CHARACTER THEORY OF FINITE GROUPS — EXERCISE SHEET 4 — **TU KAISERSLAUTERN**

JUN.-PROF. DR. CAROLINE LASSUEUR — **BIRTE JOHANSSON**

Due date: **Thursday the 4th of June, 2020, 13:00 in OpenOLAT** — SS 2020

Throughout this exercise sheet $K = \mathbb{C}$ is the field of complex numbers, $(G, \cdot)$ is a finite group, and V a finite-dimensional $\mathbb{C}$-vector space. Each Exercise is worth 4 points.

EXERCISE 13

(a) Prove that the degree formula can be read off from the 2nd Orthogonality Relations.

(b) Use the degree formula to prove again that if G is a finite abelian group, then

$$\mathrm{Irr}(G) = \{\text{linear characters of } G\}\,.$$

(b) Give the character tables of the Klein-four group $C_2 \times C_2$ and of $C_2 \times C_2 \times C_2$.

EXERCISE 14
Let G be a finite group. Set $X := X(G)$ and

$$C := \begin{bmatrix} |C_G(g_1)| & 0 & \cdots & 0 \\ 0 & |C_G(g_2)| & \ddots & \vdots \\ \vdots & \ddots & \ddots & 0 \\ 0 & \cdots & 0 & |C_G(g_r)| \end{bmatrix} \in M_r(\mathbb{C})\,.$$

Use the orbit-stabiliser theorem in order to prove that the 1st Orthogonality Relations can be rewritten under the form

$$XC^{-1}\overline{X}^{\mathrm{Tr}} = I_r$$

where $\overline{X}^{\mathrm{Tr}}$ denotes the transpose of the complex-conjugate $\overline{X}$ of the character table X of G. Deduce that the character table is invertible.

EXERCISE 15
Let G and H be two finite groups. Prove that:

(a) if $\lambda, \chi \in \mathrm{Irr}(G)$ and $\lambda(1) = 1$, then $\lambda \cdot \chi \in \mathrm{Irr}(G)$;

(b) the set $\{\chi \in \mathrm{Irr}(G) \mid \chi(1) = 1\}$ of linear characters of a finite group G forms a group for the product of characters;

(c) $\mathrm{Irr}(G \times H) = \{\chi \cdot \psi \mid \chi \in \mathrm{Irr}(G), \psi \in \mathrm{Irr}(H)\}$.

[Hint: Use Corollary 9.8(d) and the degree formula.]

EXERCISE 16
Let $N \trianglelefteq G$ and let $\rho : G/N \longrightarrow \mathrm{GL}(V)$ be a $\mathbb{C}$-representation of G/N with character χ.

(a) Prove that if ρ is irreducible, then so is $\mathrm{Inf}_{G/N}^{G}(\rho)$.

(b) Compute the kernel of $\mathrm{Inf}_{G/N}^{G}(\rho)$ provided that ρ is faithful.