(Emily)

HARISH-CHANDRA SERIES OF FINITE UNITARY GROUPS

Goal: $\boxed{\text{describe}}$ the simple modules of a finite gp of Lie type $GL_n(\mathbb{F}_q)$, $GU_n(\mathbb{F}_q)$, $Sp_{2n}(\mathbb{F}_q)$, ...

- find a natural labelling set
- how are modules constructed via induction?
- over what field?
  - over $\mathbb{C}$, these problems were solved mid-20th cent.

  e.g Green found the char. table of $GL_n(\mathbb{F}_q)$ over $\mathbb{C}$
- nice answer for specific type of rep? "unipotent rep."

  the ones which appear in the cohomology of Deligne-Lusztig varieties

In this talk: give a combinatorial description of the Harish-Chandra series of simple modules in unipotent blocks of finite unitary groups $GU_n(\mathbb{F}_q)$ in positive characteristic $\ell > 0$

in fact $\ell > \frac{n}{e}$ where e = order of $-q$ mod ℓ and $e \geq 3$ odd

Additional motivation : knowing HC-series may help to compute decomposition numbers.

Background : $Gl_n(q)$ over k of char. 0 "big enough"

$$\begin{aligned} Fr : Gl_n(\overline{\mathbb{F}}_q) &\longrightarrow Gl_n(\overline{\mathbb{F}}_q) \\ (a_{ij}) &\longmapsto (a_{ij}^q) \end{aligned}$$

$$Gl_n(\mathbb{F}_q) = Gl_n(\overline{\mathbb{F}}_q)^{Fr}$$

$$\left\{ \begin{array}{l} \text{Unipotent rep.} \\ \text{of } kGl_n(\mathbb{F}_q) \end{array} \right\} \xleftrightarrow{1:1} \{ \lambda \vdash n \}$$

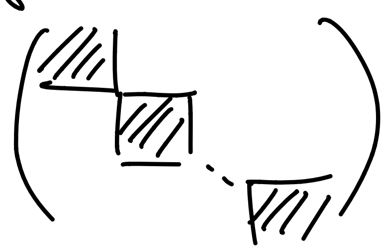
$\lambda = (\lambda_1 \geq \lambda_2 \geq \cdots \geq \lambda_s > 0)$

$\lambda_1 + \lambda_2 + \cdots + \lambda_s = n$

HC series of $Gl_n(q)$ over k

Consider conjugacy classes of levi subgroups

$$L = GL_\mu = GL_{\mu_1} \times GL_{\mu_2} \times \cdots \times GL_{\mu_s} =$$



where $\mu \vdash n$

HC induction R_L^G exact functor $KL\text{-mod} \longrightarrow KG\text{-mod}$

When does ρ_λ, a unipotent rep. of G occur as a summand of $R_L^G(\rho)$ where ρ is some unip. rep. of L

HC restriction : ${}^*R_L^G$: KG-mod $\longrightarrow$ KL-mod

$(R_L^G, {}^*R_L^G)$ a biadjoint pair

def: $M \in KG$-mod is cuspidal if ${}^*R_L^G(M) = 0$

for any $L \neq G$ of the form described above

Cuspidals are building blocks

Unipotent cuspidals of $KGl_n(q)$-mod : only when $n=1$ $\lambda=(1)$

<u>Finite general unitary groups</u>

$$F : Gl_n(\overline{\mathbb{F}}_q) \longrightarrow Gl_n(\overline{\mathbb{F}}_q)$$
$$(a_{ij}) \longmapsto Fr((a_{ji})^{-1})$$

$$GU_n(q) := Gl_n(\overline{\mathbb{F}}_q)^F \subseteq Gl_n(q^2)$$

<u>Over K</u> : [Lusztig-Srinivasan]

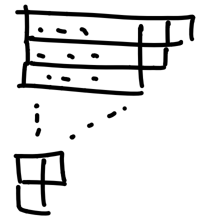
$$\left\{ \begin{array}{l} \text{unipotent rep.} \\ \text{of } KGU_n(q) \end{array} \right\} \overset{1:1}{\longleftrightarrow} \{ \lambda \vdash n \}$$

but HC-series are more complicated.

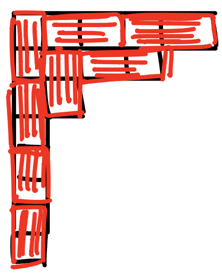
weird mix of type A & B combinatorics

Classification of cuspidals

ρ_λ is cuspidal $\iff$ λ is a "staircase partition"
$(t, t-1, t-2, \dots, 2, 1)$



$\iff$ λ is a 2-core

2-cores $\quad \mu =$ 

remove all dominoes
what remains is a staircase
partition $\Delta_t = (t, .., 1)$ for some $t \geq 0$

$$\mu \longmapsto (\Delta_t \,,\, (\mu^1, \mu^2))$$

2-core of μ

2 quotient of μ (a bipartition)
with $|\mu^1| + |\mu^2| = \#$ dominoes removed
from μ

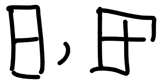
Partitions of n $\longleftrightarrow$ $\mathrm{Irr}_{\mathbb{C}}\, \mathfrak{S}_n$
Bipartitions of n $\longleftrightarrow$ $\mathrm{Irr}_{\mathbb{C}}\, W(B_n)$ Weyl gp of type B_n

$$GU_n(q) \supseteq L = GU_m(q) \times GL_{\mu_1}(q^2) \times \cdots \times GL_{\mu_s}(q^2)$$
with $n = m + 2|\mu|$
these are the Levis used for HC-theory

$$\begin{tikzcd} R_L^G(\rho_\lambda) \arrow[d, leftrightarrow] & \rho_\lambda \boxtimes \rho_{\chi_1} \boxtimes \cdots \boxtimes \rho_{\chi_s} \arrow[l, squiggly] \arrow[d, leftrightarrow] \\ \mathrm{Ind}_{W'}^{W}(\ldots) & (\lambda^1,\lambda^2) \boxtimes \chi_1 \boxtimes \cdots \boxtimes \chi_s \arrow[l, squiggly] \end{tikzcd}$$

<u>Special case</u> : $L = GU_n(q) \times GL_1(q^2) \leq GU_{n+2}(q)$

$X = \rho_\lambda \boxtimes \mathrm{triv}$, $R_L^G(X)$ computed as:

- take λ^1, λ^2 2-quotient, then sum over adding boxes in all possible ways
  e.g 

- then $R_L^G(X) = \bigoplus_\mu X_\mu$ where 2-core(μ) = 2-core(λ)
  2-quotient(μ) is one of the bipart. obtained in the previous step

<u>Positive characteristic $\ell \nmid q$</u> same questions - different answers

simple modules in unipotent blocks have same labelling as in char. 0 i.e by partitions of n

<u>$GL_n(q)$</u> : e = order of $q \bmod \ell$
$e \geq 2$ and $\ell > n$

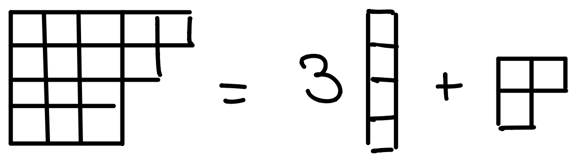
X_λ simple unipotent rep. corresponding to λ

X_λ is cuspidal $\iff$ $n = e$ and $\lambda = (1^e)$
or $n = 1$ and $\lambda = (1)$

<u>HC series</u> λ^t = transpose partition of λ

Write $\lambda^t = e\sigma + \nu$ "dividing λ by e with remainder"

<u>$e=3$</u> :



cuspidal support of X_λ = $GL_e(q)^{|\sigma|} \times GL_1(q)^{|\nu|}$
(smallest Levi L such that $R_L^G(Y) \twoheadrightarrow X_\lambda$ for some Y)

<u>HC series of $GU_n(q)$ in char. ℓ</u>

$e :=$ order of $-q$ mod ℓ $\quad \ell > n$

- $e = 1$, then X_λ is cuspidal $\iff$ λ is 2-regular [Geck-Hiss-Malle]
- e even [Geck-Hiss-Malle]
  the answer is given in terms of $GL_n(q)$ combinatorics
- $e \geqslant 3$ odd [Gerber-Hiss-Jacon] noticed that there is a crystal graph on level 2 Fock space

(has to do with bipartitions / type B Weyl group) that appears to describe HC-induction for $GU_n(q)$

$$R^{GU_{n+2}(q)}_{GU_n(q)\times GL_1(q^2)}(X_\lambda \boxtimes X_1) \twoheadrightarrow \bigoplus_\mu X_\mu$$

where μ^1, μ^2 is obtained from λ^1, λ^2 by adding a "good node" acc. to the crystal graph rule proved by [DVV], and identified the cuspidals combinatorially

Thm [N] There is a combinatorial formula in terms of the $\widehat{sl}_e$-crystal and a second crystal structure ($\underline{sl_\infty\text{-crystal}}$) that describes the HC series of any unipotent rep. X_λ of $GU_n(q)$

adds vertical strip of e boxes at a time