, where i is the column of $\gamma(k)$, and s and t are the vertical lengths of the steps before and after $\gamma(k)$. So by induction we have

$$\mu_{\beta,-}(\gamma'') = \sum_{(\beta^0,\dots,\beta^d)} \left(\prod_{j=0}^{d-1} I^{\alpha^{j+1}+\beta^{j+1}-\beta^j} \cdot \binom{\alpha^{j+1}+\beta^{j+1}}{\beta^j} \right)$$

where the conditions on the β^j are $\alpha^0+\beta^0=\beta$ and $I\alpha^j+I\beta^j=h(j)-(s-t)\delta_{i,j}$. Note that $s-t>0$ since γ makes a right turn. As in the previous case we can make the conditions on the β^j the same as in the proposition by replacing the summation variables β^i by $\beta^{i-1}+\alpha^{i+1}+\beta^{i+1}-\beta^i$. We then arrive at

$$\mu_{\beta,-}(\gamma'')=\sum_{(\beta^0,\ldots,\beta^d)}\prod_{j=0}^{d-1}\Bigg(I^{\alpha^{j+1}+\beta^{j+1}-\beta^j}\cdot \binom{\alpha^{j+1}+\beta^{j+1}+\delta_{i,j+1}(\beta^{i-1}+\alpha^{i+1}+\beta^{i+1}-2\beta^i)}{\beta^j+\delta_{i,j}(\beta^{i-1}+\alpha^{i+1}+\beta^{i+1}-2\beta^i)}\Bigg).$$

This is already the formula of the proposition except for the factors

$$\binom{\alpha^i+\beta^i}{\beta^{i-1}}\binom{\alpha^{i+1}+\beta^{i+1}}{\beta^i}$$

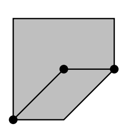
being replaced by

$$\binom{\alpha^i+\beta^i+(\beta^{i-1}+\alpha^{i+1}+\beta^{i+1}-2\beta^i)}{\beta^{i-1}} \cdot\binom{\alpha^{i+1}+\beta^{i+1}}{\beta^i+(\beta^{i-1}+\alpha^{i+1}+\beta^{i+1}-2\beta^i)}.$$

But these terms are the same by the identity $\binom{n+k+l}{n+k}\binom{n+k}{n}=\binom{n+k+l}{n+l}\binom{n+l}{n}$ with $n=\beta^{i-1}$, $k=\beta^i-\beta^{i-1}$, and $l=\alpha^{i+1}+\beta^{i+1}-\beta^i$ (note that $\alpha^i=(0)$ for our path γ).

This proves the proposition. □

Remark 3.10
Note that it is important for the second step of the proof above that the two boundary lines of Δ_d below and above γ — the line $\{y=0\}$ respectively the diagonal line from $(0,d)$ to $(d,0)$ — are indeed straight lines. We cannot generalize the proof to polygons which contain a vertex above respectively below γ, because then the heights of the three columns of $\gamma(k-1)$, $\gamma(k)$ and $\gamma(k+1)$ cannot be described as $h(i)$, $h(i)+s$ and $h(i)+s+t$. So we cannot use the identity $\binom{n+k+l}{n+k}\binom{n+k}{n}=\binom{n+k+l}{n+l}\binom{n+l}{n}$. The picture below shows a polygon for which the formula of proposition 3.8 does not hold. The column-wise subdivisions would predict 0 as negative multiplicity for the path. However, the path γ'' is a valid end path, so we get 1.



The proof can be generalized to polygons where the boundaries above and below γ are straight lines, for example to rectangles (see 3.12).

We are now ready to prove the Caporaso-Harris formula in the lattice path set-up:

Theorem 3.11
The numbers $N^{\alpha,\beta}_{\text{path}}(d,g)$ *satisfy the Caporaso-Harris formula. In particular we have* $N^{\alpha,\beta}_{\text{path}}(d,g) = N^{\alpha,\beta}_{\text{cplx}}(d,g)$ *for all* d, g, α, β.

Proof:
The idea of the proof is simply to list the possibilities of the first step of the path γ. Let γ be a λ-increasing path from $(0, I\beta)$ to $q = (d,0)$. Then we have one of the following two cases:

Case 1: The point $\gamma(1)$ is on the line $\{x = 0\}$. Then $\gamma(1)$ must be $(0, I\beta - k)$ for some $\beta_k \neq 0$ as otherwise $\mu_{\beta,-}(\gamma)$ would be 0. It follows that $\gamma\mid_{[1,2d+g+|\beta|-1]}$ is a path from $(0, d - I(\alpha + e_k))$ and with $\mu_{\alpha,\beta}(\gamma) = k \cdot \mu_{\alpha+e_k,\beta-e_k}(\gamma\mid_{[1,2d+g+|\beta|-1]})$. Therefore the paths γ with $\gamma(1) \in s$ contribute

$$\sum_{k:\beta_k>0} k \cdot N^{\alpha+e_k,\beta-e_k}_{\text{path}}(d,g)$$

to the number $N^{\alpha,\beta}_{\text{path}}(d,g)$.

Case 2: The point $\gamma(1)$ is not on $\{x = 0\}$. Then it must be on the line $\{x = 1\}$ by lemma 3.6. From proposition 3.8 it follows that both the negative and the positive multiplicity can be computed as a product of a factor coming from the first column and the (negative resp. positive) multiplicity of the restricted path $\tilde\gamma := \gamma|_{[1,2d+g+|\beta|-1]}$. More precisely, we have

$$\begin{aligned}\mu_{\alpha,\beta}(\gamma) &= \mu_{\beta,-}(\gamma) \cdot \mu_{\alpha,+}(\gamma)\\ &= \sum_{\beta'} I^{\beta'-\beta}\binom{\beta'}{\beta}\mu_{\beta',-}(\tilde\gamma) \cdot \sum_{\alpha'}\binom{\alpha}{\alpha'}\mu_{\alpha',+}(\tilde\gamma)\\ &= \sum_{\alpha',\beta'} I^{\beta'-\beta}\binom{\beta'}{\beta}\binom{\alpha}{\alpha'} \cdot \mu_{\alpha',\beta'}(\tilde\gamma).\end{aligned}$$

So the contribution of the paths with $\gamma(1) \notin s$ to $N^{\alpha,\beta}_{\text{path}}(d,g)$ is

$$\sum I^{\beta'-\beta}\binom{\beta'}{\beta}\binom{\alpha}{\alpha'} \cdot N^{\alpha',\beta'}_{\text{path}}(d-1,g')$$

where the sum is taken over all possible α', β' and g'. Let us figure out what these possible values are. It is clear that $\alpha' \leq \alpha$ and $\beta \leq \beta'$. Also, $I\alpha' + I\beta' = d - 1$ must be fulfilled. As $\tilde\gamma$ has one step less than γ we know that $2d+g+|\beta|-1-1 = 2(d-1)+g'+|\beta'|-1$ and hence $g-g' = |\beta'-\beta|-1$. A path $\epsilon : [0,n] \to \Delta$ from $(0, I\beta)$ to q that meets all lattice points of Δ has $|\beta|+d(d+1)/2$ steps. As γ has $2d+g-1+|\beta|$ steps, $|\beta|+d(d+1)/2-(2d+g-1+|\beta|) = (d-1)(d-2)/2-g$ lattice points are missed by γ. But $\tilde\gamma$ cannot miss more points, therefore $(d-2)(d-3)/2-g' \leq (d-1)(d-2)/2-g$, i.e. $d - 2 \geq g - g'$. □

Remark 3.12
The same argument can also be applied to other polygons Δ. For example, the

analogous recursion formula for $\mathbb{P}^1 \times \mathbb{P}^1$, i.e. for a $d' \times d$ rectangle $\Delta_{(d',d)}$ reads

$$N^{\alpha,\beta}_{\text{path}}((d',d),g) = \sum_{k:\beta_k>0} k \cdot N^{\alpha+e_k,\beta-e_k}_{\text{path}}((d',d),g) + \sum I^{\beta'-\beta} \binom{\alpha}{\alpha'} \binom{\beta'}{\beta} N^{\alpha',\beta'}_{\text{path}}((d'-1,d),g')$$

for all α, β with $I\alpha + I\beta = d$, where the second sum is taken over all α', β', g' such that $\alpha \leq \alpha$, $\beta' \geq \beta$, $I\alpha' + I\beta' = d$, $g - g' \leq d - 1$ and $|\beta - \beta'| = g' - g - 1$.

4. The Caporaso-Harris formula in the tropical set-up

4.1. **Relative Gromov-Witten invariants in tropical geometry.** We will now define the analogues of the numbers $N^{\alpha,\beta}_{\text{cplx}}(d,g)$ in terms of tropical curves. The definition is quite straightforward:

Definition 4.1
Let C be a simple tropical curve of degree Δ_d and genus g with α_i fixed and β_i non-fixed unbounded ends to the left of weight i for all i. We define the (α,β)-*multiplicity* of C to be

$$\text{mult}_{\alpha,\beta}(C) := \frac{1}{I^\alpha} \cdot \text{mult}(C)$$

where $\text{mult}(C)$ is the usual multiplicity as in definition 2.7 (g).

Let $d \geq 0$ and g be integers, and let α and β be sequences with $I\alpha + I\beta = d$. Then we define $N^{\alpha,\beta}_{\text{trop}}(d,g)$ to be the number of tropical curves of degree Δ_d and genus g with α_i fixed and β_i non-fixed unbounded ends to the left of weight i for all i that pass in addition through a set $\mathcal{P}$ of $2d + g + |\beta| - 1$ points in general position. The curves are to be counted with their respective (α,β)-multiplicities. By [GM05] this definition does not depend on the choice of marked points and fixed unbounded ends.

First of all we will show that this definition actually coincides with the lattice path construction of definition 3.4:

Theorem 4.2
For all d, g, α, β we have $N^{\alpha,\beta}_{\text{trop}}(d,g) = N^{\alpha,\beta}_{\text{path}}(d,g)$.

Proof:
The proof is analogous to the proof of [Mik03] theorem 2. As usual we choose $\lambda(x,y) = x - \varepsilon y$. Let $\mathcal{P}$ be a set of $2d + g + |\beta| - 1$ points on a line H orthogonal to the kernel of λ such that the distance between p_i and p_{i+1} is much bigger than the distance of p_{i-1} and p_i for all i, and such that all points lie below the fixed ends. In other words, if the fixed ends have the y-coordinates $y_1, \dots, y_{|\alpha|}$ then the y-coordinates of p_i are chosen to be less than all $y_1, \dots, y_{|\alpha|}$. Our aim is to show that the number of tropical curves through this special configuration is equal to the number of lattice paths as in section 3. Let C be a tropical curve with the right properties through this set of points. Mark the points where H intersects the fixed ends. The proof of [Mik03] theorem 2 tells us that the edges of Δ (dual to the edges of the curves where they meet $\mathcal{P}$ and the new marked points) form a λ-increasing path from $p = (0,d)$ to $q = (d,0)$. The fact that the fixed ends lie above all other

points tells us that the path starts with α_i steps of lengths i. So we can cut the first part and get a path from $(0, I\beta)$ to q with the right properties.

Next, let a path $\gamma : [0, 2d + g + |\beta| - 1] \to \Delta_d$ be given that starts at $(0, d - I\alpha)$ and ends at q. Extend γ to a path $\gamma_\alpha : [0, |\alpha| + n] \to \Delta_d$ by adding α_i steps of integer length i at $\{x = 0\}$ from $\gamma_\alpha(0) = p$ to $\gamma_\alpha(|\alpha|) = (0, I\beta)$. Add the steps of integer lengths i in an order corresponding to the order of the fixed ends. The recursive definition of $\mu_{\beta,-}(\gamma_\alpha)$ corresponds to counting the possibilities for a dual tropical curve in the half plane below H through $\mathcal{P}$. Passing from γ_α to γ'_α and γ''_α corresponds to counting the possibilities in a strip between H and a parallel shift of H. We end up with a path δ_- which begins with α_i steps of length i and continues with β_i steps of lengths i. This shows that all possible dual curves have the right horizontal ends. Furthermore, $\mu_{\beta,-}(\gamma_\alpha)$ coincides with the number of possible combinatorial types of the curve in the half plane below H times the multiplicity of the part of the curve in the half plane below H. With the same arguments we get that $\mu_{\alpha,+}(\gamma_\alpha)$ is equal to the number of possibilities for the combinatorial type times the multiplicity in the upper half plane. Altogether, we have

$$\begin{aligned} N^{\alpha,\beta}_{\text{path}}(d, g) &= \sum_\gamma \text{mult}_{\alpha,\beta}(\gamma) \\ &= \frac{1}{I^\alpha} \sum_\gamma \text{mult}_{\beta,-}(\gamma_\alpha) \cdot \text{mult}_{\alpha,+}(\gamma_\alpha) \\ &= \frac{1}{I^\alpha} \sum_C \text{mult}(C) = \sum_C \text{mult}_{\alpha,\beta}(C) \\ &= N^{\alpha,\beta}_{\text{trop}}(d, g), \end{aligned}$$

where C runs over all tropical curves with the right properties and γ runs over all paths with the right properties. □

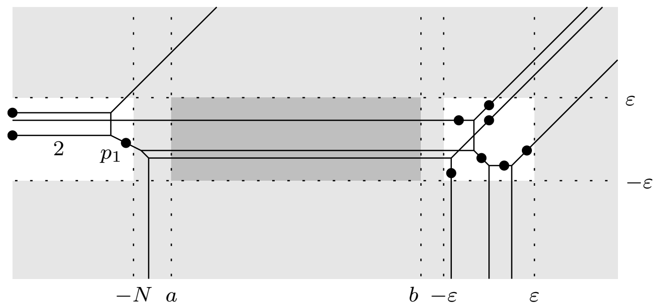
4.2. **The Caporaso-Harris formula.** Of course it follows from theorems 3.11 and 4.2 that the numbers $N^{\alpha,\beta}_{\text{trop}}(d, g)$ satisfy the Caporaso-Harris formula. In this section we will prove this statement directly without using Newton polyhedra and lattice paths. The advantage of this method is that it may be possible to generalize it to curves in higher-dimensional varieties (where the concept of Newton polyhedra cannot be used to describe tropical curves).

Theorem 4.3
The numbers $N^{\alpha,\beta}_{\text{trop}}(d, g)$ satisfy the Caporaso-Harris formula.

Proof:
Let $\varepsilon > 0$ be a small and $N > 0$ a large real number. We choose the fixed unbounded left ends and the set $\mathcal{P} = \{p_1, \ldots, p_n\}$ in tropical general position so that

- the y-coordinates of all p_i and the fixed ends are in the open interval $(-\varepsilon, \varepsilon)$;
- the x-coordinates of $p_2, \ldots, p_n$ are in the open interval $(-\varepsilon, \varepsilon)$;
- the x-coordinate of p_1 is less than $-N$.

In other words, we keep all conditions for the curves in a small horizontal strip and move p_1 very far to the left.



Let us consider a tropical curve C satisfying the given conditions. We want to show that C must "look as in the picture above", i.e. that the curve "splits" into two parts joined by only horizontal lines.

First of all we claim that no vertex of C can have its y-coordinate below $-\varepsilon$: otherwise let V be a vertex with lowest y-coordinate. By the balancing condition there must be an edge pointing downwards from V. As there is no vertex below V this must be an unbounded edge. For degree reasons this edge must have direction $(0, -1)$ and weight 1, and it must be the only edge pointing downwards. By the balancing condition it then follows that at least one other edge starting at V must be horizontal. Again by the balancing condition we can then go from V along this horizontal edge to infinity in the region $\{y \le -\varepsilon\}$. As there are no marked points in this region we could go from V to infinity without passing marked points in two different ways, which is a contradiction to remark 2.10. (This is an important argument which we will use several times in this proof.)

In the same way we see that no vertex of C can have its y-coordinate above ε.

Next, consider the rectangle

$$R := \{(x, y);\ -N \le x \le -\varepsilon, -\varepsilon \le y \le \varepsilon\}.$$

We want to study whether there can be vertices of C within R. Let C_0 be a irreducible component of $C \cap R$. Note that any end of C_0 leaving R at the top or bottom edge must go straight to infinity as we have just seen that there are no vertices of C above or below R. If there are ends of C_0 leaving R at the top *and* at the bottom then we could again go from one infinite end of C to another without passing a marked point, again in contradiction to remark 2.10. So we may assume without loss of generality that C_0 does not meet the top edge of R. With the same argument, we can see that C_0 can meet the top edge of R only in one point.

It follows by the balancing condition that all edges of $C_0 \cap R$ that are not horizontal project to the x-axis to a union of two (maybe empty) intervals $[-N, x_1] \cup [x_2, -\varepsilon]$. But note that the number of edges of C as well as the minimum slope of an edge (and hence the maximum distance an edge can have within R) are bounded by a constant that depends only on the degree of the curves. So we can find $a, b \in \mathbb{R}$ (that depend only on d) such that the interval $[a, b]$ is disjoint from $[-N, x_1] \cup [x_2, -\varepsilon]$, or in other words such that there are no non-horizontal edges in $[a, b] \times [-\varepsilon, \varepsilon]$. In particular, there are then no vertices of C in $[a, b] \times \mathbb{R}$. Hence we see that the curve C must look as in the picture above: we can "cut" it at any line $x = x_0$ with $a < x_0 < b$ and obtain curves on both sides of this line that are joined only by horizontal lines.

There are now two cases to distinguish:

(a) p_1 lies on a horizontal non-fixed end of C. Then the region where $x \leq -\varepsilon$ consists of only horizontal lines. (Otherwise we could again go from one unfixed end to another without meeting a marked point, in contradiction to remark 2.10.) We can hence consider C as having one more fixed end at p_1 and passing through $\mathcal{P} \setminus \{p_1\}$. We just have to multiply with the weight of this end, as the multiplicity of curves with fixed ends is defined as $\frac{1}{I^\alpha} \cdot \text{mult}(C)$. Therefore the contribution of these curves to $N_{\text{trop}}^{\alpha,\beta}(d,g)$ is

$$\sum_{k:\beta_k>0} k \cdot N_{\text{trop}}^{\alpha+e_k,\beta-e_k}(d,g).$$

(b) p_1 does not lie on a horizontal end of C (as in the picture above). Then C can be separated into a left and a right part as above. As the left part contains only one marked point it follows again by remark 2.10 that the left part has exactly one end in direction $(0,-1)$ and $(1,1)$ each, together with some more horizontal ends.

Hence the curve on the right must have degree $d-1$. Let us denote this curve by C'.

How many possibilities are there for C? Assume that $\alpha' \leq \alpha$ of the fixed horizontal ends only intersect the part $C \setminus C'$ and are not adjacent to a 3-valent vertex of $C \setminus C'$. Then C' has α' fixed horizontal ends. Given a curve C' of degree $d-1$ with α' fixed ends through $\mathcal{P} \setminus \{p_1\}$, there are $\binom{\alpha}{\alpha'}$ possibilities to choose which fixed ends of C belong to C'. C' has $d-1-I\alpha'$ non-fixed horizontal ends. Let β' be a sequence which fulfills $I\beta' = d-1-I\alpha'$, hence a possible choice of weights for the non-fixed ends of C'. Assume that $\beta'' \leq \beta'$ of these ends are adjacent to a 3-valent vertex of $C \setminus C'$ whereas $\beta' - \beta''$ ends intersect $C \setminus C'$. The irreducible component of $C \setminus C'$ which contains p_1 has to contain the two ends of direction $(0,-1)$ and $(1,1)$ due to the balancing condition. Also, it can contain some ends of direction $(-1,0)$ — but these have to be fixed ends then, as p_1 cannot separate more than two (nonfixed) ends. So all the β nonfixed ends of direction $(-1,0)$ have to intersect $C \setminus C'$ — therefore they have to be ends of C'. That is, $\beta' - \beta'' = \beta$ (in particular $\beta' \geq \beta$). Given C', there are $\binom{\beta'}{\beta}$ possibilities to choose which ends of C' are also ends of C. Furthermore, we have

$$\begin{aligned}\text{mult}_{\alpha,\beta}(C) &= \frac{1}{I^\alpha}\,\text{mult}(C) = \frac{1}{I^\alpha} \cdot I^{\alpha-\alpha'} \cdot I^{\beta'-\beta} \cdot \text{mult}(C') \\ &= \frac{1}{I^{\alpha'}} \cdot I^{\beta'-\beta} \cdot \text{mult}(C') = I^{\beta'-\beta} \cdot \text{mult}_{\alpha',\beta'}(C')\end{aligned}$$

where the factors $I^{\alpha-\alpha'}$ and $I^{\beta'-\beta}$ arise due to the 3-valent vertices which are not part of C'.

To determine the genus g' of C', note that C' has by $|\alpha+\beta''|$ fewer vertices than C and by $|\alpha+\beta''|-1+|\beta''|$ fewer bounded edges — there are $|\alpha+\beta''|-1$ bounded edges in $C \setminus C'$, and $|\beta''|$ bounded edges are cut. Hence $g' = 1 - \#\Gamma^0 + |\alpha+\beta''| - \#\Gamma^1_0 - |\alpha+\beta''| - |\beta''| = g - (|\beta''|-1)$. Furthermore, $g - g' \leq d-2$ as at most $d-2$ loops may be cut. Now given a curve C' with α' fixed and β' nonfixed bounded edges through $\mathcal{P} \setminus \{p_1\}$, and having chosen which of the α fixed ends of C are also fixed ends of C' and which of the β' ends of C' are also ends of C, there is only one possibility to add an irreducible component through p_1 to make it a possible curve C with α fixed ends and β nonfixed. The positions and directions of all bounded edges are prescribed by the point p_1, by the positions of the $\beta' - \beta$ ends to the left of C', and by the $\alpha - \alpha'$ fixed ends. Hence we can count the possibilities for C' (times the factor $\binom{\alpha}{\alpha'} \cdot \binom{\beta'}{\beta} \cdot I^{\beta'-\beta}$) instead of the possibilities for C (where the possible choices for α', β' and g' have to satisfy just the conditions we know from the Caporaso-Harris formula).

The sum of these two contributions gives the required Caporaso-Harris formula. □

4.3. **The tropical Caporaso-Harris formula for irreducible curves.** So far we have only considered not necessarily irreducible curves since this case is much simpler (the irreducibility condition is hard to keep track of when we split a curve into two parts as in the proof of theorem 4.3). In this section we want to show how the ideas of section 4.2 can be carried over to the case of irreducible curves.

Definition 4.4
Let $N_{\text{trop}}^{\text{irr},(\alpha,\beta)}(d,g)$ be the number of irreducible tropical curves of degree Δ_d and genus g with α_i fixed and β_i non-fixed horizontal left ends of weight i for all i that pass in addition through a set of $|\beta| + 2d + g - 1$ points in general position. Again, the curves are to be counted with their (α,β)-multiplicity as of definition 4.1. As in definition 4.1 it follows from [GM05] that these numbers do not depend on the choice of the points and the positions of the fixed ends.

Theorem 4.5
The numbers $N_{\text{trop}}^{\text{irr},(\alpha,\beta)}(d,g)$ satisfy the recursion relations

$$\begin{aligned} N_{\text{trop}}^{\text{irr},(\alpha,\beta)}(d,g) = & \sum_{k:\beta_k>0} k \cdot N_{\text{trop}}^{\text{irr},(\alpha+e_k,\beta-e_k)}(d,g) \\ & + \sum \frac{1}{\sigma} \binom{2d+g+|\beta|-2}{2d_1+g_1+|\beta^1|-1,\ldots,2d_k+g_k+|\beta^k|-1} \\ & \cdot \binom{\alpha}{\alpha^1,\ldots,\alpha^k} \\ & \cdot \prod_{j=1}^{k} \left(\binom{\beta^j}{\beta^j - \beta^{j'}} \cdot I^{\beta^{j'}} \cdot N_{\text{trop}}^{\text{irr},(\alpha^j,\beta^j)}(d_j,g_j) \right) \end{aligned}$$

with the second sum taken over all collections of integers $d_1,\ldots,d_k$ and $g_1,\ldots,g_k$ and all collections of sequences $\alpha^1,\ldots,\alpha^k$, $\beta^1,\ldots,\beta^k$ and $\beta^{1'},\ldots,\beta^{k'}$ satisfying

$$\begin{aligned} \alpha^1 + \ldots + \alpha^k &\le \alpha; \\ \beta^1 + \ldots + \beta^k &= \beta + \beta^{1'} + \ldots + \beta^{k'}; \\ |\beta^{j'}| &> 0; \\ d_1 + \ldots + d_k &= d-1; \\ g - (g_1 + \ldots + g_k) &= |\beta^{1'} + \ldots + \beta^{k'}| + k. \end{aligned}$$

Here as usual $\binom{n}{a_1,\dots,a_k}$ *denotes the multinomial coefficient*

$$\binom{n}{a_1,\dots,a_k} = \frac{n!}{a_1!\cdot\ldots\cdot a_k!(n-a_1-\ldots-a_k)!}$$

and correspondingly, for sequences $\alpha,\alpha^1,\dots,\alpha^k$ *the multinomial coefficient is*

$$\binom{\alpha}{\alpha^1,\dots,\alpha^k} = \prod_i \binom{\alpha_i}{\alpha_i^1,\dots,\alpha_i^k}.$$

The number σ *is defined as follows: Define an equivalence relation on the set* $\{1,2,\dots,k\}$ *by* $i :\sim j$ *if* $d_i = d_j$, $g_i = g_j$, $\alpha^i = \alpha^j$, $\beta^i = \beta^j$ *and* $\beta^{i'} = \beta^{j'}$. *Then* σ *is the product of the factorials of the cardinalities of the equivalence classes.*

Note that this recursion formula coincides with the corresponding Caporaso-Harris formula for irreducible curves (see [CH98] section 1.4).

Proof:
The proof is analogous to that of theorem 4.3. Fix a set $\mathcal{P} = \{p_1,\dots,p_n\}$ in tropical general position with p_1 very far left as in the proof of theorem 4.3. Let C be an irreducible tropical curve with the right properties through the points. The first term in the recursion formula (that corresponds to curves with only horizontal lines in the area where $x \le -\varepsilon$) follows in the same way as in theorem 4.3. So assume that p_1 does not lie on a horizontal end of C. As before we get a curve C' of degree $d-1$ to the right of the cut. The curve C' does not need to be irreducible however. It can split in k irreducible components $C_1,\dots,C_k$ of degree $d_1,\dots,d_k$. Then $d_1+\ldots+d_k = d-1$. As before, we would like to count the possibilities for the C_j separately, and then determine how many ways there are to make a possible curve C out of a given choice of $C_1,\dots,C_k$. So let the C_j be curves of degree d_j through the set of points $\mathcal{P}\setminus\{p_1\}$. Let C_j have α^j fixed horizontal ends and β^j nonfixed horizontal ends, satisfying $|\beta^j|+|\alpha^j| = d_j$. Then $\alpha^1+\ldots+\alpha^k \le \alpha$, and there are $\binom{\alpha}{\alpha^1,\dots,\alpha^k}$ possibilities to choose which fixed ends of C belong to which C_j. As before, the irreducible component of $C\setminus(C_1\cup\ldots\cup C_k)$ which contains p_1 is fixed by only one point, therefore it contains none of the β nonfixed ends of C. That is, all β nonfixed ends have to intersect the part $C\setminus(C_1\cup\ldots\cup C_k)$ and have to be ends of one of the C_j then. Assume that $\beta^{j'}$ of the β^j ends are adjacent to a 3-valent vertex of $C\setminus(C_1\cup\ldots\cup C_k)$ whereas $\beta^j-\beta^{j'}$ ends just intersect $C\setminus(C_1\cup\ldots\cup C_k)$. As C is irreducible we must have $|\beta^{j'}|>0$ as otherwise C_j would form a separate component of C. Then given the curves C_j through $\mathcal{P}\setminus\{p_1\}$, there are $\binom{\beta^j}{\beta^j-\beta^{j'}}$ possibilities to choose which of the nonfixed ends of C_j are also ends of C, and $\beta+\sum\beta^{j'} = \sum\beta^j$. Each C_j is fixed by $2d_j+g_j+|\beta^j|-1$ points, where g_j denotes the genus of C_j. (There cannot be fewer points on one of the C_j, since otherwise the unbounded ends or loops could not be separated by the points, in contradiction to remark 2.10.) Therefore, there are

$$\binom{2d+g+|\beta|-2}{2d_1+g_1+|\beta^1|-1,\dots,2d_k+g_k+|\beta^k|-1}$$

possibilities to distribute the points $p_2, \dots, p_n$ on the components $C_1, \dots, C_k$. Furthermore, we have

$$\begin{aligned}
\operatorname{mult}_{\alpha,\beta}(C) &= \frac{1}{I^\alpha} \operatorname{mult}(C) \\
&= \frac{1}{I^\alpha} \cdot I^{\alpha-\alpha^1-\dots-\alpha^k} \cdot I^{\beta^{1'}+\dots+\beta^{k'}} \cdot \operatorname{mult}(C_1) \cdot \ldots \cdot \operatorname{mult}(C_k) \\
&= \frac{1}{I^{\alpha^1+\dots+\alpha^k}} \cdot I^{\beta^{1'}+\dots+\beta^{k'}} \cdot \operatorname{mult}(C_1) \cdot \ldots \cdot \operatorname{mult}(C_k) \\
&= I^{\beta^{1'}+\dots+\beta^{k'}} \cdot \operatorname{mult}_{\alpha^1,\beta^1}(C_1) \cdot \ldots \cdot \operatorname{mult}_{\alpha^k,\beta^k}(C_k)
\end{aligned}$$

where the factors $I^{\alpha-\alpha^1-\dots-\alpha^k}$ and $I^{\beta^{1'}+\dots+\beta^{k'}}$ arise due to the 3-valent vertices which are not part of $C_1, \dots, C_k$. Concerning the genus, note that C has $|\alpha - \alpha^1 - \ldots - \alpha^k + \beta^{1'} + \ldots + \beta^{k'}|$ more vertices than $C_1 \cup \ldots \cup C_k$ and $|\alpha - \alpha^1 - \ldots - \alpha^k + \beta^{1'} + \ldots + \beta^{k'}| - 1 + |\beta^{1'} + \ldots + \beta^{k'}|$ more bounded edges (there are $|\alpha - \alpha^1 - \ldots - \alpha^k + \beta^{1'} + \ldots + \beta^{k'}| - 1$ bounded edges in $C \setminus (C_1 \cup \ldots \cup C_k)$ and $|\beta^{1'} + \ldots + \beta^{k'}|$ bounded edges are cut). Hence

$$\begin{aligned}
g &= 1 + g_1 + \ldots + g_k - k \\
&\quad - (|\alpha - \alpha^1 - \ldots - \alpha^k + \beta^{1'} + \ldots + \beta^{k'}|) \\
&\quad + |\alpha - \alpha^1 - \ldots - \alpha^k + \beta^{1'} + \ldots + \beta^{k'}| - 1 + |\beta^{1'} + \ldots + \beta^{k'}| \\
&= \sum g_j + \sum |\beta^{j'}| - k.
\end{aligned}$$

This proves the recursion formula except for the factor $\frac{1}{\sigma}$. This factor is simply needed because up to now we count different curves if two components C_i and C_j of C' are identical, depending on whether C_i is the i-th component or C_j is the i-th component. Therefore we have to divide by σ. □

References

[CH98] Lucia Caporaso and Joe Harris, *Counting plane curves of any genus*, Invent. Math. **131** (1998), no. 2, 345–392.

[GM05] Andreas Gathmann and Hannah Markwig, *The number of tropical plane curves through points in general position*, Journal für die reine und angewandte Mathematik (to appear), preprint math.AG/0504390.

[Mik03] Grigory Mikhalkin, *Enumerative tropical geometry in* $\mathbb{R}^2$, J. Amer. Math. Soc. **18** (2005), 313–377, preprint math.AG/0312530.

[NS04] Takeo Nishinou and Bernd Siebert, *Toric degenerations of toric varieties and tropical curves*, preprint math.AG/0409060.

Andreas Gathmann, University of Kaiserslautern, Department of Mathematics, PO Box 3049, 67653 Kaiserslautern, Germany

E-mail address: `andreas@mathematik.uni-kl.de`

Hannah Markwig, University of Kaiserslautern, Department of Mathematics, PO Box 3049, 67653 Kaiserslautern, Germany

E-mail address: `markwig@mathematik.uni-kl.de`