A NOTE ON $\mathcal{O}G$-MODULES WHICH ARE NECESSARILY FREE AS $\mathcal{O}$-MODULES

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Abstract. In this note we prove that endo-p-permutation $\mathcal{O}G$-modules are necessarily free when regarded as $\mathcal{O}$-modules, i.e. $\mathcal{O}G$-lattices. In particular, so are endo-trivial and endo-permutation modules.

Throughout, we let p denote a prime number, and G a finite group of order divisible by p. As base ring, we consider a *complete discrete valuation ring $\mathcal{O}$ of characteristic zero* with unique maximal ideal $J(\mathcal{O})$ and with residue field $\mathcal{O}/J(\mathcal{O})$ of characteristic p. Furthermore, all $\mathcal{O}G$-modules considered are assumed to be finitely generated left modules.

To begin with, let us recall the definitions and some elementary facts about endo-p-permutation modules and related classes of modules:

(a) An **$\mathcal{O}G$-lattice** is an $\mathcal{O}G$-module which is free as an $\mathcal{O}$-module.

(b) An $\mathcal{O}G$-module M is called a **permutation module** if M is free as an $\mathcal{O}$-module and admits a G-invariant $\mathcal{O}$-basis. Therefore permutation $\mathcal{O}G$-modules are $\mathcal{O}G$-lattices by definition.

(c) An $\mathcal{O}G$-module M is called a **p-permutation** module if $\operatorname{Res}^G_Q(M)$ is a permutation $\mathcal{O}G$-module for every p-subgroup Q of G. Now, let P be a Sylow p-subgroup P of G. Then, equivalently, it suffices to require that $\operatorname{Res}^G_P(M)$ is a permutation $\mathcal{O}P$-module. In other words an $\mathcal{O}G$-module is a p-permutation module if and only if M is free as an $\mathcal{O}$-module and admits a P-invariant $\mathcal{O}$-basis.
Again, p-permutation $\mathcal{O}G$-modules are by definition $\mathcal{O}G$-lattices. Furthermore, an indecomposable $\mathcal{O}G$-lattice M is a p-permutation module if and only if M is a trivial source $\mathcal{O}G$-lattice. (Cf. [The95, §27].)

(d) An $\mathcal{O}G$-module M is called an **endo-permutation** module if its endomorphism algebra $\operatorname{End}_{\mathcal{O}}(M)$ is a permutation $\mathcal{O}G$-module.
Here (and below) $\operatorname{End}_{\mathcal{O}}(M)$ is endowed with its natural $\mathcal{O}G$-module structure via the action of G by conjugation, i.e.

$$^{g}\phi(m) = g\cdot\phi(g^{-1}\cdot m) \quad \forall\, g\in G,\ \forall\,\phi\in \operatorname{End}_{\mathcal{O}}(M) \text{ and } \forall\, m\in M\,.$$

(Cf. [The95, §28].)

(e) An $\mathcal{O}G$-module M is called an **endo-p-permutation** module if its endomorphism algebra $\operatorname{End}_{\mathcal{O}}(M)$ is a p-permutation $\mathcal{O}G$-module.
(Cf. [Urf07, Definition 1.1].)

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(f) An $\mathcal{O}G$-module M is called an **endo-trivial** module if $\mathrm{End}_{\mathcal{O}}(M) \cong \mathcal{O} \oplus X$ for some projective $\mathcal{O}G$-module X.

(g) Clearly:
- projective modules are p-permutation modules;
- permutation modules are p-permutation modules;
- permutation modules are endo-permutation modules;
- p-permutation modules are endo-p-permutation modules;
- endo-trivial modules over p-groups are endo-permutation modules; and
- over arbitrary finite groups endo-trivial modules are endo-p-permutation modules.

Thus the class of all endo-p-permutation $\mathcal{O}G$-modules encompasses all the $\mathcal{O}G$-modules defined above, namely the permutation modules, the p-permutation modules, the endo-permutation modules, and the endo-trivial modules.

Proposition A. *If M is an $\mathcal{O}G$-module such that* $\mathrm{End}_{\mathcal{O}}(M)$ *is free when regarded as an $\mathcal{O}$-module, then so is M when regarded as an $\mathcal{O}$-module.*

Proof. By the assumption $\mathrm{End}_{\mathcal{O}}(M)$ is $\mathcal{O}$-free. Furthermore, since $\mathcal{O}$ is assumed to be a discrete valuation ring, by the structure theorem for finitely generated modules over principal ideal domains, the module M, regraded as an $\mathcal{O}$-module, admits a direct sum decomposition of the form

$$M \cong \bigoplus_{i=1}^{r} \left(\bigoplus_{j_i=1}^{s_i} \mathcal{O}/J(\mathcal{O})^{n_i} \right) \oplus (\mathcal{O}\text{-free summands}),$$

where r is a non-negative integer and s_i, n_i $(1 \leq i \leq r)$ are positive integers. Now, we claim that if the torsion part of M is not trivial, then neither is the torsion part of $\mathrm{End}_{\mathcal{O}}(M)$. Indeed, for every $1 \leq i \leq r$ we have that

$$\mathrm{End}_{\mathcal{O}}(\mathcal{O}/J(\mathcal{O})^{n_i}) \cong \mathcal{O}/J(\mathcal{O})^{n_i}$$

as $\mathcal{O}$-module, therefore the $\mathcal{O}$-linear endomorphisms of M have the form

$$\mathrm{End}_{\mathcal{O}}(M) \cong \bigoplus_{i=1}^{r} \left(\bigoplus_{j_i=1}^{s_i} \mathrm{End}_{\mathcal{O}} \left(\mathcal{O}/J(\mathcal{O})^{n_i}\right) \right) \oplus X \cong \bigoplus_{i=1}^{r} \left(\bigoplus_{j_i=1}^{s_i} \mathcal{O}/J(\mathcal{O})^{n_i} \right) \oplus X$$

(as $\mathcal{O}$-module) for some $\mathcal{O}$-module X. This is a contradiction. □

Corollary B. *Any endo-p-permutation $\mathcal{O}G$-module is free when regarded as an $\mathcal{O}$-module. In particular, so is any endo-trivial $\mathcal{O}G$-module and any endo-permutation $\mathcal{O}G$-module.*

Proof. If M is an endo-p-permutation $\mathcal{O}G$-module, then by definition $\mathrm{End}_{\mathcal{O}}(M)$ is a p-permutation $\mathcal{O}G$-module, hence $\mathcal{O}$-free by the above and the claim follows from Proposition A. □

Appendix

Recall that a ring $\mathcal{O}$ is by definition ([Ser68, §1]) a *discrete valuation ring* if $\mathcal{O}$ is a PID, and possesses a unique non-zero prime ideal $\mathfrak{m}(\mathcal{O})$. It follows immediately that in this case $\mathfrak{m}(\mathcal{O})$ is the unique maximal ideal of $\mathcal{O}$, so that $\mathcal{O}$ is a local ring and $\mathfrak{m}(\mathcal{O}) = J(\mathcal{O})$, the Jacobson radical of $\mathcal{O}$. Moreover, as $\mathcal{O}$ is a PID, the ideal $\mathfrak{m}(\mathcal{O})$ is of the form $\mathfrak{m}(\mathcal{O}) = \pi\mathcal{O}$, where π is an irreducible element of $\mathcal{O}$, and any non-zero ideal of $\mathcal{O}$ is of the form $\mathfrak{m}(\mathcal{O})^n = \pi^n\mathcal{O}$, where $n \in \mathbb{Z}_{>0}$.

In fact a commutative ring $\mathcal{O}$ is a discrete valuation ring if and only if it is a Noetherian local ring and its unique maximal ideal is generated by a non-nilpotent element.

References

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