CHAPTER 5: INDUCTION AND RESTRICTION

Notation: Same as in chap. 4.
In part. throughout this chapter G denotes a finite group.

We present here a fundamental method to construct characters of G from characters of subgroups $H \leq G$.

Note: From now on we work only in terms of characters. For the corresponding operations in terms of representations/ G-vector spaces see Representation Theory.

1. Induced Characters

Definition 5.1: Let $H \leq G$ and let $\varphi \in \mathcal{Cl}(H)$ be a class function on H. Then the induced class function from φ to G or the induction of φ from H to G is the function

$$\operatorname{Ind}_H^G(\varphi) := \varphi\uparrow_H^G : G \longrightarrow \mathbb{C}$$
$$g \longmapsto \frac{1}{|H|}\sum_{x\in G} \varphi^\circ(xgx^{-1})$$

where
$$\varphi^\circ(g) = \begin{cases} \varphi(g) & \text{if } g\in H \\ 0 & \text{if } g \in G\setminus H \end{cases}$$

Note w/ G-vect. space $\Rightarrow \varphi^\circ(x^{-1}gx)$ right G-vect. space give $\varphi^\circ(xgx^{-1})$

Lemma 5.2: With the notation of Def^n 5.1, $\varphi\uparrow_H^G$ is again a class function.

Proof: Clear since G is a transitive G-set for its left action on itself:
$\forall y, g \in G$ we have

$$\varphi\uparrow_H^G(ygy^{-1}) = \frac{1}{|H|}\sum_{x\in G}\varphi^\circ((xy)g(xy)^{-1}) \overset{s:=xy}{=} \frac{1}{|H|}\sum_{s\in G}\varphi^\circ(sgs^{-1}) = \varphi\uparrow_H^G(g)$$

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Definition 5.3: If $H \leq G$ and $\psi \in \mathcal{Cl}(G)$, the restricted class function from φ to H or the restriction of φ from G to H is the function $\operatorname{Res}_H^G(\ ) := \downarrow_H^G := |_H$. This is obviously a class function

Proposition 5.4: (Frobenius Reciprocity)

Let $H \leq G$ and let $\varphi \in \mathcal{Cl}(H)$ be a class function on H, and ψ a class function on G. Then

$$\langle \varphi\uparrow_H^G , \psi \rangle_G = \langle \varphi, \psi\downarrow_H^G \rangle_H .$$

Proof:

$$\langle \varphi\uparrow_H^G , \psi \rangle_G = \frac{1}{|G|} \sum_{g\in G} \varphi\uparrow_H^G(g)\psi(g^{-1}) = \frac{1}{|G|}\sum_{g\in G} \frac{1}{|H|}\sum_{x\in G} \varphi^o(xgx^{-1})\psi(g^{-1})$$

$$= \frac{1}{|G||H|} \sum_{g\in G}\sum_{x\in G} \varphi^o(g)\psi(x^{-1}g^{-1}x)$$

$$\overset{\psi\in \mathcal{Cl}(G)}{=} \frac{1}{|H|}\sum_{g\in G} \varphi^o(g)\psi(g^{-1})$$

$$= \frac{1}{|H|}\sum_{g\in H} \underbrace{\varphi^o(g)}_{=\varphi(g)}\psi(g^{-1})$$

$$= \langle \varphi, \psi\downarrow_H^G \rangle_H$$

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Example: $G = S_3$, $H = \langle (12) \rangle$ $\varphi =$ sgn character

$$\varphi\uparrow_H^G : S_3 \longrightarrow \mathbb{C}$$
$$g \longmapsto \frac{1}{2}\sum_{x\in G} \varphi^o(xgx^{-1})$$

Since $\varphi\uparrow_H^G$ is a class function we compute it on one representative of each conj. class.

- $\varphi\uparrow_H^G(id) = \frac{1}{2}\sum_{x\in G} \underbrace{\varphi^o(id)}_{=\varphi(id)=1} = \frac{1}{2}|G| = \frac{1}{2}\cdot 6 = 3$

- $\varphi\uparrow_H^G((12)) = \frac{1}{2}\sum_{x\in G} \varphi^o(x(12)x^{-1}) \quad \frac{1}{2}|C_{S_3}((12))|\varphi(12)$

  $= \frac{1}{2}\cdot 2\cdot(-1) = -1$

- $\varphi\uparrow_H^G((123)) = \frac{1}{2}\sum_{x\in G} \varphi^o(\underbrace{x(123)x^{-1}}_{\notin H}) = 0$

Corollary 5.5 Let χ be a character of $H \leq G$ with $\chi(1) =: n$. Then $\chi\uparrow_H^G(1) = n|G:H|$ and $\chi\uparrow_H^G$ is a character of G.

Proof: For $\psi \in Irr(G)$, set
$$m_\psi := \langle \chi\uparrow_H^G, \psi \rangle_G \overset{5.4}{=} \langle \chi, \psi\downarrow_H^G \rangle_H = \langle \chi, \sum_{i=1}^{s} \psi_i \rangle_H = \sum_{i=1}^{s} \langle \chi, \psi_i \rangle_H \in \mathbb{Z}_{\geq 0}$$
where $\psi\downarrow_H^G = \sum_{i=1}^{s} \psi_i$ is a decomp into irred H-characters.

$$\Rightarrow \chi\uparrow_H^G = \sum_{\psi \in Irr(G)} m_\psi \psi \quad \text{is a character}$$

with
$$\chi\uparrow_H^G(1) = \frac{1}{|H|} \sum_{x \in G} \chi^\circ(1) = \frac{1}{|H|} \sum_{x \in G} \chi(1) = \frac{|G|}{|H|} = |G:H|\chi(1). \quad \#$$

Lemma 5.6: $\chi_{reg} = 1\!\!1_{\{1\}}^G$.

Proof: Let $g \in G$. Then
$$1_{\{1\}}\uparrow_{\{1\}}^G(g) = \frac{1}{|\{1\}|} \sum_{x \in G} \underbrace{1_{\{1\}}^\circ(xgx^{-1})}_{\substack{=0 \text{ unless} \\ g = 1_G}} = \delta_{1g}|G|$$
$$= \chi_{reg}(g).$$

Theorem 5.7: Let $K, H \leq G$.

(a) (Transitivity): If $K \leq H$ and $\psi \in \mathcal{Cl}(K)$, then

$$\left(\psi\uparrow_K^H\right)\uparrow_H^G = \psi\uparrow_K^G$$

(b) (Frobenius Formula): If $\varphi \in \mathcal{Cl}(G)$ and $\psi \in \mathcal{Cl}(H)$ we have

$$\varphi \cdot \psi\uparrow_H^G = \left(\varphi\downarrow_H^G \cdot \psi\right)\uparrow_H^G$$

Proof: (a) Let $g \in G$. Then

$$\left(\psi\uparrow_K^H\right)\uparrow_H^G(g) = \frac{1}{|H||K|}\sum_{x\in G}\sum_{y\in H}\psi^\circ(xygy^{-1}x^{-1})$$

$$= \frac{1}{|H||K|}\sum_{y\in H}\sum_{x\in G}\psi^\circ(xygy^{-1}x^{-1})$$

$$\overset{z:=xy}{=} \frac{1}{|H||K|}\sum_{y\in H}\sum_{z\in G}\psi^\circ(zgz^{-1})$$

$$= \frac{1}{\cancel{|H|}|K|}\cdot\cancel{|H|}\cdot\sum_{z\in G}\psi^\circ(zgz^{-1}) = \left(\psi\uparrow_K^G\right)(g)$$

(b) Let $g \in G$. Then

$$\varphi(g)\cdot\psi\uparrow_H^G(g) = \frac{1}{|H|}\sum_{x\in G}\varphi(g)\psi^\circ(xgx^{-1})$$

$$\overset{\varphi\in\mathcal{Cl}(G)}{=} \frac{1}{|H|}\sum_{x\in G}\varphi(xgx^{-1})\psi^\circ(xgx^{-1})$$

$$= \left(\varphi\downarrow_H^G\cdot\psi\right)\uparrow_H^G(g)$$

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Example: (The character table of the simple group A_5.)

Set $G := A_5$. Conj classes: $C_1 = \{id\}$, $C_2 = [(12)(34)]$ $\quad |C_1| = 1$, $|C_2| = 15$

$C_3 = [(123)]$ $\quad |C_3| = 20$

$C_4 \sqcup C_5 = \{5\text{-cycles}\}$ ($\exists\, 24$ of them.

Easy to check $g \in C_4 \Leftrightarrow g^{-1} \in C_4$ as well

but g^2 and $g^3 \in C_5$

$|C_4| = |C_5| = 12$

- $|\mathrm{Irr}(A_5)| = 5$
- A_5 simple nonab $\Rightarrow A_5/[A_5, A_5] = 1 \Rightarrow$ there is only one linear character: $\chi_1 := 1_G$
- A_5 simple also mean that we cannot use inflation as in the case of S_4
- But we can use induction:

let $H := A_4 \leq A_5$. In part $|G:H| = 5$

We can induce 1_H to G: $1_H\uparrow_H^G(\mathrm{id}) \overset{(5.5)}{=} 1 \cdot |G:H| = 5$

$1_H\uparrow_H^G((12)(34)) = \frac{1}{12} \cdot 12 = 1$

$1_H\uparrow_H^G((123)) = \frac{1}{12} \cdot 24 = 2$

$1_H\uparrow_H^G(5\text{-cycle}) = \frac{1}{12} \cdot 0 = 0$

Compute: $\langle 1_H\uparrow_H^G, 1_G \rangle = 1$

and $\langle 1_H\uparrow_H^G - 1_G, 1_H\uparrow_H^G - 1_G \rangle = 1$

$\Rightarrow 1_H\uparrow_H^G - 1_G \in \mathrm{Irr}(G)$.

So we set $\chi_4 := 1_H\uparrow_H^G - 1_G = (4, 0, 1, -1, -1)$ of degree 4

Now by the degree formula $60 = |A_5| = \chi_2(1)^2 + \chi_3(1)^2 + \chi_5(1)^2 + 1 + 16$
$= 43 + 17$

Claim: wlog $\chi_5(1) = 5$

Recall: $\chi_5(1), \chi_2(1), \chi_3(1) \notin \{1, 2\}$ as A_5 simple

Moreover $\chi_5(1), \chi_2(1), \chi_3(1) \mid |A_5| = 60$

$\Rightarrow \chi_5(1), \chi_2(1), \chi_3(1) \in \{3, 5, 4, 6\}$ $\quad 10^2 = 100 > 60$, $12^2 = 144 > 60$, ⋮

- $n_5 = 6$ not possible as $36 + 7 = 43$.
- $n_5 < 5$ not possible
- Only possibility $n_5 = 5$, $n_2 = n_3 = 3$

- Burnside's vanishing Thm 4.10 $\Rightarrow$ $\exists$ at least one zero per non-trivial column. With Cr 4.11 we get

	id	[(12)(34)]	[(123)]	C_4	C_5
$\chi_1 = 1_G$	1	1	1	1	1
χ_2	3	$a_2 = -1$	0	c_2	d_2
χ_3	3	a_3	0	c_3	d_3
χ_4	4	0	1	-1	-1
χ_5	5	$a_5 = 1$	$b_5 = -1$	0	0

rec: $|C_2| = 15$
$|C_3| = 20$
$|C_4| = |C_5| = 12$

- Orthog. relations yield:

1st & 3rd col's: $0 = 1\cdot 1 + 3\cdot 0 + 3\cdot 0 + 4\cdot 1 + 5\cdot b_5 \Rightarrow \boxed{b_5 = -1}$

~~1st & 2nd col's:~~ $\cancel{0 = 1\cdot 1 + 3\cdot a_2 + 3\cdot a_3 + 4\cdot 0 + 5\cdot a_5 \Rightarrow}$

2nd & 3rd col's: $0 = 1\cdot 1 + a_5\cdot(-1) \Rightarrow \boxed{a_5 = 1}$

- Inducing from $H = \langle (12345) \rangle \cong C_5$ a non-trivial character ψ = "χ_2" see Example 4 yields

$$\psi\uparrow_H^G = (12, 0, 0, \varsigma^2 + \varsigma^3, \varsigma + \varsigma^4) \qquad (|G:H| = 12)$$

where ς primitive 5-th root of unity

get $\langle \psi\uparrow_H^G, \chi_4 \rangle = 1$

$\langle \psi\uparrow_H^G, \chi_5 \rangle = 1$

$$\langle \underbrace{\psi\uparrow_H^G - \chi_4 - \chi_5}_{''}, \psi\uparrow_H^G - \chi_4 - \chi_5 \rangle = 1 \Rightarrow \text{irreducible}$$

$(3, -1, 0, -\varsigma - \varsigma^4, -\varsigma^2 - \varsigma^3) =: \chi_2$

- Orthog relations:

Col's 1&2 $\Rightarrow \boxed{a_3 = -1}$

Col's 3&4 $\Rightarrow \boxed{c_3 = -\varsigma^2 - \varsigma^3}$

Col's 3&5 $\Rightarrow \boxed{d_3 = -\varsigma - \varsigma^4}$

2. Clifford Theory

We now want to investigate induction from normal subgroups.

Lemma 5.8 Let $N \trianglelefteq G$ and let $g \in G$.

(a) If $R: N \to GL_n(\mathbb{C})$ is a matrix representation of N, then so is the map
$$^gR: N \to GL_n(\mathbb{C}), \quad n \mapsto R(gng^{-1})$$

(b) If R affords the character ψ, then ^gR affords the character
$$^g\psi : N \to \mathbb{C}, \quad n \mapsto \psi(gng^{-1})$$

(c) The map
$$G \times \mathcal{Cl}(N) \to \mathcal{Cl}(N), \quad (g, \varphi) \mapsto {}^g\varphi$$
is a (left) action of G on $\mathcal{Cl}(N)$.

Proof: easy check ! #

Definition 5.9: If $N \trianglelefteq G$ and ψ is a character of N, then we denote by
$$I_G(\psi) := \{ g \in G \mid {}^g\psi = \psi \}$$
the stabiliser of ψ in G and call this group the inertia(l) group of ψ in G.

Lemma 5.10: Let $N \trianglelefteq G$, ψ be a character of N. Then:

(a) $\forall x \in G$, we have: $\psi \in \mathrm{Irr}(N) \iff {}^x\psi \in \mathrm{Irr}(N)$

(b) $$(\psi\uparrow_N^G)\downarrow_N^G = |I_G(\psi) : N| \sum_{x \in [G/I_G(\psi)]} {}^x\psi$$

Proof: (a) $$\langle {}^x\psi, {}^x\psi \rangle = \frac{1}{|N|} \sum_{n \in N} {}^x\psi(n)\, {}^x\psi(n^{-1}) = \frac{1}{|N|} \sum_{n \in N} \psi(xnx^{-1})\psi(xn^{-1}x^{-1})$$
$$\overset{h := xnx^{-1}}{=} \frac{1}{|N|} \sum_{h \in N} \psi(h)\psi(h^{-1}) = \langle \psi, \psi \rangle$$

Hence $\langle {}^x\psi, {}^x\psi \rangle = 1 \iff \langle \psi, \psi \rangle = 1$

and ${}^x\psi \in \mathrm{Irr}(N) \iff \psi \in \mathrm{Irr}(N)$ by 2.18(d)/(a)

(b) Let $n \in N$. Compute:

$$(\psi\uparrow_N^G)\downarrow_N^G(n) = (\psi\uparrow_N^G)(n) = \frac{1}{|N|}\sum_{x\in G} \psi^\circ(\underbrace{xnx^{-1}}_{\in N}) = \frac{1}{|N|}\sum_{x\in G}\psi(xnx^{-1})$$

$$= \frac{1}{|N|}\sum_{x\in G} {}^x\psi(n)$$

$x = \tilde{x}\cdot a$, $a \in I_G(\psi)$

$x\, I_G(\psi)$

$$= \frac{1}{|N|}\sum_{\tilde{x}\in[G/I_G(\psi)]} |I_G(\psi)|\, {}^{\tilde{x}}\psi(n)$$

$$= |I_G(\psi):N| \sum_{\tilde{x}\in[G/I_G(\psi)]} {}^{\tilde{x}}\psi(n)$$

(b) #

Notation: Given $\psi \in \mathrm{Irr}(N)$ we let

$$\mathrm{Irr}(G\,|\,\psi) := \{\chi \in \mathrm{Irr}(G) \mid \langle \chi\downarrow_N^G, \psi\rangle \neq 0\}$$

i.e. ψ is a constituent of $\chi\downarrow_N^G$.

Theorem 5.11 ("Clifford Theory")

Let $N \trianglelefteq G$. Let $\chi \in \mathrm{Irr}(G)$ and let $\psi \in \mathrm{Irr}(N)$.

(a) If ψ is a constituent of $\chi\downarrow_N^G$, then

$$\chi\downarrow_N^G = e \sum_{g\in[G/I_G(\psi)]} {}^g\psi$$

where $e := \langle \chi\downarrow_N^G, \psi\rangle$ $(\neq 0)$ is the <u>ramification index</u> of χ in N

(b) All the constituents of $\chi\downarrow_H^G$ have the same degree.

(c) If $\eta \in \mathrm{Irr}(I_G(\psi)\,|\,\psi)$, then $\eta\uparrow_{I_G(\psi)}^G \in \mathrm{Irr}(G)$.

(d) If $\chi \in \mathrm{Irr}(G\,|\,\psi)$, then $\exists!\, \eta \in \mathrm{Irr}(I_G(\psi)\,|\,\psi)$ s.t. $\langle \chi\downarrow_{I_G(\psi)}^G, \eta\rangle_{I_G(\psi)} \neq 0$

(e) Induction from $I_G(\psi)$ to G induces a bijection

$$\mathrm{Ind}_{I_G(\psi)}^G : \mathrm{Irr}(I_G(\psi)\,|\,\psi) \longrightarrow \mathrm{Irr}(G\,|\,\psi)$$
$$\eta \longmapsto \eta\uparrow_{I_G(\psi)}^G$$

Proof: (a) We have $0 \neq \langle \chi\downarrow_N^G, \psi\rangle_N \overset{\text{Frob rec.}}{=} \langle \chi, \psi\uparrow_N^G\rangle_G$

$\Rightarrow$ χ is a constituent of $\psi\uparrow_N^G$ $\Rightarrow$ $\chi\downarrow_N^G \mid (\psi\uparrow_N^G)\downarrow_N^G$

Now if $\eta \in \mathrm{Irr}(N)$ is a constituent of $\chi\downarrow_N^G$ $(\Rightarrow \chi\downarrow_N^G = \eta + \nu)$
then $0 \neq \langle \chi\downarrow_N^G, \eta\rangle$ and $\langle \chi\downarrow_N^G, \eta\rangle \leq \langle \psi\uparrow_N^G\downarrow_N^G, \eta\rangle$.
Moreover η must be of the form ${}^x\psi$ for some $x \in G$, since by Lemma 5.10, the constituents of $(\psi\uparrow_N^G)\downarrow_N^G$ are $\{{}^x\psi \mid x \in [G/I_G(\psi)]\}$.
Furthermore, $\forall g \in G$:

$$e = \langle \chi\downarrow_N^G, \psi\rangle_N = \frac{1}{|N|}\sum_{h\in N}\chi(h)\psi(h^{-1}) \overset{N \trianglelefteq G}{=} \frac{1}{|N|}\sum_{h\in N}\chi(ghg^{-1})\psi(gh^{-1}g^{-1})$$

$$= \frac{1}{|N|}\sum_{h\in N}\chi(h)\,{}^g\psi(h) = \langle \chi\downarrow_N^G, {}^g\psi\rangle_N$$

Therefore, every conjugate ${}^g\psi$ of ψ occurs as a constituent of $\mathrm{Res}_N^G(\chi) = \chi\downarrow_N^G$ with the same multiplicity e.

Hence the formula. /(a)

(b) follows from (a) since ${}^x\psi(1_N) = \psi(x1_Nx^{-1}) = \psi(1_N)$ $\forall x \in G$.

(c) Since $I_G(\psi) = I_{I_G(\psi)}(\psi)$, by (a) we have $\eta\downarrow_N^{I_G(\psi)} = e'\psi$
for some $e' \in \mathbb{Z}_{>0}$. $\left(e' = \frac{\eta(1)}{\psi(1)}\right)$

Let $\tilde{\chi} \in \mathrm{Irr}(G)$ with $0 \neq \langle \tilde{\chi}, \eta\uparrow_{I_G(\psi)}^G\rangle_G \overset{\text{Frobenius}}{=} \langle \tilde{\chi}\downarrow_{I_G(\psi)}^G, \eta\rangle_{I_G(\psi)}$

Then $e := \langle \tilde{\chi}\downarrow_N^G, \psi\rangle_N = \langle \tilde{\chi}\downarrow_{I_G(\psi)}^G\downarrow_N^{I_G(\psi)}, \psi\rangle_N$

$$\geq \langle \eta\downarrow_N^{I_G(\psi)}, \psi\rangle_N = e' > 0$$

$\Rightarrow \tilde{\chi} \in \mathrm{Irr}(G|\psi)$

Moreover by (a), $e = \langle \tilde{\chi}\downarrow_N^G, {}^g\psi\rangle_N \geq e'$ $\forall g \in G$, so that

$$\begin{aligned}\tilde{\chi}(1) = e\sum_{g\in[G/I_G(\psi)]}{}^g\psi(1) &\overset{(b)}{=} e|G:I_G(\psi)|\psi(1)\\ &\geq e'|G:I_G(\psi)|\psi(1)\\ &= e'|G:I_G(\psi)|\psi(1)\\ &= |G:I_G(\psi)|\eta(1)\\ &= \eta\uparrow_{I_G(\psi)}^G(1)\\ &\geq \tilde{\chi}(1)\end{aligned}$$

$\Rightarrow \eta\uparrow_{I_G(\psi)}^G = \tilde{\chi} \in \mathrm{Irr}(G)$. /(c)

(d) Let $\chi \in \mathrm{Irr}(G|\psi)$, s.t. $\chi\downarrow_N^G = e \sum_{g\in[G/I_G(\psi)]} {}^g\psi$ for some $e \in \mathbb{N}$
(e)

Since $\chi\downarrow_N^G = \chi\downarrow_{I_G(\psi)}^G \downarrow_N^{I_G(\psi)}$, $\exists\ \eta \in \mathrm{Irr}(I_G(\psi))$ s.t.

$$\langle \chi\downarrow_{I_G(\psi)}^G, \eta \rangle_{I_G(\psi)} \neq 0 \neq \langle \eta\downarrow_N^{I_G(\psi)}, \psi \rangle_N$$

In part. $\eta \in \mathrm{Irr}(I_G(\psi)|\psi)$
and by Frobenius reciprocity, $0 \neq \langle \chi, \eta\uparrow_{I_G(\psi)}^G \rangle_G$

$\overset{(c)}{\Longrightarrow}$ $\chi = \eta\uparrow_{I_G(\psi)}^G$ and $\eta\downarrow_N^{I_G(\psi)} = e\psi$, so e is the ram. index of ψ in I.

We have $\chi\downarrow_{I_G(\psi)}^G = \sum_{\lambda\in \mathrm{Irr}(I_G(\psi))} a_\lambda \lambda$ for coefficients $a_\lambda \in \mathbb{Z}_{\geq 0}$ with $a_\eta > 0$.

Moreover $(a_\eta - 1)\,\mathrm{Res}_N^{I_G(\psi)}(\eta) + \sum_{\lambda\neq\eta} a_\lambda \lambda\downarrow_N^{I_G(\psi)} = \chi\downarrow_N^G - \eta_N^{I_G(\psi)}$

$$= e \sum_{\substack{g\neq 1 \\ g\in[G/I_G(\psi)]}} {}^g\psi$$

so ψ does not occur in this sum, but in $\eta\downarrow_N^{I_G(\psi)}$, so that we must have $a_\eta = 1$ and $\lambda \notin \mathrm{Irr}(I_G(\psi)|\psi)$ $\forall \lambda\neq\eta$.
$\Rightarrow$ η is uniquely determined as the only constituent of $\chi\downarrow_{I_G(\psi)}^G$ in $\mathrm{Irr}(I_G(\psi)|\psi)$
The bijection of (e) follows. #

3. Subgroups of index 2

Let $N \leq G$ with $|G:N| = 2$ ($\Rightarrow N \trianglelefteq G$ as $G = N \sqcup gN = N \sqcup Ng$ $\forall g \in G\setminus N$)

In this case the character tables of G and N are closely related.
Typical examples: • $N = A_n$, $G = S_n$
• $N = PSL_2(q)$, $G = SL_2(q)$ with q odd.
• D_{2m}, D_{4m}

Lemma 5.12

If $N \trianglelefteq G$ with $|G:N|=2$ and $\chi \in \mathrm{Irr}(G)$, then either

(1) $\chi\downarrow_N^G \in \mathrm{Irr}(N)$, or

(2) $\chi\downarrow_N^G = \psi + {}^g\psi$ for a $\psi \in \mathrm{Irr}(N)$ and a $g \in G\setminus N$.

<u>Proof</u>: Let $\psi \in \mathrm{Irr}(N)$ be a constituent of $\chi\downarrow_N^G$.

Since $|G:N|=2$, we have two possibilities for $I_G(\psi)$:

- <u>Case 1</u>: $I_G(\psi)=N$. $\Rightarrow$ $\mathrm{Irr}(I_G(\psi)|\psi)=\{\psi\}$

  so that by Thm 5.11(e) we have
  $$\chi = \psi\uparrow_N^G \quad \Rightarrow \chi(1) = 2\psi(1)$$
  and Thm 5.11(a) yields $\chi\downarrow_N^G = \psi + {}^g\psi$ with $g \in G\setminus N$.

- <u>Case 2</u>: $I_G(\psi)=G$. $\Rightarrow$ $[G/I_G(\psi)]=\{1\}$

  so that by Thm 5.11(a) we have
  $$\chi\downarrow_N^G = e\psi \quad \text{with } e = \langle \chi\downarrow_N^G, \psi\rangle_N = \langle \chi, \psi\uparrow_N^G\rangle_G$$
  Moreover by Lemma 5.10(b)
  $$\psi\uparrow_N^G\downarrow_N^G = 2\psi$$
  $$\Rightarrow 2\psi(1) = \psi\uparrow_N^G(1) = \psi\uparrow_N^G\downarrow_N^G(1) \geq \chi\downarrow_N^G(1) \,(=\chi(1)) = e\psi(1)$$

  $\Rightarrow e \leq 2$

  Were $e=2$, then we would have $\psi\uparrow_N^G(1) = 2\psi(1)$

  hence $\chi = \psi\uparrow_N^G$ and then
  $$1 = \langle \chi, \psi\uparrow_N^G\rangle_G = \langle \chi\downarrow_N^G, \psi\rangle_N = e = 2 \;\lightning$$
  Whence $e=1$ $\Rightarrow$ $\chi\downarrow_N^G = \psi \in \mathrm{Irr}(N)$

  and $\psi\uparrow_N^G = \chi + \chi'$ for some $\chi' \in \mathrm{Irr}(G)$ with $\chi \neq \chi'$. #

Since $G/N \cong \underset{\langle \bar{g} \mid \bar{g}^2=1\rangle}{(C_2,\cdot)}$ and $\mathrm{Irr}(C_2)=\{1_{C_2}, \mathrm{sgn}\}$ where $\mathrm{sgn}: C_2 \to \mathbb{C}^\times,\ \bar{g} \mapsto -1$

we may inflate the sign character sgn to G and set
$$\lambda_{\mathrm{sgn}} := \mathrm{Inf}_{G/N}^G(\mathrm{sgn}) : G \longrightarrow \mathbb{C}$$
$$g \longmapsto \begin{cases} 1 & \text{if } g \in N \\ -1 & \text{if } g \notin N \end{cases}$$

Recall from Cor. 2.25, that $\forall\, \chi \in \mathrm{Irr}(G)$, $\chi\lambda_{\mathrm{sgn}} \in \mathrm{Irr}(G)$ and $\chi, \chi\lambda_{\mathrm{sgn}}$ have the same degree.
In addition $\chi\downarrow_N^G$ and $(\chi\lambda_{\mathrm{sgn}})\downarrow_N^G$ are equal by def^n.

Lemma 5.13 Let $N \trianglelefteq G$ with $|G:N| = 2$ and let $\chi \in \mathrm{Irr}(G)$. TFAE:

(1) $\chi\downarrow_N^G \in \mathrm{Irr}(N)$;

(2) $\exists\, g \in G \setminus N$ s.t. $\chi(g) \neq 0$;

(3) $\chi \neq \chi\lambda_{\mathrm{sgn}}$.

Proof: (1) => (2): $\chi \in \mathrm{Irr}(G) \Rightarrow 1 = \langle \chi, \chi \rangle$

$$= \frac{1}{|G|} \sum_{g \in G} \chi(g)\chi(g^{-1})$$

$$= \underbrace{\frac{1}{|G|} \sum_{g \in H} \chi(g)\chi(g^{-1})}_{|H| \cdot \langle \chi\downarrow_N^G, \chi\downarrow_N^G \rangle} + \underbrace{\frac{1}{|G|} \sum_{g \in G \setminus N} \chi(g)\chi(g^{-1})}_{\geq 0}$$

$$= \underbrace{\frac{|H|}{|G|}}_{= \frac{1}{2}} \cdot 1 + \underbrace{\frac{1}{|G|} \sum_{g \in G \setminus N} \chi(g)\chi(g^{-1})}_{\therefore\ = \frac{1}{2}}$$

$\Rightarrow \exists\, g \in G \setminus N$ s.t. $\chi(g) \neq 0$.

(2) => (3): Let $g \in G \setminus N$ s.t. $\chi(g) \neq 0$

$\Rightarrow \chi\lambda_{\mathrm{sgn}}(g) = \chi(g)\lambda_{\mathrm{sgn}}(g) = -\chi(g)$

Hence $\chi \neq \chi\lambda_{\mathrm{sgn}}$.

(3) => (1): $\chi \neq \chi\lambda_{\mathrm{sgn}} \Rightarrow \exists\, g \in G \setminus N$ s.t. $\chi(g) \neq \chi\lambda_{\mathrm{sgn}}(g) = -\chi(g)$

$\Rightarrow \chi(g) \neq 0$

So $1 = \langle \chi, \chi \rangle \overset{\text{see above}}{=} \frac{1}{2} \underbrace{\langle \chi\downarrow_N^G, \chi\downarrow_N^G \rangle}_{\geq 1} + \underbrace{\frac{1}{|G|} \sum_{g \in G \setminus N} \chi(g)\chi(g^{-1})}_{\neq 0}$

This forces $\langle \chi\downarrow_N^G, \chi\downarrow_N^G \rangle = 1$

$\Rightarrow \chi\downarrow_N^G \in \mathrm{Irr}(N)$.

#

Lemma 5.14: Let $N \trianglelefteq G$ with $|G:N|=2$.
Assume $\chi \in \mathrm{Irr}(G)$ is s.t. $\chi\downarrow_N^G \in \mathrm{Irr}(N)$.
If $\varphi \in \mathrm{Irr}(G)$ satisfies $\varphi\downarrow_N^G = \chi\downarrow_N^G$, then $\varphi \in \{\chi, \chi\lambda_{sgn}\}$.

Proof: We have
$$(\chi + \chi\lambda_{sgn})(g) = \begin{cases} 2\chi(g) & \text{if } g \in N \\ 0 & \text{if } g \in G \setminus N \end{cases}$$
Hence
$$\langle \chi + \chi\lambda_{sgn}, \varphi \rangle_G = \frac{1}{|G|}\sum_{g\in N} \overset{=\frac{|G|}{|N|}}{2}\,\chi(g)\varphi(g^{-1})$$
$$= \frac{1}{|H|}\sum_{g\in N}\chi(g)\varphi(g^{-1}) = \langle \chi\downarrow_N^G, \varphi\downarrow_N^G \rangle_N \overset{\text{hypothesis}}{=} 1$$
As $\chi, \chi\lambda_{sgn} \in \mathrm{Irr}(G)$ this forces $\varphi \in \{\chi, \chi\lambda_{sgn}\}$. #

Lemma 5.15: Let $N \trianglelefteq G$ with $|G:N|=2$.
Assume $\chi \in \mathrm{Irr}(G)$ is s.t. $\chi\downarrow_N^G = \psi + {}^g\psi$ with $\psi \in \mathrm{Irr}(N)$ and $g \in G\setminus N$ a in Lemma 5.12.
If $\varphi \in \mathrm{Irr}(G)$ satisfies $\varphi\downarrow_N^G$ has ψ or ${}^g\psi$ as a constituent, then $\varphi = \chi$.

Proof: Lemma 5.13 $\Rightarrow$ $\chi(g) = 0 \ \forall g \in G\setminus N$ as $\chi\downarrow_N^G$ is reducible.
$$\Rightarrow \langle \varphi, \chi \rangle_G = \frac{1}{|G|}\sum_{g\in G}\varphi(g)\chi(g^{-1}) = \frac{1}{|G|}\sum_{g\in N}\varphi(g)\chi(g^{-1})$$
$$= \frac{1}{2}\langle \varphi\downarrow_N^G, \chi\downarrow_N^G \rangle$$
$$= \frac{1}{2}\langle \underbrace{\varphi\downarrow_N^G}_{\substack{=:\psi+\eta \\ (\text{resp. } {}^g\psi+\eta)}}, \psi + {}^g\psi \rangle \neq 0$$
$$\Rightarrow \langle \varphi, \chi \rangle_G \neq 0 \Rightarrow \langle \varphi, \chi \rangle_G = 1 \Rightarrow \varphi = \chi. \ \#$$

Conclusion: If $N \trianglelefteq G$ with $|G:N|=2$ and the character table of G is known, then we can list the irreducible characters of N as follows

• Step 1: Each $\chi \in \mathrm{Irr}(G)$ which is non-zero somewhere outside N restricts to N irreducibly : i.e. $\chi\downarrow_N^G \in \mathrm{Irr}(N)$. Such characters of G occure in pairs $\chi, \chi\lambda_{sgn}$ s.t. $\chi\downarrow_N^G = (\chi\lambda_{sgn})\downarrow_N^G$

• Step 2: Each $\chi \in \mathrm{Irr}(G)$ which vanishes outside N restricts to N as a sum of two conjugate irreducible characters of N : $\chi\downarrow_N^G = \psi + {}^g\psi$ $(g \in G \setminus N)$
ψ and ${}^g\psi$ do not occure as constituents of $\chi'\downarrow_N^G$ for $\chi \neq \chi' \in \mathrm{Irr}(G)$

• Every irred. character of N occurs as a constituent of $\chi\downarrow_N^G$ for some $\chi \in \mathrm{Irr}(G)$, hence we are done.

Exercise: Compute the character table of A_5 from that of S_5 using the above method.

4. Extending characters

Using Clifford Theory we understand that irreducibility of characters is preserved between $I_G(\psi)$ and G under induction.
We need to understand the step $\psi\uparrow_N^{I_G(\psi)}$.
In particular, what happens when $I_G(\psi) = G$ itself?
Can we say sth. about $\mathrm{Irr}(G \mid \psi)$?

Lemma 5.16: Let $N \trianglelefteq G$ and $\psi \in \mathrm{Irr}(N)$ s.t. $I_G(\psi) = G$.
Then: (a) $\psi\uparrow_N^G = \sum_{\chi \in \mathrm{Irr}(G)} e_\chi \chi$ where $e_\chi = \langle \chi\downarrow_N^G, \psi \rangle$, and

(b) $\sum_{\chi \in \mathrm{Irr}(G)} e_\chi^2 = |G : N|$.

Proof: Write $\psi\uparrow_N^G = \sum_{\chi\in \mathrm{Irr}(G)} a_\chi \chi$ with $\{a_\chi\}_{\chi\in\mathrm{Irr}(G)} \subseteq \mathbb{Z}_{\geq 0}$

Frob. reciprocity: $a_\chi \neq 0 \iff \chi \in \mathrm{Irr}(G|\psi)$

$(a_\chi = \langle \psi\uparrow_N^G, \chi\rangle = \langle \psi, \chi\downarrow_N^G\rangle)$

But by Clifford theory: if $\chi\in\mathrm{Irr}(G|\psi)$, then $\chi\downarrow_N^G = e_\chi \psi$

$\Rightarrow \quad a_\chi = e_\chi \quad \forall \chi \in \mathrm{Irr}(G|\psi)$

Therefore
$$|G:N|\psi(1) = \psi\uparrow_N^G(1) = \sum_{\chi\in\mathrm{Irr}(G|\psi)} e_\chi \underbrace{\chi(1)}_{=e_\chi\psi(1)}$$
$$= \sum_{\chi\in\mathrm{Irr}(G|\psi)} e_\chi^2 \psi(1)$$
$$= \psi(1) \sum_{\chi\in\mathrm{Irr}(G|\psi)} e_\chi^2$$

Hence $|G:N| = \sum_{\chi\in\mathrm{Irr}(G)} e_\chi^2$. #

Therefore the multiplicities $\{e_\chi\}_{\chi\in\mathrm{Irr}(G)}$ behave like the degrees of the irreducible characters of G/N.
This is not a coincidence in many cases.

Definition 5.17:

Let $N \trianglelefteq G$ and let $\chi \in \mathrm{Irr}(G)$. If $\psi\in\mathrm{Irr}(N)$ is such that $\chi\downarrow_N^G = \psi$, then we say that χ is an extension of ψ or that ψ extends to G.

Proposition 5.18: Let $N\trianglelefteq G$ and $\chi\in\mathrm{Irr}(G)$. Then
$$\chi\downarrow_N^G\uparrow_N^G = \mathrm{Inf}_{G/N}^G(\chi_{reg})\cdot\chi$$

Proof: Exercise, Sheet 7.

Theorem 5.19 [Gallagher] (1962)

Let $N\trianglelefteq G$ and $\chi\in\mathrm{Irr}(G)$ s.t. $\psi := \chi\downarrow_N^G \in \mathrm{Irr}(N)$. Then the characters $\{\mathrm{Inf}_{G/N}^G(\lambda)\cdot\chi \mid \lambda\in\mathrm{Irr}(G/N)\}$ of G are pairwise distinct and irreducible. Moreover

$$\psi\uparrow_N^G = \sum_{\lambda \in \mathrm{Irr}(G/N)} \lambda(1)\, \mathrm{Inf}_{G/N}^G(\lambda)\chi$$

<u>Proof:</u> Let χ_{reg} be the regular character of G/N.

$$\overset{\text{Prop. 5.18}}{\Longrightarrow} \quad \psi\uparrow_N^G = \mathrm{Inf}_{G/N}^G(\chi_{reg})\chi .$$

$$\overset{\text{Lem 2.14}}{=} \mathrm{Inf}_{G/N}^G\Big(\sum_{\lambda\in\mathrm{Irr}(G/N)} \lambda(1)\lambda\Big)\cdot\chi$$

$$= \sum_{\lambda\in\mathrm{Irr}(G/N)} \lambda(1)\,\mathrm{Inf}_{G/N}^G(\lambda)\cdot\chi$$

• Now $|G:N| \overset{L.5.16}{=} \sum_{\chi\in\mathrm{Irr}(G)} e_\chi^2 \overset{5.16}{=} \langle \psi\uparrow_N^G, \psi\uparrow_N^G\rangle_G$

$$= \sum_{\lambda,\mu\in\mathrm{Irr}(G/N)} \lambda(1)\mu(1)\langle \mathrm{Inf}_{G/N}^G(\lambda)\chi, \mathrm{Inf}_{G/N}^G(\mu)\chi\rangle_G$$

$$\geq \sum_{\lambda\in\mathrm{Irr}(G/N)} \lambda(1)^2 = |G:N|$$

hence equality throughout.

$\Longrightarrow \quad \langle \mathrm{Inf}_{G/N}^G(\lambda)\chi, \mathrm{Inf}_{G/N}^G(\mu)\chi\rangle = \delta_{\lambda\mu}$

(the $\mathrm{Inf}_{G/N}^G(\lambda)\chi$ are characters by Cor. 2.25)

$\Longrightarrow \quad \mathrm{Inf}_{G/N}^G(\lambda)\chi \in \mathrm{Irr}(G)$ and they are distinct as claimed.

#

<u>Example</u> :

(a) See § on normal subgps of index 2
We saw that if $\psi\in\mathrm{Irr}(N)$ extends to G then $\psi\uparrow_N^G = \chi + \chi'$ with $\chi' = \chi\lambda_{sgn}$, where in fact $\lambda_{sgn} = \mathrm{Inf}_{G/N}^G(sgn)$ where sgn is the sgn character of $G/N \cong C_2$.

(b) $G = D_8$: See Ex Sheet 5.
$N := Z(G) \cong C_2$. But we see from the character table of D_8 that the sgn character of C_2 does not extend to D_8
(there is no linear character of D_8 taking value -1 on the central involution)

Proposition 5.20: Let $N \trianglelefteq G$ and let $\psi \in \mathrm{Irr}(N)$. If $I_G(\psi)/N$ has prime order, then ψ extends to $I_G(\psi)$.

Proof: W.l.o.g. we may assume that $G = I_G(\psi)$.

Set $|G/N| =: p$. Then $p \in \mathbb{P}$ by (H).

$$\Rightarrow \quad G/N \cong C_p$$

To show: $\boxed{e=1}$

Let $\chi \in \mathrm{Irr}(G|\psi) \overset{\text{Prop. 5.11}}{\Longrightarrow} \chi\downarrow^G_N = e\psi$ for some $e \in \mathbb{Z}_{\geq 1}$

- Now because G/N is abelian, any $\lambda \in \mathrm{Irr}(G/N)$ is linear and
$$\mathrm{Inf}^G_{G/N}(\lambda)\chi \in \mathrm{Irr}(G) \text{ by Cor. 2.25.}$$

Moreover, if $\mathrm{Inf}^G_{G/N}(\lambda)\chi = \mathrm{Inf}^G_{G/N}(\mu)\chi$ for two characters $\lambda \neq \mu \in \mathrm{Irr}(G/N)$, then consider

$$U/N = \ker(\lambda\bar{\mu}), \quad \text{~~for that~~ } N \leq U \lneq G$$

which forces $N = U$ as $|G/N| = p \in \mathbb{P}$

$$\Rightarrow \chi(g) = 0 \quad \forall g \in G \setminus U = G\setminus N$$

$$\Rightarrow \quad e^2 = \langle e\psi, e\psi\rangle_N = \langle \chi\downarrow^G_N, \chi\downarrow^G_N\rangle_N$$
$$= |G:N| \underbrace{\langle \chi, \chi\rangle_G}_{=1} = |G:N| = p \in \mathbb{P}$$
$$\lightning$$

Hence: $\lambda \neq \mu \in \mathrm{Irr}(G/N) \Rightarrow \mathrm{Inf}^G_{G/N}(\lambda)\chi \neq \mathrm{Inf}^G_{G/N}(\mu)\chi$

with $\langle \mathrm{Inf}^G_{G/N}(\lambda)\chi, \psi\uparrow^G_N\rangle_G \overset{\text{Frob. rec.}}{=} \langle (\mathrm{Inf}^G_{G/N}(\lambda)\chi)\downarrow^G_N, \psi\rangle_N$

$\left(\mathrm{Inf}^G_{G/N}(\lambda)\downarrow^G_N = 1_N\right)$

$$= \langle \chi\downarrow^G_N, \psi\rangle_N = e$$

$$\Rightarrow |G:N|\psi(1) = \psi\uparrow^G_N(1) \geq e \cdot \sum_{\lambda \in \mathrm{Irr}(G/N)} \overbrace{(\mathrm{Inf}^G_{G/N}(\lambda)\chi)(1)}^{=\chi(1)}$$
$$= e \cdot |G:N|\chi(1)$$
$$\geq e|G:N|\psi(1)$$

$$\Rightarrow \quad e = 1 \text{ and } \chi \text{ extends to } \psi.$$