CHAPTER 6: BRAUER'S CHARACTERISATION OF CHARACTERS

Aim: Given a class function θ on G, decide whether θ is a "true" character or not!

1. Elementary subgroups

Notation 6.1: Let R be a ring (unital) with $\mathbb{Z} \subseteq R \subseteq \mathbb{C}$.
Set:

$R\,\mathrm{Irr}(G) := \{R\text{-linear combinations of irred. characters of } G\}$

For $\mathcal{H}$ a family of subgroups of G, we let:

$\mathcal{R}_R(G,\mathcal{H}) := \{\text{class fct. } \theta : G \to R \mid \theta|_H \in R\,\mathrm{Irr}(H)\ \forall H \in \mathcal{H}\}$

$\mathcal{I}_R(G,\mathcal{H}) := \{R\text{-lin. comb's of characters } \psi\uparrow_H^G \mid \psi \in \mathrm{Irr}(H), H \in \mathcal{H}\}$

For $R = \mathbb{Z}$, we simply write $\mathcal{R}(G,\mathcal{H})$ and $\mathcal{I}(G,\mathcal{H})$ (drop the index R).

Lemma 6.2: (a) $\mathcal{I}(G,\mathcal{H}) \subseteq \mathcal{I}_R(G,\mathcal{H}) \subseteq R\,\mathrm{Irr}(G) \subseteq \mathcal{R}_R(G,\mathcal{H})$

(b) $\mathcal{R}_R(G,\mathcal{H})$ is a ring with identity element 1_G, and $\mathcal{I}_R(G,\mathcal{H})$ an ideal thereof.

In particular: $\mathcal{I}(G,\mathcal{H}) = \mathcal{R}_R(G,\mathcal{H}) \iff 1_G \in \mathcal{I}(G,\mathcal{H})$

Proof: (a) is clear by definitions.

(b) • $\mathcal{R}_R(G,\mathcal{H})$ is a ring is one elt 1_G: easy check!

• Let $\rho \in \mathcal{I}_R(G,\mathcal{H})$ and $\theta \in \mathcal{R}_R(G,\mathcal{H})$. Then

$$\rho = \sum_{H\in\mathcal{H}} a_H \psi\uparrow_H^G \quad \text{for some } a_H \in R,\ \psi \in \mathrm{Irr}(H)$$

$$\Rightarrow \rho\cdot\theta = \sum_{H\in\mathcal{H}} a_H \psi\uparrow_H^G\,\theta \underset{\substack{\text{Frobenius}\\ \text{Formula}}}{=} \sum_{H\in\mathcal{H}} a_H(\underbrace{\psi\,\theta|_H^G}_{\in \mathbb{Z}_+\mathrm{Irr}(H)})\uparrow_H^G \in \mathcal{I}_R(G,\mathcal{H})$$

so that $\mathcal{I}_R(G,\mathcal{H}) \trianglelefteq \mathcal{R}_R(G,\mathcal{H})$ #

$R = \mathbb{Z}$ yields $\mathcal{I}(G,\mathcal{H}) \trianglelefteq \mathcal{R}_R(G,\mathcal{H})$

Definition 6.3: (a) Let $p \in \mathbb{P}$. A finite group E is <u>p-elementary</u> iff $E = P \times Z$ with:
- P a p-group
- Z cyclic
- $p \nmid |Z|$

(b) A finite group E is called <u>elementary</u> iff E is p-elementary for some $p \in \mathbb{P}$.

<u>We want to prove:</u>

If $\mathcal{H} := \{\text{elementary subgrps of } G\}$, then $\mathcal{I}(G, \mathcal{H}) = \mathcal{R}(G, \mathcal{H})$

Lemma 6.4: Let S be a non-empty finite set, R a ring of functions $f : S \to \mathbb{Z}$ (with pointwise addition/multiplication). If $1_S : S \to \mathbb{Z}$, $s \mapsto 1$, is not in R, then there exists an element $x \in S$ and $p \in \mathbb{P}$ such that $p \mid f(x)$ $\forall f \in R$.

<u>Proof:</u> Given $x \in S$, let $I_x := \{f(x) \mid f \in R\}$

Easy check: $(I_x, +) \leq (\mathbb{Z}, +)$ (additive subgrp), hence an ideal of $\mathbb{Z}$.

- Now if $I_x \lneq \mathbb{Z}$ for some $x \in S$, then $I_x \subseteq \langle p \rangle_{\mathbb{Z}}$ for some $p \in \mathbb{P}$ $\Rightarrow$ claim holds ✓
- If $I_x = \mathbb{Z}$ $\forall x \in S$, then $\forall x \in S$, $\exists f_x \in R$ s.t. $f_x(x) = 1$

$$\Rightarrow (f_x - 1_S)(x) = 0 \Rightarrow \prod_{x \in S} (f_x - 1_S) = 0$$

Expanding this product (finite!), we obtain that 1_S is a $\mathbb{Z}$-linear combination of products of f_x's in R

$\Rightarrow 1_S \in R$. ↯

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We apply this to the following character theoretic situation:

Proposition 6.5: Let $H, K \leq G$. Then $\exists \{a_U\}_{U \leq H} \subseteq \mathbb{Z}_{\geq 0}$ s.t.

$$1\!\!1_H\uparrow_H^G \cdot 1_K\uparrow_K^G = \sum_{U \leq H} a_U 1_U\uparrow_U^G .$$

Proof: Set $\theta := 1_K\uparrow_K^G$.

$$\Rightarrow \quad 1\!\!1_H\uparrow_H^G\, 1_K\uparrow_K^G = \theta \cdot 1_H\uparrow_H^G \overset{\text{Frob.}}{\underset{\text{Form}}{=}} (\theta\downarrow_H^G \cdot 1_H)\uparrow_H^G = (\theta\downarrow_H^G)\uparrow_H^G$$

Now ~~so that~~ θ is the permutation character of G on the cosets of K

$\Rightarrow$ $\theta\downarrow_H^G$ is the permutation character of H

so that $\theta\downarrow_H^G = \sum a_U 1_U\uparrow_U^H$ for suitable subgrps $U \leq H$ and coefficients of $a_U \in \mathbb{Z}_{\geq 0}$.

But $1_U\uparrow_U^H\uparrow_H^G \cong 1_U\uparrow_U^G$

Hence $1\uparrow_H^G\, 1\uparrow_K^G = \sum_{U \leq H} a_U 1_U\uparrow_U^G$. #

Corollary 6.6: The $\mathbb{Z}$-linear combinations of characters of G of the form $1_H\uparrow_H^G$, $H \leq G$ form a ring, which we denote by $P(G)$.
Let $\underline{\mathcal{H}}$ be a set a subgrps of G s.t.: $K \leq H, H \in \underline{\mathcal{H}} \Rightarrow K \in \underline{\mathcal{H}}$
Let $P(G, \mathcal{H})$ be the set of $\mathbb{Z}$-linear combinations of characters $1_H\uparrow_H^G$ with $H \in \underline{\mathcal{H}}$.
Then $P(G, \underline{\mathcal{H}}) \trianglelefteq P(G)$.

Proof: straightforward by the above. #
(Proof of 6.5)

Definition 6.7: Let $p \in \mathbb{P}$. A finite group H is called <u>p-quasi-elementary</u> iff $\exists Z \trianglelefteq H$ cyclic s.t. H/Z is a p-group.
H is called <u>quasi-elementary</u> $\Leftrightarrow$ $\exists p \in \mathbb{P}$ s.t. H is p-quasi-elementary.

Clear: H p-elementary $\Rightarrow$ H p-quasi-elementary.

Proposition 6.8: Let $p \in \mathbb{P}$ and let $x \in G$. Then there exists a p-quasi-elementary subgroup $H \leq G$ s.t. $1_H \uparrow_H^G (x)$ is not divisible by p.

Proof: • Decompose $\langle x \rangle = C \times P$ with C p'-group (cyclic), P p-group (cyclic)

Set $N := N_G(C)$. Clearly $x \in N_G(C)$

Now $P \cong \langle x \rangle / C$ is a p-group $\Rightarrow \exists$ a subgrp $H \geq \langle x \rangle$ s.t. $H/C \in Syl_p(N_G(C)/C)$ (correspondence thm)

C cyclic $\trianglelefteq H$ with H/C p-grp $\overset{\text{def}^{\text{n}}}{\Rightarrow}$ H is p-quasi-elementary.

• Now $1_H \uparrow_H^G (x) = |\{ Hy \mid Hyx = Hy, y \in G \}|$ (def$^{\text{n}}$ induction)

but $Hy = Hyx \Rightarrow yxy^{-1} \in H \Rightarrow yCy^{-1} \leq H$

As $C \trianglelefteq H$, $yCy^{-1} = C$, so that $y \in N_G(C)$ and vice versa

So we need to count the fixed points of $\langle x \rangle$ on the cosets of H in $N_G(C)$.

But as $C \trianglelefteq N_G(C)$ and $C \trianglelefteq H$, C fixes all cosets of H

Since $\langle x \rangle / C$ is a p-grp, each orbit length is divisible by p or 1

$\Rightarrow 1_H \uparrow_H^G (x) \equiv |N_G(C) : H| \pmod p$

By the choice of H: $p \nmid |N_G(C) : H|$. ✓ #

Theorem 6.9: Let $\mathcal{H} := \{ H \leq G \mid H \text{ quasi-elementary} \}$. Then $1_G \in P(G, \mathcal{H})$.

Proof: Cor. 6.6 $\Rightarrow$ $P(G, \mathcal{H})$ ring of $\mathbb{Z}$-valued functions of G

If $1_G \notin P(G, \mathcal{H})$, then $\exists x \in G, \exists p \in \mathbb{P}$ s.t. $p \mid \varphi(x)$ $\forall \varphi \in P(G, \mathcal{H})$ ↯ Prop. 6.8. #

§2. The Theorem of Brauer

Lemma 6.10: Let $G = CP$ with $C \trianglelefteq G$, P a p-group and $p \nmid |C|$.
Let λ be a linear character of C invariant under G, with $C_C(P) \leq \ker\lambda$. Then $\lambda = 1_C$

Proof: We may see λ as a character of $C/\ker\lambda$ with all distinct values

Let $x \in C$. λ G-invariant $\Rightarrow$ P fixes the coset $\ker(\lambda)x \in C/\ker\lambda$

$\Rightarrow$ Each non-trivial orbit of P on $\ker(\lambda)x$ has length divisible by p

But $p \nmid |\ker(\lambda)|$ $\Rightarrow$ $p \nmid |\ker(\lambda)x|$

$\Rightarrow$ $\ker(\lambda)x \cap C_C(P) \neq \emptyset$

But $C_C(P) \subseteq \ker(\lambda)$ by Assumption

$\Rightarrow$ $\ker(\lambda)x \cap \ker\lambda \neq \emptyset$ $\Rightarrow$ $\ker(\lambda)x = \ker(\lambda)$

$\Rightarrow$ $C \subseteq \ker\lambda$

$\Rightarrow$ $\lambda = 1_C$ #

Theorem 6.11 (Brauer) G finite group.
Let $\mathbb{Z} \subseteq R \subseteq \mathbb{C}$ be a Ring and let $\mathcal{E} := \{\text{elementary subgrps of } G\}$.
Then:

(a) $R\,\mathrm{Irr}(G) = \mathcal{R}_R(G, \mathcal{E})$

(b) $\mathbb{Z}\,\mathrm{Irr}(G) = \mathcal{I}(G, \mathcal{E})$.

Thus any $\chi \in \mathrm{Irr}(G)$ is a $\mathbb{Z}$-linear combination of characters induced from elementary subgrps of G.

Proof: Lemma 6.2 $\Rightarrow$ enough to prove: $1_G \in \mathcal{I}(G,\mathcal{E})$.

Induction nach $|G|$:

- clear if $G \in \mathcal{E}$, i.e. G elementary.
- Now assume $1_H \in \mathcal{I}(H, \mathcal{E}_H)$ $\forall H < G$ (where $\mathcal{E}_H = \{Q \leq H \text{ elementary}\}$)

By lemma 6.2: $\mathbb{Z}\,\mathrm{Irr}(H) = \mathcal{I}(H, \mathcal{E}_H)$

Claim 1: $\varphi \in \mathrm{Irr}(H) \Rightarrow \varphi\uparrow_H^G \in \mathcal{I}(G,\mathcal{E})$ $\forall H<G$.

Indeed: $H<G$ and $\varphi \in \mathrm{Irr}(H)$

$\Rightarrow \varphi \in \mathbb{Z}\,\mathrm{Irr}(H) = \mathcal{I}(H,\mathcal{E}_H)$

$\Rightarrow \varphi\uparrow_H^G \in \mathcal{I}(G,\mathcal{E}_H) \subseteq \mathcal{I}(G,\mathcal{E})$ ✓

$\Rightarrow$ Remains to prove: 1_G is a $\mathbb{Z}$-linear combination of characters induced from proper subgrps of G.

Case 1: G is not quasi-elementary: done by Thm 6.9.

Case 2: G is quasi-elementary.

Let C be the cyclic normal p-complement ($p \in \mathbb{P}$)
(i.e. $C \trianglelefteq G$ and G/C p-grp)

Let $P \in \mathrm{Syl}_p(G) \Rightarrow G = CP$, and $Z := C_C(P)$

G not elementary $\Rightarrow Z < C$, $E := PZ < G$

Write $1_E\uparrow_E^G = 1_G + \xi$, where ξ is a character of G

To prove: each constituent of ξ is induced from a proper subgrp.

($\Rightarrow 1_G = 1_E\uparrow_E^G - \xi \in \mathcal{I}(G,\mathcal{E})$ as claimed)

So let χ be a constituent of ξ.

As $CE = CZP = G$, $C \cap E = C \cap PZ = Z$

$\Rightarrow 1_C + \xi\downarrow_C^G = 1_E\uparrow_E^G\downarrow_C^G = 1_Z\uparrow_Z^C$

$\Rightarrow 1 = \langle 1_Z\uparrow_Z^C, 1_C\rangle_C = \langle 1_C + \xi\downarrow_C^G, 1_C\rangle_C$

$\Rightarrow \langle \xi\downarrow_C^G, 1_C\rangle_C = 0$

Hence: if λ is an irred const. of $\chi\downarrow_C^G$, then $\lambda \neq 1_C$

But $Z \trianglelefteq G \Rightarrow Z \leq \mathrm{Ker}(1_E\uparrow_E^G)$
$\Rightarrow Z \leq \mathrm{Ker}(\chi)$
$\Rightarrow Z \leq \mathrm{Ker}(\lambda)$

$\overset{\text{Lem. 6.10}}{\Longrightarrow}$ λ is not G-invariant.

Let $T := I_G(\lambda) < G$ be the inertia group of λ

Then $\chi = \psi\uparrow_C^G$ for some $\psi \in \mathrm{Irr}(T)$

by Clifford Theory, as required. #

Theorem 6.13

Let $p \in \mathbb{P}$ and let $\chi \in \mathrm{Irr}(G)$ with $p \nmid \frac{|G|}{\chi(1)}$.

Then $\chi(g) = 0 \ \forall g \in G$ such that $p \mid o(g)$.

Proof: Define a class function θ on G by $\theta(G) := \begin{cases} \chi(g) & p \nmid o(g) \\ 0 & p \mid o(g) \end{cases}$

Claim: $\theta \in \mathbb{Z}\mathrm{Irr}(G)$.

Indeed: let $E \leq G$ be an elementary subgrp i.e.

$E = P \times Z$ with $\begin{cases} P & p\text{-grp} \\ Z & \text{cyclic of order prime to } p. \end{cases}$

- If $g \in E$, $p \nmid o(g)$, then $g \in Z \Rightarrow \begin{cases} \theta\downarrow^G_{E \setminus Z} = 0 \\ \theta\downarrow^G_Z = \chi\downarrow^G_Z \end{cases}$

For $\psi \in \mathrm{Irr}(E)$, we have

$$|E| \langle \theta\downarrow^G_E, \psi \rangle_E = \sum_{x \in Z} \chi(x)\overline{\psi}(x)$$

$$= |Z| \langle \chi\downarrow^G_Z, \psi\downarrow^E_Z \rangle_Z \in |Z|\mathbb{Z}$$

As $|E| = |P|\cdot|Z|$, we actually have

$$|P| \langle \theta\downarrow^G_E, \psi \rangle_E \in \mathbb{Z}$$

Now consider ω_χ the central character ~~of~~ assoc. to χ,

i.e. $\omega_\chi : Z(\mathbb{C}G) \longrightarrow \mathbb{C}$ with

$$\hat{C}_i \longmapsto \frac{|C_i|}{\chi(1)}\chi(g_i) \quad \text{where } [g_i] = C_i$$

$$\Rightarrow \chi(g_i) = \chi(1)\omega_\chi(\hat{C}_i)\frac{1}{|C_i|} = \chi(1)\omega_\chi(\hat{C}_i)\frac{|C_G(g_i)|}{|G|}$$

$$\Rightarrow |E| \langle \theta\downarrow^G_E, \psi \rangle = \sum_{x \in Z} \chi(x)\overline{\psi}(x) = \frac{\chi(1)}{|G|}\sum_{x \in Z} \omega_\chi(x)\overline{\psi}(x)|C_G(x)|$$

$$\Rightarrow \frac{|G||Z|}{\chi(1)} \langle \theta\downarrow^G_E, \psi \rangle = \sum_{x \in Z} \omega_\chi(x)\overline{\psi}(x)\left|C_G(x) : P\right| \quad \text{as } P \leq C_G(x) \ \forall x \in Z$$

an algebraic integer.

But $\underbrace{\langle \theta\downarrow^G_E, \psi \rangle_E \, |P| \in \mathbb{Z}}_{①}$ $\Rightarrow$ $\langle \theta\downarrow^G_E, \psi \rangle_E \in \mathbb{Q}$ $\quad (\forall \psi \in Irr(E))$

$\Rightarrow \underbrace{\frac{|G||Z|}{\chi(1)} \langle \theta\downarrow^G_E, \psi \rangle_E \in \mathbb{Z}}_{②}$

Now since $\gcd\left(\frac{|G||Z|}{\chi(1)}, |P|\right) = 1$, ① + ② yield $\langle \theta\downarrow^G_E, \psi \rangle_E \in \mathbb{Z}$

$\Rightarrow$ $\theta\downarrow_E$ is a generalised character

$\overset{\text{Thm 6.11}}{\Rightarrow}$ θ is a generalised character / claim

$\Rightarrow$ $\langle \theta, \chi \rangle \in \mathbb{Z}$, $\langle \theta, \theta \rangle = \langle \theta, \chi \rangle = \frac{1}{|G|} \sum_{\substack{g \in G \\ p \nmid o(g)}} |\chi(g)|^2$

$\Rightarrow$ $0 < \langle \theta, \chi \rangle \leq 1$

$\Rightarrow$ $\langle \theta, \chi \rangle = \langle \chi, \chi \rangle = \langle \theta, \theta \rangle = 1$

$\Rightarrow$ $\langle \chi - \theta, \chi - \theta \rangle = 0$

$\Rightarrow$ $\chi = \theta$

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