UNITRIANGULAR SHAPE OF DECOMPOSITION MATRICES

[jt with O. Brunat & J. Taylor]

H finite group, $\ell > 0$ prime number

$$\begin{tikzcd}
K \arrow[r, hookleftarrow, no head] & \mathcal{O} \arrow[r, two heads] \arrow[l, hook] & k \\
\mathbb{Q}_\ell \arrow[u, phantom, "\cup" description] & \mathbb{Z}_\ell \arrow[u, phantom, "\cup" description] \arrow[l, hook] \arrow[r, two heads] & \mathbb{F}_\ell \arrow[u, phantom, "\cup" description]
\end{tikzcd}$$

ordinary representations $\quad \mathrm{Irr}_K H \dashleftarrow\dashrightarrow \mathrm{IBr}_k H \quad$ modular representations

decomposition map

Ex: $H = \mathfrak{S}_3$ and $\ell = 3$ $\qquad \# \mathrm{Irr}\,\mathfrak{S}_3 = 3$

$\# \mathrm{IBr}\,\mathfrak{S}_3 = 2$

decomposition matrix

$$\begin{bmatrix} 1 & \cdot \\ 1 & 1 \\ \cdot & 1 \end{bmatrix}$$

unitriangular shape

Consequences:
* parametrisation of $\mathrm{IBr}(H)$
* natural lattices for $\mathrm{Irr}(H)$

Geck conjectured that property for finite reductive groups $H = G(\mathbb{F}_q)$ (e.g $GL_n(q), Sp_{2n}(q), \dots, E_8(q)$)
and $\ell \nmid q$, $\ell \gg 0$.

I - Motivating example : $GL_n(q)$

$$\left\{ \begin{matrix} \text{unipotent} \\ \text{characters} \end{matrix} \right\} \overset{1:1}{\longleftrightarrow} \left\{ \begin{matrix} \text{partitions} \\ \text{of } n \end{matrix} \right\} \overset{1:1}{\longleftrightarrow} \left\{ \begin{matrix} \text{unipotent} \\ \text{conj. classes} \end{matrix} \right\}$$

$$\rho_\lambda \longleftarrow\!\!\shortmid \quad \lambda \quad \longmapsto \mathcal{O}_\lambda$$

s.t $\rho_{(n)}$ = trivial char. $\qquad \mathcal{O}_{(n)}$ = regular class
$\rho_{(1^n)}$ = Steinberg char. $\qquad \mathcal{O}_{(1^n)} = \{1\}$ trivial class

Fact : $\boxed{\rho_\lambda(\mathcal{O}_\mu) = 0 \text{ if } \mathcal{O}_\mu \not\subseteq \overline{\mathcal{O}_\lambda}}$ (i.e. $\mu \not\trianglelefteq \lambda$)

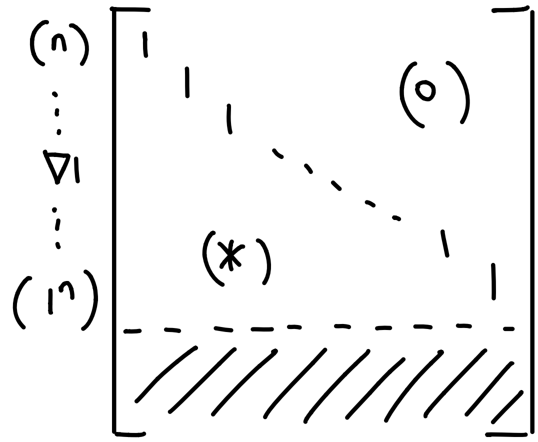
On the other hand, there are unipotently supported char. Γ_λ of projective modules s.t

$$\boxed{\Gamma_\lambda^*(\mathcal{O}_\mu) = 0 \text{ if } \mathcal{O}_\lambda \not\subseteq \overline{\mathcal{O}_\mu}}$$

Consequence $\langle \Gamma_\lambda^* ; \rho_\mu \rangle = 0$ if $\lambda \not\trianglelefteq \mu$

and $\langle \Gamma_\lambda^* ; \rho_\lambda \rangle = 1$

$\Rightarrow$ the decomposition matrix has the following shape



II - Generalisation

G connected reductive group / $\mathbb{F}_q$

Unipotent characters fall into families, associated to (special) unipotent orbits of G

$$\mathrm{Uch}(G(\mathbb{F}_q)) = \bigsqcup_{\mathcal{O}} \mathrm{Uch}(\mathcal{O})$$

Ex: $\mathrm{Uch}(\{1\}) = \{\mathrm{St}\}$ $\qquad$ $\mathrm{Uch}(\mathcal{O}_\lambda) = \{\rho_\lambda\}$

$\mathrm{Uch}(\mathcal{O}_{reg}) = \{1\}$ $\qquad$ for $G = GL_n$

Lusztig: $\mathcal{O} \rightsquigarrow$ "small" finite group $A_{\mathcal{O}}$

$$\mathrm{Uch}(\mathcal{O}) \longleftrightarrow \{(a,\psi) \mid a \in A_{\mathcal{O}}, \psi \in \mathrm{Irr}\, C_{A_{\mathcal{O}}}(a)\}/\sim$$

$$\rho(\mathcal{O},a,\psi) \longleftarrow (a,\psi)$$

Ex: $G = GL_n$ $\quad A_{\mathcal{O}} = 1$ for all $\mathcal{O}$

$$\rho_\lambda = \rho(\mathcal{O}_\lambda, 1, 1)$$

Kawanaka defines projective characters

$$\Gamma_{(\mathcal{O}, a, \psi)}$$

which are NOT unipotently valued as before but conj that

$$\langle \Gamma_{(\mathcal{O},a,\psi)}^* ; \rho(\mathcal{O}', b, \varphi) \rangle = \begin{cases} 0 & \text{if } \mathcal{O} \not\subseteq \overline{\mathcal{O}'} \\ \delta_{(a,\psi),(b,\varphi)} & \text{otw} \end{cases}$$ ✓

$\Rightarrow$ Geck's conjecture

Pb: values of $\rho(\mathcal{O}, a, \psi)$ and $\Gamma_{(\mathcal{O},a,\psi)}$ are complicated

Solution: look at the Fourier transform w.r.t $A_{\mathcal{O}}$

$\rightsquigarrow$ simpler values, more vanishing!

Thm [Brunat-D-Taylor] Given $\mathcal{O}, a, \psi$ there is a unique $\rho \in \mathrm{Uch}(\mathcal{O})$ s.t

$$\langle \Gamma^*_{(\mathcal{O},a,\psi)} ; \rho \rangle = 1$$

all other occur with multiplicity zero

In part. Geck's conjecture holds