JUN.-PROF. DR. CAROLINE LASSUEUR **BERNHARD BÖHMLER**
Due date: **Thursday 23.05.2019, 12:00** SS 2019

Throughout this exercise sheet G denotes a finite group, V a $\mathbb{C}$-vector space and all vector spaces are assumed to be finite-dimensional. Each Exercise is worth 4 points.

EXERCISE 9
Prove the following assertions:

(a) Two equivalent (matrix) $\mathbb{C}$-representations of G afford the same character.

(b) The set $\mathfrak{Cl}(G)$ of class functions on G is a $\mathbb{C}$-subspace of $\mathbb{C}^G$ and $\dim_{\mathbb{C}}(\mathfrak{Cl}(G)) = |C(G)|$.

EXERCISE 10
Let $Z(G)$ denote the centre of G. Let $\rho : G \longrightarrow \mathrm{GL}(V)$ be an irreducible $\mathbb{C}$-representation of G of degree n affording the character χ. Prove that:

(a) If $z \in Z(G)$, then $\chi(z) = n\zeta$, where ζ is an $o(z)$-th root of unity.

(b) If ρ is faithful and $g \in G$, then: $g \in Z(G)$ if and only if $|\chi(g)| = \chi(1)$.

EXERCISE 11 (The dual representation)
Let V be a G-vector space and write $V^* := \mathrm{Hom}_{\mathbb{C}}(V, \mathbb{C})$ for the dual space. Prove that:

(a) V^* is endowed with the structure of a G-vector space via

$$\begin{array}{ccc} G \times V^* & \longrightarrow & V^* \\ (g, f) & \mapsto & g.f \end{array}$$

where $g.f(v) := f(g^{-1}.v)\ \forall\, v \in V$.

(b) If $\rho_V : G \longrightarrow \mathrm{GL}(V)$ is a $\mathbb{C}$-representation of G decomposing as a direct sum $\rho_{V_1} \oplus \rho_{V_2}$ of two subrepresentations, then $\rho_{V^*} = \rho_{V_1^*} \oplus \rho_{V_2^*}$.

(c) The character of ρ_{V^*} is $\chi_{V^*} = \overline{\chi_V}$.

(d) Compute the dual representations of the representations of S_3 and S_4 of Exercise 8.

EXERCISE 12

(a) Let V be a G-vector space and write χ_V for the corresponding character. Write $V^G := \{v \in V \mid g.v = v\, \forall\, g \in G\}$ for the subspace of fixed points of V under the action of G. Prove that

$$\dim_{\mathbb{C}} V^G = \frac{1}{|G|} \sum_{g \in G} \chi_V(g)\,.$$

(b) Let X be a G-set and let χ_X denote that character of the associated permutation representation.

(i) Prove that $\chi_X(g) = |\mathrm{Fix}_X(g)|$ for every $g \in G$.

(ii) Use character theory to prove that the number of orbits of G on X is

$$\frac{1}{|G|} \sum_{g \in G} |\mathrm{Fix}_X(g)|\,.$$