(Noelia)

GALOIS ACTION ON THE PRINCIPAL BLOCK

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1- Introduction

G finite group, p prime $p \mid |G|$
$P \in Syl_p(G)$, $|G|_p = p^a = |P|$
$\mathrm{Irr}\,G$ irred. characters (complex valued)
$X(G)$: character table of G

<u>Problem 12</u> [Brauer] Does $X(G)$ determine if P is abelian ?

A) 1995 [Kimmerle - Sandling]
$X(G) = X(H)$ $\quad Q \in Syl_p(G)$
then P abelian $\Leftrightarrow$ Q abelian and in that case $P \cong Q$

<u>Problem 23</u> (BHZ) Brauer's height zero conjecture

$\mathrm{Irr}\,B = \mathrm{Irr}_0\,B \iff D$ is abelian

- $(\Leftarrow)$ Kessar-Malle '13
- $(\Rightarrow)$ Reduced, Navarro-Späth '14

$B \in \mathrm{Bl}(G)$ p-block of G , $\mathrm{Irr}\, G = \bigsqcup_{B \in \mathrm{Bl}(G)} \mathrm{Irr}\, B$

$B \rightsquigarrow \{D^g\}$ D defect group , p-subgroup of G

$|D| = p^{d(B)}$ $d(B)$ = defect of B

Then $\min\{\chi(1)_p \mid \chi \in \mathrm{Irr}\, B\} = p^{a-d(B)}$

$\rightsquigarrow$ $\forall\, \psi \in \mathrm{Irr}\, B$, $\psi(1)_p = p^{a-d(B)+h}$
where $h \geq 0$ is the height of ψ

$\mathrm{Irr}_0(B) = \{\psi \in \mathrm{Irr}\, B \mid h = 0\}$ height zero chars.

$B_0(G)$ principal block (s.t $1_G \in \mathrm{Irr}\, B_0(G)$)
$1 = 1_G(1)_p = p^{a-d(B_0)+h}$ $\Rightarrow a = d(B_0)$
$\Rightarrow D \in \mathrm{Syl}_p(G)$

2- Global-local conjecture

Conj [Mckay] $|\mathrm{Irr}_{p'}(G)| = |\mathrm{Irr}_{p'}(N_G(P))|$

- Reduced '07 by Isaacs-Malle-Navarro
- '16 Malle-Späth for $p=2$

* Blocks $\quad \mathrm{Bl}(G|D)$ blocks of G with defect D

$$\begin{array}{ccc} \mathrm{Bl}(G|D) & & B \\ \updownarrow 1{:}1 & \text{Bauer's correspondence} & \downarrow \\ \mathrm{Bl}(N_G(D)|D) & & b \end{array}$$

Conj [Alperin-Mckay] $\quad |\mathrm{Irr}_0 B| = |\mathrm{Irr}_0 b|$

- reduced, Späth '13
  $\rightsquigarrow$ inductive A.M condition

* Galois action $\quad |G| = n$, ξ primitive nth root of 1

$\chi \in \mathrm{Irr}\, G$ then $\forall g \in G \quad \chi(g) \in \mathbb{Q}(\xi)$

$\sigma \in \mathrm{Gal}(\mathbb{Q}(\xi)/\mathbb{Q}) =: \mathcal{G}$ acts on $\mathrm{Irr}(G)$ by

$$\chi^\sigma(g) := (\chi(g))^\sigma$$

$\mathcal{H} := \{\sigma \in \mathcal{G} \mid \exists\, r \geq 0 \ \forall \eta \ p' \text{ root of } 1 \text{ then } \sigma(\eta) = \eta^{p^r}\}$

Galois-Mckay conj [Navarro] $\quad$ for all $\sigma \in \mathcal{H}$

$$|(\mathrm{Irr}_{p'} G)^\sigma| = |(\mathrm{Irr}_{p'} N_G(P))^\sigma|$$

- p-solvable [Turull]

- cyclic Sylow [Navarro]
- alternating groups [Brunat - Nath]
- Lie type in defining char. [Ruhstorfer]
- reduced [Navarro - Späth - Vallejo]

* Blocks and Galois

Blockwise Galois-McKay [Navarro '04] $\sigma \in \mathcal{H}$

$$|(\mathrm{Irr}_0 B)^\sigma| = |(\mathrm{Irr}_0 b)^\sigma|$$

- p-solvable [Turull]
- cyclic defect [Navarro]

3 - A particular element of $\mathcal{H}$

$e \geq 1$ σ_e fixes p' roots of 1
sends η p-power root of 1 to η^{p^e+1}

$\rightsquigarrow \sigma_e \in \mathcal{H}$

Thm [Navarro-Tiep] $\mathrm{Irr}_{p'}(B_0(G)) = \mathrm{Irr}_{p'}(B_0(G))^{\sigma_e}$
$\Rightarrow \exp(P/[P,P]) \leq p^e$

$\underline{\text{Conj}}$ [N-T] $\mathrm{Irr}_{p'}(G) = \mathrm{Irr}_{p'}(G)^{\sigma_e} \iff \exp(P/[P,P]) \leq p^e$

- Malle $p = 2$

4- Our work

$\underline{\text{Thm A}}$ $p = 2, 3$

$|\mathrm{Irr}_{p'}(B_0(G))^{\sigma_1}| = p \iff P$ cyclic

- $p > 3$ $G = D_{2p}$ $(\Leftarrow)$ doesn't hold
- $p = 5, 7, 11$ $(\Rightarrow)$ consequence of blockwise Galois-McKay

$\underline{\text{Lemma}}$ $\mathrm{Lin}(P)^{\sigma_1} = \mathrm{Irr}(P/\Phi(P))$

Frattini group

$\underline{\text{proof}}$: "$\leq$" $\lambda \in \mathrm{Lin}(P)^{\sigma_1}$, $\lambda \neq 1_P$

$$\lambda(x)^{\sigma_1} = \lambda(x)^p \lambda(x)$$

$$\lambda(x)^{\sigma_1} = \lambda(x) \quad \text{hence } \lambda^p = 1$$

i.e $[P : \mathrm{Ker}\,\lambda] = o(\lambda) = p$

$\Rightarrow P/\mathrm{Ker}\,\lambda \simeq \mathbb{Z}/p\mathbb{Z}$ and therefore $\mathrm{Ker}\,\lambda \supseteq \Phi(P)$

"$\geq$":
$$\lambda(x)^{\sigma_1} = \lambda(x)^p \lambda(x) = \lambda(x\Phi(P))^p \lambda(x\Phi(P)) = \lambda(x) \qquad \square$$

* P cyclic iff $|P/\Phi(P)| = p$
  iff $|(\mathrm{Lin}\, P)^{\sigma_1}| = p$
  iff $|\mathrm{Irr}_{p'}(B_0(P))^{\sigma_1}| = p$

Thm [Navarro, RSV] If $P \trianglelefteq G$

$$\mathrm{Irr}_{p'}(B_0(G))^{\sigma_1} = \mathrm{Irr}(G/\Phi(P)O_{p'}(G))$$

[Fong] G p-solvable $\Rightarrow$ $\mathrm{Irr}(B_0(G)) = \mathrm{Irr}(G/O_{p'}(G))$

$\overline{G} = G/\Phi(P)O_{p'}(G)$, $\overline{P} = P/\Phi(P)$

if $\mathrm{Irr}(\overline{G}) = p \in \{2,3\}$

- $p=2$: $\overline{G} = C_2 = \overline{P}$
- $p=3$: $\overline{G} = C_3 = \overline{P}$
  or $\overline{G} = S_3$ $\overline{P} = C_3$

* CFSG
  - $p = 2$

$$\begin{tikzcd}
& G \arrow[dl, dash] \arrow[dr, equal] & \\
PN & & M \\
& N \arrow[ul, dash] \arrow[ur, dash] &
\end{tikzcd}$$

Cyclic 2-Sylow
$\Rightarrow \exists$ normal complement

  - ($p=3$) [Herzog-Brauer] and P cyclic
    - p-solvable
    - if $M \trianglelefteq G$ then $p \nmid |M|$ or $p \nmid [G:M]$

Thm B S non-abelian simple

- $p=2$ $\quad 1_S, \varphi_1, \varphi_2 \in \mathrm{Irr}_{2'}(B_0(S))^{\sigma_1}$
  φ_1, φ_2 are not $\mathrm{Aut}(S)$-conjugate
  $X/S \in \mathrm{Syl}_2(\mathrm{Aut}(S)/S)$ $\quad \varphi_1$ is X-invariant
- $p=3$
  * P cyclic $\quad |\mathrm{Irr}_{3'}(B_0(S))^{\sigma_1}| = 3$
    $1_S \neq \varphi_1, \varphi_2 \in \mathrm{Irr}_{3'}(B_0(S))^{\sigma_1}$ not $\mathrm{Aut}(S)$-conj.
    and φ_1 X-invariant
  * P non-cyclic $\quad 1_S, \varphi_1, \varphi_2, \varphi_3 \in \mathrm{Irr}_{3'}(B_0(S))^{\sigma_1}$
    $\varphi_1, \varphi_2, \varphi_3$ not $\mathrm{Aut}(S)$-conj. and φ_1 X-invariant

Conj: $p = 2, 3$
$|\mathrm{Irr}_0(B)^{\sigma_1}| = p \iff D$ is cyclic