CHAPTER 1 : LINEAR REPRESENTATIONS

Throughout this chapter we let : G be a finite group
K be a field (of arbitrary characteristic)
V be a K-vector space with $\dim_K V < \infty$.

1. Definitions and Examples:

Definition 1.1: • A representation of G (or a representation of G over K, or a K-representation of G) is a group homomorphism

$$\rho : G \longrightarrow GL(V),$$

where V is a K-vector space. The degree of ρ is $\dim_K V$.

(• We also say that G acts linearly on V, and that V is a G-vector space.)

Definition 1.2: A matrix representation of G over K is a group homomorphism

$$R : G \longrightarrow GL_n(K)$$
$$g \longmapsto R(g) = (r_{ij}(g))_{1\leq i,j\leq n}$$

for a certain $n \in \mathbb{Z}_{>0}$, called the degree of R.

Remark: • Each representation $\rho : G \longrightarrow GL(V)$ and each choice of a K-basis of B of V induces a matrix representation

$$G \xrightarrow{\rho} GL(V) \xrightarrow{\cong} GL_n(K)$$
$$g \longmapsto \rho(g)$$
$$\alpha \longmapsto (\alpha)_B = \text{"matrix of } \alpha \text{ in the basis } B\text{"}$$

where $n = \dim_K V$.

⚠ Two different K-bases of V give rise to two different matrix representations.

• Conversely, each matrix representation $R : G \rightarrow GL_n(K)$ induces a representation $G \xrightarrow{R} GL_n(K) \xrightarrow{\cong} GL(K^n)$

$$(a_{ij})_{ij} \longmapsto \alpha : K^n \longrightarrow K^n, \quad e_j \mapsto \sum_{i=1}^{n} a_{ij} e_i$$

where $e_j = \begin{pmatrix} 0 \\ \vdots \\ 1 \\ 0 \\ \vdots \\ 0 \end{pmatrix} \leftarrow j\text{-th line}$

Example 1: (a) G arbitrary, $V = K$:

$$\rho : G \longrightarrow GL(K) \qquad g \longmapsto \rho(g) = Id_K$$

is a representation called the "trivial representation"

Choosing $B = \{1_K\}$ as K-basis of K induces the matrix representation $R : G \longrightarrow GL_1(K) \cong K^\times$, $g \longmapsto 1_K$

(b) If $G \leq GL(V)$ is a subgroup, then the canonical inclusion $G \hookrightarrow GL(V)$, $g \longmapsto g$ is a representation of G called the tautological representation of G.

(c) $G = S_n$ $(n \geq 1)$, $V = K^n$ with its canonical ordered basis $(e_1, \dots, e_n)$:

$$\rho : G \longrightarrow GL(K^n)$$
$$g \longmapsto \rho(g) : K^n \to K^n, \quad e_i \longmapsto \rho(g)e_i := e_{g(i)}$$

is a representation called the natural representation of S_n.

(d) More generally, if X is a G-set, i.e. X is a set endowed with a left action $\cdot : G \times X \longrightarrow X$, s.t. $|X| < \infty$, and V is a K-vector space with basis $\{e_x \mid x \in X\}$, then

$$\rho_X : G \longrightarrow GL(V)$$
$$g \longmapsto \rho(g) : V \to V, \quad e_x \longmapsto e_{g \cdot x}$$

is a representation of G, called permutation representation.

→ (c) is a particular case of (d) with $G = S_n$, $X = \{1, \dots, n\}$

Moreover, if $X = G$ and $\cdot\, G \times \cancel{X} \to X$ is left multiplication in G, then the representation

$$\rho_{reg} : G \longrightarrow GL(V)$$
$$g \longmapsto \rho(g) : V \to V, \quad e_h \longmapsto e_{g \cdot h}$$

($\{e_h \mid h \in G\}$ Basis of V)

is called the regular representation of G.

Remark:

- A representation $\rho: G \longrightarrow GL(V)$ gives rise to a left action of G on V:

$$\begin{aligned} \cdot : G \times V &\longrightarrow V \\ (g, v) &\longmapsto g.v := \rho(g).v := \rho(g)(v) \end{aligned}$$

such that $\forall g \in G$, $\forall x, y \in V$ and $\forall \lambda \in K$: (i) $g.(x+y) = g.x + g.y$; and (ii) $g.(\lambda x) = \lambda(g.x)$.

- Conversely an action $\cdot : G \times V \longrightarrow V$ satisfying (i) and (ii) gives rise to a representation

$$\begin{aligned} \rho: G &\longrightarrow GL(V) \\ g &\longmapsto \rho(g): V \longrightarrow V \\ & \qquad\qquad v \longmapsto \rho(g)(v) := g.v \end{aligned}$$

See Exercise 1, Sheet 1. Therefore the data of a K-representation of G is equivalent to the data of a G-vector space.

- This together with the previous Remark allows us to use terminology defined for representations for matrix representations and G-vector spaces as well, and conversely.

Definition 1.3: Let V be a G-vector space with corresponding representation ρ.

(a) $V' \leq V$ is called a <u>G-invariant subspace</u> of V

$:\Leftrightarrow \quad g.V' = \rho(g)(V') \subseteq V' \quad \forall g \in G.$

(in fact then $\rho(g)(V') = V'$ since $\rho(g)$ is bijective)

(b) If there exists a G-invariant subspace $0 \subsetneq V' \subsetneq V$, then ρ (or V itself) is called <u>reducible</u>; else <u>irreducible</u>.

Remark: If $\rho: G \to GL(V)$ is a representation and $V' \leq V$ is G-invariant, then

$$\begin{aligned} \rho_{V'}: G &\longrightarrow GL(V') \\ g &\longmapsto \left(\rho_{V'}(g) = \rho(g)|_{V'} : V' \longrightarrow V'\right) \end{aligned}$$

is clearly a representation of G. We say that $\rho_{V'}$ is a <u>subrepresentation</u> of ρ.

Now if we choose a k-basis $B_{V'}$ of V', which we complete to a k-basis B of V, then the corresponding matrix representation is of the form

$$(\rho(g))_B = \left(\begin{array}{c|c} (\rho(g)|_{V'})_{B'} & * \\ \hline 0 & * \end{array}\right) \quad \forall g \in G$$

$$\qquad\qquad \underbrace{\qquad\quad}_{B'} \quad \underbrace{\qquad\quad}_{B \setminus B'}$$

Example 2: (a) V G-vector space with $\dim_K V = 1$

$\Rightarrow$ corresponding representation $\rho: G \to GL(V) \cong K^\times$ is irreducible

(since $0 \subsetneq V' \subsetneq V$ G-invariant subspace $\Rightarrow \begin{cases} \dim_K V' > 0 \text{ and} \\ \dim_K V' < 1 \end{cases}$ $\lightning$)

(b) Let $\rho: S_n \longrightarrow GL(K^n)$ be the natural representation of S_n $(n \geq 1)$ as in Exp. 1.

$\Rightarrow \quad \rho(g)\left(\sum_{i=1}^n e_i\right) = \sum_{i=1}^n e_i \quad \forall g \in S_n$

$\Rightarrow \quad V' := \langle \sum_{i=1}^n e_i \rangle_K$ is an S_n-invariant subspace of K^n

$\Rightarrow \quad \rho$ is reducible if $n > 1$.

Definition 1.4: (a) A K-homomorphism $\varphi: V \to V'$ between two G-vector spaces is called a <u>G-homomorphism</u> $:\Leftrightarrow$ the corresponding representations ρ, ρ' satisfy the condition:

$$\rho'(g) \circ \varphi = \varphi \circ \rho(g) \quad \forall g \in G.$$

$$\begin{tikzcd} V \arrow[r, "\rho(g)"] \arrow[d, "\varphi"'] \arrow[dr, phantom, "\circlearrowright"] & V \arrow[d, "\varphi"] \\ V' \arrow[r, "\rho'(g)"'] & V' \end{tikzcd}$$

(b) If, moreover, φ is an isomorphism, then ρ and ρ' are called <u>equivalent</u> (or isomorphic). Nota: $\rho \sim \rho'$

(c) Two matrix representations $R, R': G \longrightarrow GL_n(K)$ are called <u>equivalent</u> $:\Leftrightarrow \exists\, T \in GL_n(K)$ such that

$$R'(g) = T R(g) T^{-1} \quad \forall g \in G.$$

Nota: $R \sim R'$.

Definition 1.5: An injective representation $\rho: G \to GL(V)$ is termed <u>faithful</u>.

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Proposition 1.6: Let V, V' be G-vector spaces and $\varphi: V \to V'$ be a G-homomorphism

(a) $V_1 \leq V$ G-invariant $\Rightarrow \varphi(V_1) \leq V'$ G-invariant

(b) $V_1' \leq V'$ G-invariant $\Rightarrow \varphi^{-1}(V_1') \leq V$ G-invariant

(c) in particular, $\mathrm{Ker}(\varphi)$, $\mathrm{Im}(\varphi)$ are G-invariant subspaces

(d) V irreducible $\Rightarrow \mathrm{Im}(\varphi)$ irreducible.

Proof: Write $\rho: G \to GL(V)$, $\rho': G \to GL(V')$ for the corresponding representations

(a) $x' \in \varphi(V_1) \Rightarrow \exists x \in V_1$ s.t. $x' = \varphi(x)$

$$\Rightarrow \forall g \in G: \rho'(g).x' = \rho'(g).\varphi(x) = (\rho'(g)\circ\varphi)(x)$$
$$= (\varphi \circ \rho(g))(x) = \varphi(\underbrace{\rho(g).x}_{\in V_1}) \in \varphi(V_1)$$

Hence $\varphi(V_1)$ is G-invariant.

(b) $x \in \varphi^{-1}(V_1') \Rightarrow \varphi(\rho(g).x) = (\varphi\circ\rho(g))(x) = (\rho'(g)\circ\varphi)(x)$
$$= \rho'(g).\underbrace{\varphi(x)}_{\in V_1'} \in \rho'(g).V_1' = V_1' \quad \forall g \in G$$

hence $\rho(g).x \in \varphi^{-1}(V_1')$ $\forall g \in G$, i.e. $\varphi^{-1}(V_1')$ is G-invariant.

(c) Obvious from (a) and (b) since $\ker(\varphi) = \varphi^{-1}(\{0\})$ and $\operatorname{Im}\varphi = \varphi(V)$.

(d) If $0 \subsetneq V' \subsetneq \underbrace{\varphi(V)}_{=\operatorname{Im}\varphi}$ is G-invariant, then $0 \subsetneq \varphi^{-1}(V') \subsetneq V$ is also G-invariant by (b). Therefore: $\operatorname{Im}(\varphi)$ reducible $\Rightarrow$ V reducible.

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Exercise 3, Sheet 1

$p \in \mathbb{P}$, $G := C_p = \langle g \mid g^p = 1\rangle$, $K := \mathbb{F}_p$, $V := \mathbb{F}_p^2$ with canonical basis $B = (e_1, e_2)$

Consider the matrix representation

$$R: G \longrightarrow GL_2(K)$$
$$g^b \longmapsto \begin{pmatrix} 1 & b \\ 0 & 1 \end{pmatrix}$$

Prove that: (a) Ke_1 is G-invariant ($\Rightarrow$ R is reducible)

(b) there is no direct sum decomposition of V into G-invariant subspaces.

Week 17

2. Maschke's Theorem and Schur's Lemma

Definition 1.7 Let $\rho: G \to GL(V)$ be a K-representation, and let $W_1, W_2 \leq V$ be two G-invariant subspaces. If $V = W_1 \oplus W_2$, then we say that ρ is the direct sum of ρ_{W_1} and ρ_{W_2} and we write $\rho = \rho_{W_1} \oplus \rho_{W_2}$.

Notice that if we choose K-bases B_i of W_i $(i=1,2)$ and consider the K-basis $B_1 \sqcup B_2$ of V, the corresponding matrix representation is of the form

$$\left(\rho(g)\right)_{B_1\sqcup B_2} = \left(\begin{array}{c|c} (\rho_{W_1}(g))_{B_1} & 0 \\ \hline 0 & (\rho_{W_2}(g))_{B_2} \end{array}\right) \quad \forall g \in G.$$

Theorem 1.8 [Maschke's Theorem]

Let G be a finite group, and let V be a G-vector space over a field K such that $\mathrm{char}(K) \nmid |G|$.
If $W \leq V$ is a G-invariant subspace, then there exists a G-invariant complement $U \leq V$ to W, i.e. $V = W \oplus U$ and $\rho_V = \rho_W \oplus \rho_U$.

<u>Proof:</u> • Note that $\mathrm{char}(K) \nmid |G| \Rightarrow |G| \cdot 1_K$ is invertible in K.

• Let $U_0 \leq V$ be a complement to W, i.e. $V = W \oplus U_0$ as K-vector spaces. (U_0 is possibly not G-invariant!)

Let
$$\pi : \begin{array}{ccc} V = W \oplus U_0 & \longrightarrow & W \\ v = \underset{\in W}{w} + \underset{\in U_0}{u} & \longmapsto & w \end{array}$$
be the projection onto W along U_0. ($\mathrm{Im}\,\pi = W$ and $\ker \pi = U_0$)

Define
$$\tilde{\pi} : \begin{array}{ccc} V & \longrightarrow & V \\ v & \longmapsto & \frac{1}{|G|} \sum_{g \in G} g.(\pi(g^{-1}.v)) = \frac{1}{|G|} \sum_{g \in G} \rho(g)(\pi(\rho(g^{-1})(v))) \end{array}$$

Clearly $\tilde{\pi} \in \mathrm{End}_K(V)$, as g, g^{-1} act linearly $\forall g \in G$ and $\pi \in \mathrm{End}_K(V)$.

<u>$\mathrm{Im}\,\tilde{\pi} = W$:</u> $v \in V \Rightarrow \tilde{\pi}(v) = \frac{1}{|G|} \sum_{g \in G} \underbrace{g.(\underbrace{\pi(\underbrace{g^{-1}.v}_{\in V})}_{\in W})}_{\in W \text{ (G-invariant)}} \in W$ ✓

<u>$\tilde{\pi}|_W = \mathrm{Id}_W$:</u> $w \in W \Rightarrow \tilde{\pi}(w) = \frac{1}{|G|} \sum_{g \in G} g.(\underbrace{\pi(\underbrace{g^{-1}.w}_{\in W \text{ (G-invariant)}})}_{= g^{-1}.w})$

$$= \frac{1}{|G|} \sum_{g \in G} g.(g^{-1}.w) = \frac{1}{|G|} \sum_{g \in G} (\underbrace{gg^{-1}}_{=1_G}).w = w$$

$\Rightarrow \tilde{\pi}$ is a projection onto W

$\Rightarrow V = W \oplus \mathrm{Ker}(\tilde{\pi})$ (Grundlagen)

<u>$\ker \tilde{\pi}$ is G-invariant:</u>

First $\forall h \in G$ and $\forall v \in V$ we have

$$(\tilde{\pi} \circ \rho(h))(v) = \tilde{\pi}(\rho(h).v) = \tilde{\pi}(h.v) = \frac{1}{|G|} \sum_{g \in G} \overset{hh^{-1}g}{\overset{\parallel}{g}} . \pi(g^{-1}.(h.v))$$

with $(h^{-1}g)^{-1}.v = (g^{-1}h).v$

$$= \frac{1}{|G|} \sum_{g \in G} h.((h^{-1}g).\pi((h^{-1}g)^{-1}.v)) = h.\left(\frac{1}{|G|} \sum_{g \in G} (h^{-1}g).(\pi((h^{-1}g)^{-1}.v))\right)$$

$$\overset{s := h^{-1}g}{=} h. \sum_{s \in G} s.(\pi(s^{-1}.v)) = h.\tilde{\pi}(v) = (\rho(h) \circ \tilde{\pi})(v)$$

$\Rightarrow \tilde{\pi}$ is a G-homomorphism

Therefore $\ker \tilde{\pi}$ is G-invariant by Prop. 1.6.
$\Rightarrow$ Set $U := \ker \tilde{\pi}$.

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Sheet 2, Ex 1: 2nd proof of Maschke for $K = \mathbb{C}$ using a G-inv. scalar product.

Corollary 1.9: If G is a finite group and K a field s.t. $\mathrm{char}(K) \nmid |G|$, then every K-representation of G decomposes into a direct sum of irreducible subrepresentations.

Proof: Let $\rho : G \to GL(V)$ be a K-representation, with $\dim_K V \geq 1$.

- Case 1: ρ is irreducible: nothing to do ✓
- Case 2: ρ is reducible. In part. $\dim_K V \geq 2$ and (since we assume that $\dim_K V < \infty$,) $\exists\, V_1 \lneq V$ irreducible G-invariant subspace. (Note: $\dim_K V_1 \geq 1$)
$\overset{\text{Maschke}}{\Longrightarrow} \exists\, U \leq V$ G-invariant s.t. $V = V_1 \oplus U$
Now $\dim_K U \lneq \dim_K V$, therefore an induction argument yields:
$$V = V_1 \oplus V_2 \oplus \cdots \oplus V_r \quad (r \geq 2),$$
where $V_2, \dots, V_r \leq U$ are irred. G-invariant subspaces. #

Remark: The converse to Maschke's Theorem holds as well.
See Lecture "Representation Theory"
[You can also try to figure out a proof by yourself: consider the regular repres. of G and a subrepres of dim $|G| - 1$ ...]

Next we investigate morphisms between irreducible G-vector spaces.

Theorem 1.10 (Schur's Lemma)

Let V, V' be irreducible G-vector spaces and let $\varphi : V \to V'$ be a G-homomorphism. Then either $\varphi = 0$, or φ is an isomorphism (and hence the corresponding representations are equivalent).

Proof: If $\varphi = 0$, we are done. Else $\varphi \neq 0 \Rightarrow \varphi(V) \neq 0$ and is a G-invariant subspace of V' by Prop. 1.6. But V' is

irreducible, hence $\varphi(V) = V'$ $\Rightarrow$ φ is surjective.
Now $\ker(\varphi) \underset{\neq}{\leq} V$ is a G-invariant subspace by Prop. 1.6 again
$\overset{V \text{ irred}}{\Longrightarrow}$ $\ker(\varphi) = 0$ $\Rightarrow$ φ is injective $\Rightarrow$ φ is bijective. #

Corollary 1.11: Let V be a G-vector space over K. If V is irreducible, then:

(a) $\mathrm{End}_G(V) := \{\varphi \in \mathrm{End}_K(V) \mid \varphi \text{ is a } G\text{-homomorphism}\}$ is a skew-field.

(b) If K is algebraically closed, then $\mathrm{End}_G(V) \cong K$ (as rings!)

Proof: (a) 1st notice that $\mathrm{End}_G(V)$ is a subring of $\mathrm{End}_K(V)$ (check it!)
Now let $\varphi \in \mathrm{End}_G(V)\setminus\{0\}$ $\overset{\text{Thm 1.10}}{\Longrightarrow}$ φ is invertible, as V is irred.
$\Rightarrow$ $\mathrm{End}_G(V)$ is a skew-field. /

(b) Let $\varphi \in \mathrm{End}_G(V)$
K alg. closed $\Rightarrow$ characteristic polynomial of φ has a root in K $\Rightarrow$ $\exists$ an eigenvalue $\lambda \in K$ of φ

$$\Rightarrow \underbrace{\{0\} \neq \ker(\underbrace{\varphi - \lambda \mathrm{Id}_V}_{\in \mathrm{End}_G(V)}) \leq V}_{G\text{-invariant by Prop 1.6}} \Rightarrow \ker(\varphi - \lambda \mathrm{Id}_V) = V.$$

$\Rightarrow$ $\varphi - \lambda \mathrm{Id}_V = 0$ $\Rightarrow$ $\varphi = \lambda \mathrm{Id}_V$
and we get an isomorphism

$$\mathrm{End}_G(V) \xrightarrow{\cong} K$$
$$\varphi = \lambda \mathrm{Id}_V \longmapsto \lambda$$

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Week 2

Corollary 1.12 If G is a finite abelian group, then every irreducible $\mathbb{C}$-representation of G has degree one.

Proof: [Exercise 2, Sheet 2] Hint: use Cor. 1.11 (b).

Corollary 1.13 [Schur's Lemma, matrix version]

Let R, R' be two irreducible matrix representations of G of degree n, n' resp.
If there is $0 \neq F \in M_{n' \times n}(K)$ such that $R'(g)F = FR(g)$ $\forall g \in G$,
then: (i) $n = n'$;
(ii) $\det F \neq 0$;
(iii) $R \sim R'$.

Proof: This is the translation of Schur's Lemma to matrix representations (w.r.t. a chosen ordered basis). #

Theorem 1.14 [Schur's relations]

Let R, R' be two irreducible matrix representations of G of degrees n, n' respectively. Then the following holds:

(a) If $R \not\sim R'$, then $\sum_{g\in G} R(g)_{ij} R'(g^{-1})_{kl} = 0 \quad \forall\, 1\le i,j\le n$, $\forall\, 1\le k,l\le n'$

(b) If $R = R'$, $K = \overline{K}$, and $\operatorname{char}(K) \nmid n$, then

$$\sum_{g\in G} R(g)_{ij} R'(g^{-1})_{kl} = \frac{|G|}{n}\delta_{il}\delta_{kj} \quad \forall\, 1\le i,j,k,l\le n.$$

Proof: Given $A \in M_{n\times n'}(K)$, define $\tilde{A} := \sum_{g\in G} R(g^{-1}) A R'(g)$.

Then $\forall h\in G$, we have:

$$\tilde{A}R'(h) = \sum_{g\in G} R(g^{-1}) A \overbrace{R'(g)R'(h)}^{R'(gh)} = \sum_{g\in G} \overbrace{R(h)R((gh)^{-1})}^{R(g^{-1})} A R'(gh)$$
$$= R(h)\sum_{g\in G} R((gh)^{-1}) A R'(gh)$$
$$= R(h)\tilde{A} \qquad (*)$$

In particular for $A_0 = (\delta_{jj_0}\delta_{kk_0})_{\substack{1\le j\le n\\ 1\le k\le n'}}$, $\quad \begin{matrix}1\le j_0\le n\\ 1\le k_0\le n\end{matrix}$ fixed

we have

$$\tilde{A}_0 = \sum_{g\in G} R(g^{-1})A_0R'(g) = \sum_{g\in G}\Big(\sum_{j=1}^{n}\sum_{k=1}^{n'} R(g^{-1})_{ij}\,\delta_{jj_0}\delta_{kk_0} R'(g)_{kl}\Big)_{il}$$
$$\overset{\substack{j=j_0\\k=k_0}}{=} \Big(\sum_{g\in G} R(g^{-1})_{ij_0} R'(g)_{k_0 l}\Big)_{il} \qquad (**)$$

- If $R \not\sim R'$, then it follows from $(*)$ and Cor. 1.13 that $\tilde{A}_0 = 0$.
  Hence $(**) \Rightarrow$ (a) ✓

- Now assume that $R = R'$ (In part $n = n'$)

  By swapping the rôles of R and R' in Cor. 1.13 we get that $\det\tilde{A}_0 \neq 0$, or $\tilde{A}_0 = 0$.
  Since $K = \overline{K}$, we can diagonalise $\tilde{A}_0$ as $\tilde{A}_0 = \lambda_{j_0k_0} I_n$ where $\lambda_{j_0k_0} \in K^\times$.
  using the same argument as in Cor. 1.11(b) for matrices

Alt: see $\tilde{A}_0$ as the matrix of a G-endo of K^n, thanks to $(*)$. Then Cor. 1.11 yields $\lambda_{j_0k_0}$

Rewriting $j_0 = j$, $k_0 = k$, we get $\sum_{g\in G} R(g^{-1})_{ij} R'(g)_{kl} = \lambda_{jk}\delta_{il}$ by $(**)$

$(***)$

Note that when g runs of G, so does g^{-1}, therefore we also have the equality

$$\sum_{g\in G} R(g^{-1})_{ke} R(g)_{ij} = \lambda_{ei}\delta_{kj} \quad \text{(which equals the } 1^{st} \text{ sum as well)}$$

Now if $i \neq l$, $k = j$ $\Rightarrow$ the 1st equation is 0 and the 2^{nd} λ_{ei}

$(\delta_{il} = 0)$ $(\delta_{kj} = 1)$ $\Rightarrow$ $\lambda_{ei} = 0$

$\Rightarrow$ we can write $\lambda_{ei} = \delta_{il}\lambda_i$ for some $\lambda_i \in K$

$\Rightarrow \sum_{g\in G} R(g^{-1})_{ij} R(g)_{ke} = \delta_{il}\delta_{kj}\lambda_i$ and it remains to determine the coefficient λ_i.

For this choose $l = i$ and sum over all $j = k$ as follows:

$$\underbrace{\sum_{j=1}^{n} \lambda_i \delta_{ii}\delta_{jj}}_{= n\lambda_i} = \sum_{g\in G}\sum_{j=1}^{n} R(g^{-1})_{ij} R(g)_{ji} = \sum_{g\in G} R(\underbrace{g^{-1}g}_{I_n})_{ii} = \sum_{g\in G} 1 = |G|,$$

hence $\lambda_i = \frac{|G|}{n}$ and we are done. #

Corollary 1.15: Assume $\mathrm{char}(K) = 0$ and $K = \overline{K}$.
If $R^i : G \to GL_{n_i}(K)$ are pairwise non-equivalent irreducible matrix representations of G $(1 \le i \le r)$, then

$$\sum_{g\in G} R^i(g)_{ke} R^j(g^{-1})_{st} = \frac{|G|}{n_i}\delta_{ij}\delta_{kt}\delta_{ls} \quad (\forall i,j,k,l,s,t)$$

$\forall 1 \le i,j \le r$, $\forall 1 \le k,l \le n_i$, $\forall 1 \le s,t \le n_j$.

Proof: Apply Thm 1.14 with $R = R^i$, $R' = R^j$.

Theorem 1.16: Assume $\mathrm{char}(K) = 0$ and $K = \overline{K}$.
Let $R^i : G \to GL_{n_i}(K)$ $(1 \le i \le r)$ be pairwise non-equivalent irreducible matrix representations of G.
Then the $\sum_{i=1}^{r} n_i^2$ functions $R^i_{ke} : G \to K$, $g \mapsto R^i(g)_{ke}$

are K-linearly independent.

Proof: Assume $\sum_{i=1}^{r}\sum_{k,l=1}^{n_i} c^i_{kl} R^i_{kl} = 0$ for coefficients $c^i_{kl} \in K$. To see: $c^i_{kl}=0$ $\forall i,k,l$.

$$\Rightarrow \forall g \in G,\ \sum_{i=1}^{r}\sum_{k,l}^{n_i} c^i_{kl} R^i_{kl}(g) = 0$$

$\Rightarrow$ $\forall\, 1 \le j \le r$, $\forall 1 \le s,t \le n_j$, we have

$$0 = \sum_{g\in G}\sum_{i=1}^{r}\sum_{k,l=1}^{n_i} c^i_{kl} R^i_{kl}(g) R^j_{st}(g^{-1})$$
$$= \sum_{i=1}^{r}\sum_{k,l=1}^{n_i} c^i_{kl} \sum_{g\in G} R^i_{kl}(g) R^j_{st}(g^{-1})$$
$$\overset{\text{Cor. 1.15}}{=} \sum_{i=1}^{r}\sum_{k,l=1}^{n_i} c^i_{kl} \frac{|G|}{n_i}\delta_{ij}\delta_{kt}\delta_{ls} \overset{\substack{l=s\\k=t\\i=j}}{=} \frac{|G|}{n_j} c^j_{ts}$$

$\Rightarrow c^j_{ts} = 0$ $\forall\, 1 \le j \le r$, $\forall\, 1 \le t,s \le n_j$ $\Rightarrow$ the R^i_{kl} are K-linearly independent. #

Corollary 1.17: With the hypotheses of Thm. 1.16, we have:

(a) $\sum_{i=1}^{r} n_i^2 \le |G|$;

(b) $r \le |G|$.

("There are at most $|G|$ pairwise non-equivalent irreducible representations of G")

Proof: The space $K^G = \{f: G \to K \mid f \text{ function}\}$ of K-valued functions of G has K-dimension $|G|$, since $\{\delta_g : G \to K,\ h \mapsto \delta_{gh} \mid g \in G\}$ is a K-basis of K^G (see GDM).

Now Thm 1.16 says that the $R^i_{kl} \in K^G$ are K-lin. indep and there are $\sum_{i=1}^{r} n_i^2$ of them $\Rightarrow \sum_{i=1}^{r} n_i^2 \le |G|$. $\to$ (a)

$\Rightarrow r \le |G|$ $\to$ (b)

Theorem 1.18: [Diagonalisation Theorem]

Let $\rho: G \to GL(V)$ be a $\mathbb{C}$-representation of G. Fix $g \in G$.
Then, there exists a $\mathbb{C}$-basis B of V w.r.t. which $(\rho(g))_B$ has the form $\begin{pmatrix}\varepsilon_1 & & 0\\ & \ddots & \\ 0 & & \varepsilon_n\end{pmatrix}$, where each ε_i $(1 \le i \le n)$ is an $o(g)$-th root of unity in $\mathbb{C}$.

Proof: Let $m := o(g)$. Consider the restriction $\rho|_{\langle g \rangle} : \langle g \rangle \to GL(V)$ of ρ to the cyclic subgrp generated by g.
By the corrollary to Maschke's Theorem, we can decompose $\rho|_{\langle g \rangle}$ as a direct sum of irreducible $\mathbb{C}$-representations, say $\rho|_{\langle g \rangle} = \rho_{V_1} \oplus \cdots \oplus \rho_{V_n}$, where $V_1, \dots, V_n \leq V$ are $\langle g \rangle$-invariant.
Now $\langle g \rangle$ is abelian $\overset{\text{Cor. 1.12}}{\Longrightarrow}$ $\dim_{\mathbb{C}} V_1 = \dots = \dim_{\mathbb{C}} V_n = 1$
Choose a $\mathbb{C}$-basis $\{x_i\}$ of V_i $\forall\, 1 \leq i \leq n$
$\Rightarrow$ $B := (x_1, \dots, x_n)$ is a basis of V s.t.

$$(\rho(g))_B = \begin{pmatrix} \varepsilon_1 & & 0 \\ & \ddots & \\ 0 & & \varepsilon_n \end{pmatrix}$$

for some coeff's $\varepsilon_1, \dots, \varepsilon_n \in \mathbb{C}$.
But $g^m = 1_G \Rightarrow \varepsilon_i^m = \rho_{V_i}(g)^m = \rho_{V_i}(g^m) = \rho_{V_i}(1_G) = 1_{\mathbb{C}}$

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Corollary 1.19: If $\rho : G \to GL(V)$ is a $\mathbb{C}$-representation of an abelian group G, then the linear transformations $\{\rho(g)\}_{g \in G}$ are simultaneously diagonalisable.

Proof: Same argument as in the previous proof for G instead of $\langle g \rangle$.

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