Throughout this exercise sheet $K = \mathbb{C}$ is the field of complex numbers, $(G, \cdot)$ is a finite group.

Exercise 13

(a) Prove that the degree formula can be read off from the 2nd Orthogonality Relations.

(b) Use the degree formula to prove again that if G is a finite abelian group, then

$$\mathrm{Irr}(G) = \{\text{linear characters of } G\}.$$

Exercise 14 (**Exercise to hand in / 8 points**)

(a) Let G be a finite group. Set $X := X(G)$ and

$$C := \begin{bmatrix} |C_G(g_1)| & 0 & \cdots & 0 \\ 0 & |C_G(g_2)| & \ddots & \vdots \\ \vdots & \ddots & \ddots & 0 \\ 0 & \cdots & 0 & |C_G(g_r)| \end{bmatrix} \in M_r(\mathbb{C}).$$

Prove that the 1st Orthogonality Relations can be rewritten under the form

$$XC^{-1}\overline{X}^{\mathrm{Tr}} = I_r$$

where $\overline{X}^{\mathrm{Tr}}$ denotes the transpose of the complex-conjugate $\overline{X}$ of the character table X of G.

(b) Prove that the character table is invertible.

(c) Compute the matrix C for $G = S_3$ and $G = C_{10}$, and verify that the formula in (a) holds.

(d) Compute the character tables of the Klein-four group $C_2 \times C_2$ and of the elementary abelian 2-group $C_2 \times C_2 \times C_2$ of rank 3.