JUN.-PROF. DR. CAROLINE LASSUEUR **ANNIKA BARTELT AND MARIE ROTH**
Due date: Wednesday, the 24th of May 2023, 17:00 SS 2023

Throughout, unless otherwise stated, K denotes a field of arbitrary characteristic, $(G, \cdot)$ a finite group with neutral element 1_G, and $(V, +, \cdot)$ a finite-dimensional K-vector space. Each Exercise is worth 4 points.

EXERCISE 1 (From K-representations of G to KG-modules and back)
The goal of this exercise is to better understand Proposition 4.3.

(a) (Proof of Remark 4.4). We define an external composition law on V by the elements of KG through a left action $G \times V \longrightarrow V, (g, v) \mapsto g \cdot v$ of G on V which we extend by K-linearity to the whole of KG. We now have a KG-module $(V, +, \cdot)$, where the new external composition law $\cdot : KG \longrightarrow V$ extends the initial external composition law on V by the elements of K.

Prove that checking the KG-module axioms (Appendix A, Definition A.1) for $(V, +, \cdot)$ is equivalent to checking the following axioms:

(GV1) $(gh) \cdot v = g \cdot (h \cdot v)\,,$

(GV2) $1_G \cdot v = v\,,$

(GV3) $g \cdot (u + v) = g \cdot u + g \cdot v\,,$

(GV4) $g \cdot (\lambda v) = \lambda(g \cdot v) = (\lambda g) \cdot v\,,$

for all $g, h \in G$, $\lambda \in K$ and $u, v \in V$.

(b) Check the details of the proof of Proposition 4.3.

(c) Use Proposition 4.3 to express the trivial representation in terms of KG-modules.

(d) Use Proposition 4.3 to express the regular representation in terms of KG-modules. Prove that the KG-module you have obtained is isomorphic to KG (the group algebra) seen as a left KG-module over itself.

From now on, $K = \mathbb{C}$ is the field of complex numbers and V a finite-dimensional $\mathbb{C}$-vector space.

EXERCISE 2 (On the existence of faithful representations)
Prove the following assertions.

(a) The regular $\mathbb{C}$-representation of any finite group is faithful.

(b) Every finite simple group G admits a faithful irreducible $\mathbb{C}$-representation.

(c) If $G = C_{n_1} \times \cdots \times C_{n_r}$ is a product of finite cyclic groups of order $n_1, \ldots, n_r$ ($r \in \mathbb{Z}_{>0}$), then G admits a faithful $\mathbb{C}$-representation of degree r.

Exercise 3 (**Values of characters**)
Let $\rho_V : G \longrightarrow \mathrm{GL}(V)$ be a $\mathbb{C}$-representation and let χ_V be its character. Prove the following statements.

(a) If $g \in G$ is conjugate to g^{-1}, then $\chi_V(g) \in \mathbb{R}$.

(b) If $g \in G$ is an element of order 2, then $\chi_V(g) \in \mathbb{Z}$ and $\chi_V(g) \equiv \chi_V(1) \pmod 2$.

Exercise 4 (**The dual representation**)
Let $\rho_V : G \longrightarrow \mathrm{GL}(V)$ be a $\mathbb{C}$-representation.

(a) Prove that:

(i) the dual space $V^* := \mathrm{Hom}_{\mathbb{C}}(V, \mathbb{C})$ is endowed with the structure of a $\mathbb{C}G$-module via

$$\begin{array}{ccc} G \times V^* & \longrightarrow & V^* \\ (g, f) & \mapsto & g.f \end{array}$$

where $(g.f)(v) := f(g^{-1}v)\ \forall\, v \in V$;

(ii) the character of the associated $\mathbb{C}$-representation ρ_{V^*} is then $\chi_{V^*} = \overline{\chi_V}$; and

(iii) if ρ_V decomposes as a direct sum $\rho_{V_1} \oplus \rho_{V_2}$ of two subrepresentations, then $\rho_{V^*} = \rho_{V_1^*} \oplus \rho_{V_2^*}$.

(b) Determine the duals of the three irreducible representations of S_3 given in Example 2(d).