Throughout this exercise sheet K denotes a field of arbitrary characteristic, $(G, \cdot)$ a finite group with neutral element 1_G, and V a finite-dimensional K-vector space.

Exercise 5 (**Alternative proof of Maschke's Theorem over the field $\mathbb{C}$.**)
Assume $K = \mathbb{C}$ and let $\rho : G \longrightarrow \mathrm{GL}(V)$ be a $\mathbb{C}$-representation of G.

(a) Prove that there exists a G-invariant scalar product $\langle\ ,\ \rangle : V \times V \longrightarrow \mathbb{C}$, i.e. such that

$$\langle \rho(g)(u), \rho(g)(v)\rangle = \langle u, v\rangle \quad \forall\, g \in G, \forall\, u, v \in V\,.$$

[Hint: consider an arbitrary scalar product on V, say $(\ ,\ ) : V \times V \longrightarrow \mathbb{C}$, which is not necessarily G-invariant. Use a sum on the elements of G weighted by the group order in order to produce a new G-invariant scalar product on V.]

(b) Deduce that every G-invariant subspace W of V admits a G-invariant complement.
[Hint: consider the orthogonal complement of W.]

Exercise 6
Assume we are in the situation of Proposition 4.3. Namely, we are given a K-vector space $(V, +, \cdot)$ and we define an external multiplication on V by the elements of KG through a left action $G \times V \longrightarrow V, (g, v) \mapsto g \cdot v$ of G on V which we extend by K-linearity to the whole of KG. Thus, we now have a KG-module $(V, +, \cdot)$, where the new external multiplication $\cdot : KG \longrightarrow V$ extends the initial external multiplication on V by the elements of K.
Prove that checking the KG-module axioms (Appendix A, Definition A.1) for $(V, +, \cdot)$ is equivalent to checking the following axioms:

(1) $(gh) \cdot v = g \cdot (h \cdot v)\,,$

(2) $1_G \cdot v = v\,,$

(4) $g \cdot (u + v) = g \cdot u + g \cdot v\,,$

(3) $g \cdot (\lambda v) = \lambda(g \cdot v) = (\lambda g) \cdot v\,,$

for all $g, h \in G$, $\lambda \in K$ and $u, v \in V$.

Exercise 7 (**Exercise to hand in / 8 points**)

(a) Check the details of the proof of Proposition 4.3.
[Hint: use Exercise 6.]

(b) Use Proposition 4.3 to express the trivial representation in terms of KG-modules.

(c) Use Proposition 4.3 to express the regular representation in terms of KG-modules. Prove that the KG-module you have obtained is isomorphic to KG (the group algebra) seen as a left KG-module over itself.

(d) **Schur's Lemma for matrix representations.**
Let $n, n' \in \mathbb{N}$. Let $R : G \longrightarrow \mathrm{GL}_n(K)$ and $R' : G \longrightarrow \mathrm{GL}_{n'}(K)$ be two irreducible matrix representations. Prove that if there exists $A \in M_{n\times n'}(K)\setminus\{0\}$ such that $AR'(g) = R(g)A$ for every $g \in G$, then $n = n'$ and A is invertible (in particular $R \sim R'$).

Exercise 8
Prove the following assertions.

(a) The regular $\mathbb{C}$-representation of any finite group is faithful.

(b) Every finite simple group G admits a faithful irreducible $\mathbb{C}$-representation.

(c) If $G = C_{n_1} \times \cdots \times C_{n_r}$ is a product of finite cyclic groups of order $n_1, \ldots, n_r$ ($r \in \mathbb{Z}_{>0}$), then G admits a faithful $\mathbb{C}$-representation of degree r.