# Chapter 3: The Character Table

**Notation:** (unless otherwise stated) $G$ finite group, $K = \mathbb{C}$

$\mathrm{Irr}(G) = \{\chi_1, \dots, \chi_r\}$ irred characters of $G$

$C_i = [g_i]$ $(1 \le i \le r\}$ conjugacy classes of $G$ with set of representatives $g_1, \dots, g_r$ and we assume that $g_1 = 1_G$.

## 1. The Character Table of a finite group.

**Definition 3.1:** The character table of $G$ is the matrix

$$X(G) := (\chi_i(g_j))_{\substack{1\le i\le r\\ 1\le j\le r}} \in M_r(\mathbb{C}).$$

**Remark:** The proof of the 2nd Orthogonality Relations shows that the character table is an invertible matrix, i.e. $X(G) \in GL_r(\mathbb{C})$.

**Example 4:** $G = \langle g \mid g^n = 1 \rangle$ cyclic group of order $n$.

$G$ abelian $\overset{\S 2.4}{\Longrightarrow}$ $\mathrm{Irr}(G) = \{\text{linear characters of } G\}$, also $C_i = \{g_i\} =: \{g^i\}$ $\forall 1 \le i \le r$ $\Big\}$ there are exactly $|G|$ irreducible characters

Let $\zeta$ be a primitive $n$-th root of unity

$\Rightarrow \{\zeta^i \mid 1 \le i \le n\}$ is the set of all $n$-th root of unity.

Now each $\chi_i : G \to \mathbb{C}^*$ is a group hom and is determined by $\chi_i(g)$.

Moreover $g^n = 1 \Rightarrow \chi_i(g)^n = 1 \Rightarrow \chi_i(g)$ is an $n$-th root of $1_{\mathbb{C}}$.

$\Rightarrow$ we have have $n$ choices for $\chi_i(g)$

We set $\chi_i(g) = \zeta^i$ $\forall 1 \le i \le n \Rightarrow \mathrm{Irr}(G) = \{\chi_i \mid 1 \le i \le n\}$

$\Rightarrow$ The character table of $G$ is

$$X(G) = (\chi_i(g^j))_{ij} = (\zeta^{ij})_{\substack{1\le i\le n\\ 1\le j\le n}}$$

(here: $1_G = \chi_n$ and not $\chi_1$)

| | $1$ | $g$ | $g^2$ | $\dots$ | $g^{n-1}$ |
|---|---|---|---|---|---|
| $\chi_1$ | $1$ | $\zeta$ | $\zeta^2$ | $\dots$ | $\zeta^{n-1}$ |
| $\chi_2$ | $1$ | $\zeta^2$ | $\zeta^4$ | $\dots$ | $\zeta^{2n-2}$ |
| $\vdots$ | $\vdots$ | $\vdots$ | $\vdots$ | | $\vdots$ |
| $\chi_n$ | $1$ | $1$ | $1$ | $\dots$ | $1$ |

**Remark:** Often the convention is that $\chi_1 = 1_G$ the trivial character (= character of the trivial representation).

**Example 5:** $G = S_3 = \{ id, (12), (13), (23), (123), (132) \}$

Recall (AGS): the conjugacy classes are given by cycle types

we set $C_1 = [\underbrace{id}_{=:g_1}]$, $C_2 = [\underbrace{(12)}_{=:g_2}]$, $C_3 = [\underbrace{(123)}_{=:g_3}]$

$\Rightarrow \quad |Irr(S_3)| = 3$

We know $\chi_1 = 1_G$ has degree one

Degree formula yields $6 = |S_3| = \sum_{i=1}^{3} \chi_i(1)^2 = 1^2 + \chi_2(1)^2 + \chi_3(1)^2$

Only possibility: $\chi_2(1) = 1$, $\chi_3(1) = 2$ (up to ordering)

Ex 8, Sheet 3 gives the two corresponding representations

$$\rho_2 : S_3 \to \mathbb{C}^*, \quad \sigma \mapsto sgn(\sigma)$$ (the signature homomorphism)

and

$$\rho_3 : S_3 \to GL_2(\mathbb{C})$$
$$(12) \mapsto \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}$$ (up to equivalence)
$$(123) \mapsto \begin{pmatrix} 0 & -1 \\ 1 & -1 \end{pmatrix}$$

Therefore the character table of $G$ is

| | id | (12) | (123) |
|---|---|---|---|
| $\chi_1 = 1_G$ | 1 | 1 | 1 |
| $\chi_2$ | 1 | -1 | 1 |
| $\chi_3$ | 2 | 0 | -1 |

degrees

E.g. the 1st Orthogonality Relation(s) for $\chi_2$ and $\chi_3$ reads

$$\langle \chi_2, \chi_3 \rangle = \frac{1}{|G|} \left( \sum_{i=1}^{3} |C_i| \chi_2(g_i) \chi_3(g_i^{-1}) \right) \quad (\text{note } [g_i] = [g_i^{-1}])$$

$$= \frac{1}{6} \left( 1 \cdot 1 \cdot 2 + 3 \cdot (-1) \cdot 0 + 2 \cdot 1 \cdot (-1) \right) = 0 \checkmark$$

for $\chi_2$ and $\chi_2$ it reads: $\langle \chi_2, \chi_2 \rangle = \frac{1}{6} \left( 1 \cdot 1 \cdot 1 + 3 \cdot (-1)(-1) + 2 \cdot 1 \cdot 1 \right) = \frac{6}{6} = 1$

The 2nd Orth. Relation(s) for columns 1 and 2 reads:

$$0 = \delta_{1,2} \frac{|G|}{|C_1|} = \sum_{k=1}^{3} \chi_k(g_2) \chi_k(g_1^{-1}) = 1 \cdot 1 + 1 \cdot (-1) + 2 \cdot 0$$

for ~~the~~ 2nd column with itself: $\delta_{2,2} \frac{|G|}{|C_2|} = \frac{6}{3} = 1 \cdot 1 + (-1) \cdot (-1) + 0 \cdot 0 = 2 \checkmark$

<u>Example 6:</u> $G = S_4$. Again conj. classes are given by cycle shapes

We set $C_1 := [\underset{=: g_1}{1}]$, $C_2 := [\underset{=: g_2}{(12)}]$, $C_3 := [\underbrace{(12)(34)}_{=: g_3}]$

$C_4 := [\underbrace{(1234)}_{=: g_4}]$ and $C_5 = [\underbrace{(123)}_{=: g_5}]$

$\Rightarrow |C_1| = 1,\ |C_2| = 6,\ |C_3| = 3,\ |C_4| = 6,\ |C_5| = 8$

<u>Recall:</u> $V_4 = \{1, (12)(34), (13)(24), (14)(23)\} \trianglelefteq S_4$ with $S_4/V_4 \cong S_3$

Ex 7, Sheet 2 $\Rightarrow$ any irreducible representation of $S_3 = S_4/V_4$ becomes an irreducible representation of $S_4$ via the "Inflation operation"

Moreover the irreducible repres. of $S_3$ which are faithful become repres. of $S_4$ with kernel $V_4$ through this procedure.

This gives us a 1st part of the character table of $S_4$:

| | 1 | 6 | 3 | 6 | 8 |
|---|---|---|---|---|---|
| | $1$ | $(12)$ | $(12)(34)$ | $(1234)$ | $(123)$ |
| $\chi_1 = 1_G$ | 1 | 1 | 1 | 1 | 1 |
| $\chi_2$ | 1 | -1 | 1 | $a$ | 1 |
| $\chi_3$ | 2 | 0 | 2 | $b$ | -1 |
| $\chi_4$ | $n_4$ | $c$ | $d$ | $e$ | $f$ |
| $\chi_5$ | $n_5$ | $c'$ | $d'$ | $e'$ | $f'$ |

$\chi_2 = \mathrm{Inf}_{S_4/V_4}^{S_4}(\chi_2)$ from Ex. 5

$\chi_3 = \mathrm{Inf}_{S_4/V_4}^{S_4}(\chi_3)$

In part $\chi_2(\underbrace{(12)(34)}_{\in V_4}) = \chi_2(1)$

$\chi_3((12)(34)) = \chi_3(1)$

as $V_4$ is the kernel of the inflated representations $\rho_2$ and $\rho_3$ of Ex. 5

To obtain $a$ and $b$ we note that $\overline{(1234)} = \overline{(13)} \in S_4/V_4$ $\quad (1234)(12)(34) = (13)$

$\Rightarrow \begin{cases} a = \chi_2((12)) = -1 \\ b = \chi_3((12)) = 0 \end{cases}$

It remains to find $\chi_4$ and $\chi_5$

Degree formula: $24 = |S_4| = \underbrace{1 + 1 + 2^2}_{6} + \chi_4(1)^2 + \chi_5(1)^2$

$\Rightarrow \quad n_4 = n_5 = 3$

The orthogonality relations give us:

- columns 1 & 2: $0 = 1\cdot 1 + 1\cdot(-1) + 2\cdot 0 + 3\cdot c + 3c' = 3c + 3c' \Rightarrow \boxed{c' = -c}$
- columns 1 & 4: $0 = 1\cdot 1 + (1)\cdot(-1) + 2\cdot 0 + 3\cdot e + 3e' = 3e + 3e' = \boxed{e' = -e}$

- columns 5 & 5: $3 = 1\cdot\frac{|24|}{8} = \underbrace{1\cdot 1 + 1\cdot 1 + (-1)(-1)}_{3} + f^2 + f'^2 \Rightarrow \boxed{f = f' = 0}$

- rows: 1 & 4: $0 = 3 + 6\cdot 1\cdot c + 3\cdot 1\cdot d + \overset{6\cdot 1\cdot e}{8\cdot 1\cdot 0} = 3 + 6c + \underbrace{3d}_{-3} + 6e \Rightarrow \boxed{c + e = 0}$ ①

- rows 3 & 4: $0 = 2\cdot 3 + 3\cdot 2\cdot d \Rightarrow \boxed{d = -1} \Rightarrow d' = \pm 1$
- rows 3 & 5: $0 = 2\cdot 3 + 3\cdot 2\cdot d' \Rightarrow \boxed{d' = -1}$

- columns 2 & 4: $0 = 1\cdot 1 + (-1)(-1) + 0\cdot 0 + c\cdot e + \underbrace{c'\cdot e'}_{= c\cdot e} \Rightarrow -2 = 2c\cdot e$
$\Rightarrow \boxed{c\cdot e = -1}$

(column 2 & 3: $0 = 1\cdot 1 + (-1)\cdot 1 + 0\cdot 2 + c\cdot(-1) + c'\cdot(-1)$) $\Rightarrow c^2 \overset{①}{=} 1$

$\Rightarrow$ we may assume (up to changing the numbering of $\chi_5$ and $\chi_4$) that
$c = 1$, $e = -1$
$c' = -1$, $e' = 1$
as character values for $S_n$ are real numbers (see ExSheet3).

<u>Remark:</u> ⚠ two non-isomorphic groups can afford the same character table.
See [Exercise , ExSheet ] $Q_8$ and $D_8$
(Take e.g. $D_8 = \langle \sigma, \rho \mid \rho^4 = 1, \sigma^2 = 1, \sigma\rho\sigma^{-1} = \rho^{-1} \rangle$ and
$Q_8 = \langle a, b \mid a^4 = 1, a^2 = b^2, b a^{-1} b^{-1} = a \rangle$)

⚠ In particular the character table does not determine:
- groups up to isomorphism
- the full lattice of subgrps
- orders of group elements

$\leadsto$ ATLAS
$\leadsto$ GAP

## 2. Normal subgroups and kernels of characters

In Example 6, we have seen that normal subgroups and factor groups are very useful for the determination of a character table. We want to make this procedure more precise.

**Definition 3.2:** The kernel of a character $\chi$ of $G$ is
$$\ker(\chi) := \{g \in G \mid \chi(g) = \chi(1)\}.$$

**Example 7:** (a) $\chi = 1_G$ (trivial character) $\Rightarrow \ker(\chi) = G$

(b) $G = S_3$, $\chi = \chi_2$ the signature character (see Ex. 5)

$\Rightarrow \ker(\chi) = C_1 \sqcup C_3 = \langle (123) \rangle$,

whereas $\ker \chi_3 = \{1_G\}$

**Proposition 3.3:** Let $\rho_V : G \to GL(V)$ be a $\mathbb{C}$ representation of $G$ of degree $n$ with character $\chi_V$.
Then $\ker(\chi_V) = \ker(\rho_V) \trianglelefteq G$.

*Proof:* "$\supseteq$" $g \in \ker(\rho_V) \Rightarrow \rho_V(g) = Id_V \Rightarrow \chi_V(g) = Tr(Id_V)$
$= n = \chi_V(1)$

$\Rightarrow \ker(\rho_V) \subseteq \ker(\chi_V)$.

"$\subseteq$": let $g \in \ker(\chi_V) \Rightarrow \chi_V(g) = \chi_V(1) = n$

Now by Prop. 2.2.(b), we have

$\chi_V(g) = \varepsilon_1 + \dots + \varepsilon_n$, where $\varepsilon_1, \dots, \varepsilon_n$ are $o(g)$-th roots of unity

In part. $n = |\chi_V(g)| = |\varepsilon_1 + \dots + \varepsilon_n| \leq |\varepsilon_1| + \dots + |\varepsilon_n| = n$

$\Rightarrow$ all the $\varepsilon_i$'s are equal, say to $\varepsilon_1$

So $\varepsilon_1 \chi_V(1) = \chi_V(g) \underset{(\star)}{=} \chi_V(1) \Rightarrow \varepsilon_i = 1 \ \forall 1 \leq i \leq n$

$\Rightarrow g \in \ker(\rho_V)$.

Finally $\ker(\chi_V) \trianglelefteq G$ since $\rho_V$ is a grp hm. #

**Definition 3.4:** Let $N \trianglelefteq G$. If $\rho_V : G/N \to GL(V)$ is a $\mathbb{C}$-representation of $G/N$, then we write $Inf_{G/N}^{G}(\chi_V)$ for the character afforded by $Inf_{G/N}^{G}(\rho_V) : G \to GL(V)$ and call this character the <u>inflation</u> of $\chi_V$ from $G/N$ to $G$.

**Remark:** If $\pi : G \twoheadrightarrow G/N$ is the quotient hom, recall that, then

$$\mathrm{Inf}_{G/N}^{G}(\rho_V) = \rho_V \circ \pi : G \twoheadrightarrow G/N \longrightarrow GL(V)$$

$$\Rightarrow \mathrm{Inf}_{G/N}^{G}(\chi_V)(g) = \mathrm{Tr}\big((\rho_V \circ \pi)(g)\big)$$
$$= \mathrm{Tr}\big(\rho_V(gN)\big) = \chi_V(gN) \quad \forall g \in G$$

**Theorem 3.5:** Let $N \trianglelefteq G$.

Then $\mathrm{Inf}_{G/N}^{G} : \{\text{characters of } G/N\} \xrightarrow{\sim} \{\text{characters of } G \text{ with } N \text{ in their kernel}\}$

$$\chi \longmapsto \mathrm{Inf}_{G/N}^{G}(\chi)$$

is a bijection. Moreover,

$$\mathrm{Inf}_{G/N}^{G} : \mathrm{Irr}(G/N) \xrightarrow{\sim} \{\chi \in \mathrm{Irr}(G) \mid N \leq \ker \chi\}$$

is also a bijection.

**Proof:**

- Let $\chi$ be a character of $G/N$ afforded by the repres. $\rho : G/N \to GL(V)$
  $\Rightarrow$ by definition $N \leq \ker(\mathrm{Inf}_{G/N}^{G}(\rho)) \overset{\text{Prop 3.3}}{=} \ker(\mathrm{Inf}_{G/N}^{G}(\chi))$

  Hence the 1st map $\mathrm{Inf}_{G/N}^{G}$ is well-defined.
- Now if $\psi$ is a character of $G$ with $N \leq \ker(\psi)$ and afforded by $\rho : G \to GL(V)$, again by 3.3 $\ker(\rho) = \ker(\psi) \geq N$
  $\Rightarrow \rho$ induces a representation $\tilde{\rho} : G/N \to GL(V)$

$$\begin{tikzcd} G \arrow[r, "\rho"] \arrow[d, "\pi"'] \arrow[dr, phantom, "\circlearrowright"] & GL(V) \\ G/N \arrow[ur, dashed, "\exists!\tilde{\rho}"'] & \end{tikzcd}$$

  s.t. $\tilde{\rho} \circ \pi = \rho$ by the universal property of the quotient.

  $\Rightarrow \rho = \mathrm{Inf}_{G/N}^{G}(\tilde{\rho})$ and $\psi = \mathrm{Inf}_{G/N}^{G}(\chi_{\tilde{\rho}})$

  $\Rightarrow \mathrm{Inf}_{G/N}^{G}$ is surjective

  The injectivity is clear $\Rightarrow \mathrm{Inf}_{G/N}^{G}$ is bijective.
- Finally $\chi \in \mathrm{Irr}(G/N) \overset{\text{Ex. 7}}{\underset{\text{Sheet 2}}{\Rightarrow}} \mathrm{Inf}_{G/N}^{G}(\chi) \in \mathrm{Irr}(G)$

  and $\psi \in \mathrm{Irr}(G \mid N \leq \ker \chi) \Rightarrow \chi_{\tilde{\rho}} \in \mathrm{Irr}(G/N)$ is obvious.
  (in the above notation)

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We now come to properties of finite groups that can be read off their character tables.

**Corollary 3.6:** $\{\chi \in \mathrm{Irr}(G) \mid \chi(1) = 1\} \overset{\sim}{\longleftrightarrow} \mathrm{Irr}(G/G')$

In particular $G$ has exactly $|G:G'|$ linear characters, and $G$ is abelian $\Leftrightarrow$ all its irreducible characters are linear.

Proof: • $\chi$ linear character $\Rightarrow$ $\chi$ is already a representation of $G$
$\Rightarrow$ $\chi$ is a grp homomorphism $G \to \mathbb{C}^*$
$\Rightarrow \forall g,h \in G,\ \chi(ghg^{-1}h^{-1}) = \chi(g)\chi(h)\chi(g^{-1})\chi(h^{-1}) = 1$ since $\mathbb{C}^*$ is abelian
$\Rightarrow \forall g,h \in G,\ [g,h] \in \ker(\chi) \Rightarrow G' = \langle [g,h] \mid g,h \in G \rangle \leq \ker\chi$

• If $\psi \in \mathrm{Irr}(G/G') \overset{\S 2.4}{\Longrightarrow} \psi$ is linear since $G/G'$ is abelian
So Thm 3.5 yields a bijection

$$\mathrm{Irr}(G/G') \underset{\text{via } \mathrm{Inf}^G_{G/G'}}{\overset{\sim}{\longleftrightarrow}} \{\chi \in \mathrm{Irr}(G) \mid G' \leq \ker(\chi)\} \underset{\text{arg}}{\overset{\text{above}}{=}} \{\chi \in \mathrm{Irr}(G) \mid \chi(1) = 1\}.$$

By Ex. 3, $|\mathrm{Irr}(G/G')| = |G/G'| = |G:G'|$
$\Rightarrow$ $G$ has $|G:G'|$ linear characters. #
(If $G$ is abelian, the $G/G' = G$)

**Corollary 3.7:** $G$ is simple $\Leftrightarrow$ $\chi(g) \neq \chi(1)$ $\forall g \in G\setminus\{1\}\ \forall \chi \in \mathrm{Irr}(G)\setminus\{1_G\}$.

Proof: Exercise.

**Remark:** Determining which properties of finite groups can be read off the character table is an active research topic!

E.g. [Navarro-Solomon-Tiep, 2016]: The character table determines whether a finite group has an abelian Sylow $p$-subgroup $(p \mid |G|)$.

## 3. Central characters

**Definition 3.8:** The class multiplication constants of $G$ are the numbers
$$m_{jk\ell} := |\{(g,h) \in C_j \times C_k \mid gh = g_\ell\}| \quad 1 \leq j,k,\ell \leq r.$$

Note: this definition is independent of the choice of $g_\ell \in C_\ell$

**Definition 3.9:** The group algebra of $G$ over the field $K$ is the set
$$KG = \{ \sum_{g\in G} a_g g \mid a_g \in K \},$$
that is the $K$-vector space $\bigoplus_{g\in G} Kg$ with basis $G$, on which we define an addition and a multiplication:

$$+ : KG \times KG \to KG, \quad \left(\sum_{g\in G} a_g g, \sum_{g\in G} b_g g\right) \mapsto \sum_{g\in G}(a_g+b_g)g$$
, and

$$\cdot : KG \times KG \to KG$$
$$\left(\sum_{g\in G} a_g g, \sum_{h\in G} b_h h\right) \mapsto \left(\sum_{g\in G} a_g g\right)\cdot\left(\sum_{h\in G} b_h h\right) := \sum_{g,h\in G} a_g b_h gh$$
$$= \sum_{x\in G}\Big(\sum_{\substack{g,h\in G\\ gh=x}} a_g b_h\Big)x$$

**Clearly:** $KG$ is a $K$-algebra with:

- $1_{KG} = 1_G$
- $\dim_K KG = |G|$
- $G \subseteq (KG)^\times$ via $g \mapsto 1_K g$

**Notation:** Set $\hat{C}_i := \sum_{g\in C_i} g \in KG \quad \forall 1 \le i \le r$
– the class sums.

**Lemma 3.10:** In $KG$, $\hat{C}_j \cdot \hat{C}_k = \sum_{\ell=1}^{r} m_{jk\ell}\hat{C}_\ell \quad \forall 1 \le j,k \le r$.

Proof: # of times a fixed $g_\ell$ occurs in $\hat{C}_j \cdot \hat{C}_k$ is exactly $m_{jk\ell}$ by def$^n$. #

**Proposition 3.11:** $\bigoplus_{j=1}^{r} K\hat{C}_j = Z(KG)$.

Proof. "$\subseteq$": $\forall 1 \le j \le r$ and $\forall g \in G$, we have
$$g\cdot\hat{C}_j = g(g^{-1}\hat{C}_j g) = \hat{C}_j g \implies \bigoplus_{j=1}^{r} K\hat{C}_j \subseteq Z(KG)$$

"$\supseteq$": Let $a \in Z(KG)$ and write $a = \sum_{g\in G} a_g g$

Now $\forall h \in G$: $\sum_{g\in G} a_g g = a \underset{\substack{\uparrow \\ a\in Z(KG)}}{=} hah^{-1} = \sum_{g\in G} a_g hgh^{-1} = \sum_{g\in G} a_{h^{-1}gh}\, g$

hence comparing coefficients $\Rightarrow$ $a_g = a_{h^{-1}gh}$ $\forall h \in G$

$\Rightarrow$ the coefficients $a_g$ of $a$ are constants on the conjugacy classes of $G$.

$\Rightarrow$ $a = \sum_{j=1}^{r} a_{g_j} \hat{C}_j \in \bigoplus_{j=1}^{r} K\hat{C}_j$ / #

This yields:

Corollary 3.12: $\hat{C}_j \cdot \hat{C}_k = \hat{C}_k \cdot \hat{C}_j$ $\forall 1 \le j,k \le r$, so that $m_{jkl} = m_{kjl}$ $\forall 1 \le j,k,l \le r$.

If $R: G \longrightarrow GL_n(\mathbb{C})$ is a matrix representation, then we can extend it by $\mathbb{C}$-linearity to a $K$-algebra homomorphism

$$\tilde{R}: \mathbb{C}G \longrightarrow M_n(\mathbb{C}) \qquad \text{and } \tilde{R}(a) \in \mathrm{End}_{\mathbb{C}}(\mathbb{C}^n)\ \forall a \in \mathbb{C}G.$$
$$\sum_{g\in G} a_g g \longmapsto \sum_{g\in G} a_g R(g)$$

If moreover $R^i: G \to GL_n(\mathbb{C})$ is an irred. matrix repres. of $G$, then $\tilde{R}^i(g)\tilde{R}^i(\hat{C}_j) = \tilde{R}^i(g\hat{C}_j) \underset{\hat{C}_j \in Z(KG)}{=} \tilde{R}^i(\hat{C}_j g) = \tilde{R}^i(\hat{C}_j)\tilde{R}^i(g)$ $\forall g \in G, \forall 1 \le j \le r$

$\Rightarrow$ $\tilde{R}^i(\hat{C}_j) \in \mathrm{End}_G(\mathbb{C}^{n_i})$ $\forall 1 \le j \le r$

$\Rightarrow$ by the corollary to Schur's Lemma, $\exists$ a scalar $\omega_j(\hat{C}_j) \in \mathbb{C}$ s.t. $\tilde{R}^i(\hat{C}_j) = \omega_i(\hat{C}_j) \cdot I_{n_i}$ $\forall 1 \le i,j \le r$

Corollary 3.13: $\forall 1 \le i,j,k \le r$: $\omega_i(\hat{C}_j)\omega_i(\hat{C}_k) = \sum_{l=1}^{r} m_{jkl}\, \omega_i(\hat{C}_l)$

Proof: Lem. 3.10 $\Rightarrow$ $\hat{C}_j \cdot \hat{C}_k = \sum_{l=1}^{r} m_{jkl} \hat{C}_l$

$\Rightarrow$ $\tilde{R}^i(\hat{C}_j) \cdot \tilde{R}^i(\hat{C}_k) = \sum_{l=1}^{r} m_{jkl} \tilde{R}^i(\hat{C}_l)$ $\forall 1 \le i \le r$

$$\begin{array}{ccc} \tilde{R}^i(\hat{C}_j) \cdot \tilde{R}^i(\hat{C}_k) & = & \sum_{l=1}^{r} m_{jkl} \tilde{R}^i(\hat{C}_l) \\ \| & & \| \\ \omega_i(\hat{C}_j)\omega_i(\hat{C}_k) I_{n_i} & & \sum_{l=1}^{r} m_{jkl}\, \omega_i(\hat{C}_l) I_{n_i} \end{array}$$

The claim follows.

**Definition 3.14:** The functions $\omega_i : Z(\mathbb{C}G) \longrightarrow \mathbb{C}$
$$\hat{C}_j \longmapsto \omega_i(\hat{C}_j)$$
are called the central characters of $\mathbb{C}G$.

**Remark:** If $z \in Z(G)$, then $[z] = \{z\}$ => its class sum is $z$ itself => we may see the functions $\omega_i|_{Z(G)}$ as representations of $Z(G)$ of degree 1, and thus as linear characters of $Z(G)$

**Proposition 3.15:** If $\chi_i$ denotes the character of $R^i$, then we have

(a) $\omega_i(\hat{C}_j) = \frac{|C_j|}{n_i}\chi_i(g_j) \quad \forall 1 \le i,j \le r.$

(b) $m_{jkl} = \frac{|C_j|\cdot|C_k|}{|G|}\sum_{i=1}^{r}\frac{\chi_i(g_j)\chi_i(g_k)\chi_i(g_l^{-1})}{\chi_i(1)} \quad \forall 1 \le j,k,l \le r$

**Proof:** (a) Let $\tilde{\chi}_i$ denote the character of $\tilde{R}^i$ (i.e. the trace ...), so
$$\hat{C}_j = \sum_{g \in C_j} g \Rightarrow \tilde{\chi}_i(\hat{C}_j) = \mathrm{Tr}(\tilde{R}^i(\hat{C}_j)) = \mathrm{Tr}(\omega_i(\hat{C}_j)I_{n_i}) = \omega_i(\hat{C}_j)n_i$$
but also $\tilde{\chi}_i(\hat{C}_j) = \sum_{g \in C_j}\chi_i(g_j) = |C_j|\chi_i(g_j)$

The claim follows. /

(b) By Cor. 3.13: $\omega_i(\hat{C}_j)\,\omega_i(\hat{C}_k) = \sum_{s=1}^{r} m_{jks}\,\omega_i(\hat{C}_s)$
$$\overset{(a)}{=}\ \frac{|C_j|}{n_i}\chi_i(g_j)\frac{|C_k|}{n_i}\chi_i(g_k) \quad \overset{(a)}{=}\ \sum_{s=1}^{r} m_{jks}\frac{|C_s|}{n_i}\chi_i(g_s)$$
Multiplying by $n_i\chi_i(g_l^{-1})$ and summing over $1 \le i \le r$
$$\sum_{i=1}^{r}|C_j|\frac{|C_k|}{n_i}\chi_i(g_j)\chi_i(g_k)\chi_i(g_l^{-1}) = \sum_{s,i=1}^{r} m_{jks}\frac{|C_s|}{n_i}\chi_i(g_s)n_i\chi_i(g_l^{-1})$$
$$= \sum_{s=1}^{r} m_{jks}|C_s|\underbrace{\sum_{i=1}^{r}\chi_i(g_s)\chi_i(g_l^{-1})}_{\delta_{sl}\frac{|G|}{|C_l|}}$$
$$\overset{s=l}{=} m_{jkl}|G|$$
$$\Rightarrow m_{jkl} = \frac{|C_j|\cdot|C_k|}{|G|}\sum_{i=1}^{r}\frac{\chi_i(g_j)\chi_i(g_k)\chi_i(g_l^{-1})}{\chi_i(1)} \quad /$$
#

**Theorem 3.19:** The character table of $G$ is determined by the class multiplication constants and conversely.

Proof: Exercise.