Throughout this exercise sheet $K = \mathbb{C}$ is the field of complex numbers, $(G, \cdot)$ is a finite group, and V a finite-dimensional $\mathbb{C}$-vector space.

Exercise 9
Let $\rho_V : G \longrightarrow \mathrm{GL}(V)$ be a $\mathbb{C}$-representation and let χ_V be its character. Prove the following statements.

(a) If $g \in G$ is conjugate to g^{-1}, then $\chi_V(g) \in \mathbb{R}$.

(b) If $g \in G$ is an element of order 2, then $\chi_V(g) \in \mathbb{Z}$ and $\chi_V(g) \equiv \chi_V(1) \pmod 2$.

Exercise 10 (**Exercise to hand in / 8 points**)
Let $\rho_V : G \longrightarrow \mathrm{GL}(V)$ be a $\mathbb{C}$-representation.

(a) Prove that:

(i) the dual space $V^* := \mathrm{Hom}_{\mathbb{C}}(V, \mathbb{C})$ is endowed with the structure of a $\mathbb{C}G$-module via

$$\begin{array}{ccc} G \times V^* & \longrightarrow & V^* \\ (g, f) & \mapsto & g.f \end{array}$$

where $(g.f)(v) := f(g^{-1}v)\ \forall\, v \in V$;

(ii) the character of the associated $\mathbb{C}$-representation ρ_{V^*} is then $\chi_{V^*} = \overline{\chi_V}$; and

(iii) if ρ_V decomposes as a direct sum $\rho_{V_1} \oplus \rho_{V_2}$ of two subrepresentations, then $\rho_{V^*} = \rho_{V_1^*} \oplus \rho_{V_2^*}$.

(b) Determine the duals of the 3 irreducible representations of S_3 given in Example 2(d).

Exercise 11

(a) Exhibit a $\mathbb{C}$-basis of $Cl(G)$ and deduce that $\dim_{\mathbb{C}} Cl(G) = |C(G)|$.

(b) Verify that the form

$$\langle -, - \rangle_G : \mathcal{F}(G, \mathbb{C}) \times \mathcal{F}(G, \mathbb{C}) \longrightarrow \mathbb{C}, (f_1, f_2) \mapsto \langle f_1, f_2 \rangle_G := \frac{1}{|G|} \sum_{g \in G} f_1(g)\overline{f_2(g)}$$

is sesquilinear and Hermitian.

Exercise 12
Let V be a $\mathbb{C}G$-module (i.e. finite dimensional) with character χ_V. Consider the $\mathbb{C}$-subspace $V^G := \{v \in V \mid g \cdot v = v\ \forall\, g \in G\}$. Prove that

$$\dim_{\mathbb{C}} V^G = \frac{1}{|G|} \sum_{g \in G} \chi_V(g)$$

in two different ways:

1. considering the scalar product of χ_V with the trivial character $\mathbf{1}_G$;

2. seeing V^G as the image of the projector $\pi : V \longrightarrow V, v \mapsto \frac{1}{|G|} \sum_{g \in G} g \cdot v$.