Trivial source character tables of small finite groups

Abstract

Trivial source modules, also called p-permutation modules, arise naturally in the representation theory of finite groups. They are, by definition, the indecomposable direct summands of the permutation modules. Trivial source modules are building pieces for Puig equivalences, endo-permutation source equivalences, p-permutation equivalences, and splendid Rickard equivalences of blocks. They are also relevant to Alperin’s weight conjecture. For example, a simple module is a weight module if and only if it is a trivial source module.

In order to do calculations with trivial source modules the ordinary character theory of finite groups is applied. Every group listed in column $\chi$ of the trivial source character table for a given group $G$ is a representative of the $p$-classes of $G$ up to isomorphism, whereas there are in general infinitely many indecomposable modules up to isomorphism, and there are in general infinitely many indecomposable $kG$-modules up to isomorphism.

General assumptions

- $G$ is a finite group and $p$ is a prime dividing $|G|$
- $(k, G, K)$ is a p-modular system which is large enough for all quotients of all subgroups of $G$
- all modules considered are finite-dimensional
- trivial source modules are assumed to be indecomposable

Trivial source modules

Every trivial source module occurs as a direct summand of $kG^p$ for some $p$-subgroup $P$ of $G$.

All these modules have trivial sources, hence their name.

For an arbitrary (but fixed) $p$ there are only finitely many trivial source $kG$-modules up to isomorphism, whereas there are in general infinitely many indecomposable $kG$-modules, up to isomorphism.

Trivial source modules are also called p-permutation modules since a non-zero $kG$-module $V$ is a trivial source module if and only if $V$ has a $k$-basis on which a Sylow $p$-subgroup of $G$ acts as a permutation group.

Essential property: Trivial source $kG$-modules lift in a unique way to trivial source $\mathcal{O}G$-lattices wherefore we can consider the $K$-characters of these lifts.

Trivial source character tables

In ordinary representation theory the character table collects the values of character functions of group representations at conjugacy classes. Analogously, the trivial source character table collects the character values of the Brauer constructions at $p'$-elements of the normalisers of the $p$-subgroups of $G$ up to conjugacy in $G$.

In particular:
- each line corresponds to a trivial source $kG$-module
- the first column lists the dimensions of the $kG$-modules
- the first block column lists the values of the $K$-characters (of the lifts from characteristic $p$ to characteristic $0$ of the trivial source $kG$-modules) at the $p$-regular classes of $G$
- the first block line corresponds to the projective indecomposable $kG$-modules
- the entries of the diagonal blocks correspond to the Brauer character values of the PIMs of the factor groups $N_G(P)/P$, where the $P_i$ run over the $p$-subgroups of $G$ up to conjugacy in $G$
- entries in off-diagonal blocks are obtained by computing character values of the Brauer constructions at $p'$-elements of the respective normalisers wherefore all entries are zero above the main block diagonal, when ordering the vertices according to their orders.

Example: $G = S_n$, $p = 2$

In the following trivial source character table we label the $K$-characters as in the GAP character table library CTblLib.

Main objectives of the project

- Investigate and determine trivial source character tables of finite groups.
- Contribute to the development of a database of trivial source character tables.
- Publish this database in electronic form.

Methods applied

With the aid of the computer algebra systems GAP, MAGMA and SAGE we wrote an algorithm that calculates the trivial source character table for a given finite group $G$ and a prime $p$ dividing $|G|$ (provided that the ordinary character table and the decomposition matrix are known).

In order to verify the results, we use the following theory:
- the Green correspondence
- the Brauer construction
- the theory of blocks of finite groups
- $p$-Kostka numbers and the theory of Specht modules (for $G = S_n$)

Results

For example, we have obtained the trivial source character tables of the following groups:
- $G = A_i$, for $n \in \{3, \ldots, 7\}$ for arbitrary $p$ dividing $|G|$ (computationally)
- $G = S_n$, for $n \in \{3, \ldots, 7\}$ for arbitrary $p$ dividing $|G|$ (computationally)
- the infinite family of dihedral groups $G = D_{2n}$, where $2n = 4v$, $v$ odd, for $p = 2$ (both computationally and theoretically)
- the unitary group $U_3(2)$ for $p = 3$ (computationally)
- the groups $PSL_2(q)$ for $p = 3$ and $q \in \{5, 11, 13, 19\}$ (computationally)
- the case $G = M_{11}, p = 2$ (computationally)
- the case $G = M_{11}, p = 3$ (computationally, we recover [3, §4.10])

References