Jun.-Prof. Dr. Caroline Lassueur — **Annika Bartelt and Marie Roth**
Due date: Wednesday, the 21st of June 2023, 17:00 — SS 2023

Throughout, unless otherwise stated, $K = \mathbb{C}$ is the field of complex numbers and $(G, \cdot)$ a finite group with neutral element 1_G. Each Exercise is worth 4 points.

Exercise 1 (**On the Orthogonality Relations**)

(a) Prove that the degree formula can be read off from the 2nd Orthogonality Relations.

(b) Use the degree formula to prove again that if G is a finite abelian group, then

$$\mathrm{Irr}(G) = \mathrm{Lin}(G).$$

(c) Set $X := X(G)$ and

$$C := \begin{bmatrix} |C_G(g_1)| & 0 & \cdots & 0 \\ 0 & |C_G(g_2)| & \ddots & \vdots \\ \vdots & \ddots & \ddots & 0 \\ 0 & \cdots & 0 & |C_G(g_r)| \end{bmatrix} \in M_r(\mathbb{C}).$$

Prove that the 1st Orthogonality Relations can be rewritten under the form

$$XC^{-1}\overline{X}^{\mathrm{Tr}} = I_r$$

where $\overline{X}^{\mathrm{Tr}}$ denotes the transpose of the complex-conjugate $\overline{X}$ of the character table X of G.

(d) Prove that the character table is invertible.

Exercise 2
Let G and H be two finite groups. Prove that:

(a) if $\lambda, \chi \in \mathrm{Irr}(G)$ and $\lambda(1) = 1$, then $\lambda \cdot \chi \in \mathrm{Irr}(G)$;

(b) the set $\mathrm{Lin}(G)$ of linear characters of G forms a group for the product of characters;

(c) $\mathrm{Irr}(G \times H) = \{\chi \cdot \psi \mid \chi \in \mathrm{Irr}(G), \psi \in \mathrm{Irr}(H)\}$.

[Hint: Use Corollary 9.8(d) and the degree formula.]

Exercise 3 (**Faithful representations and simplicity**)

(a) Let $N \trianglelefteq G$ and let $\rho : G/N \longrightarrow \mathrm{GL}(V)$ be a $\mathbb{C}$-representation of G/N with character χ. Compute the kernel of $\mathrm{Inf}_{G/N}^{G}(\rho)$ provided that ρ is faithful.

(b) Let $\rho : G \longrightarrow \mathrm{GL}(V)$ be a $\mathbb{C}$-representation of G with character χ. Prove that

$$\ker(\chi) = \ker(\rho),$$

thus is a normal subgroup of G.

(c) Prove that if $N \trianglelefteq G$, then

$$N = \bigcap_{\substack{\chi \in \mathrm{Irr}(G) \\ N \subseteq \ker(\chi)}} \ker(\chi)\,.$$

(d) Prove that G is simple if and only if $\chi(g) \neq \chi(1)$ for each $g \in G \setminus \{1\}$ and each $\chi \in \mathrm{Irr}(G) \setminus \{\mathbf{1}_G\}$.

Exercise 4 (**Does the character table determine the group?**)

(a) Compute the character tables of the dihedral group D_8 of order 8 and of the quaternion group Q_8.

[Hint: In each case, determine the commutator subgroup and deduce that there are 4 linear characters.]

(b) If $\rho : G \longrightarrow \mathrm{GL}(V)$ is a $\mathbb{C}$-representation of G and $\det : \mathrm{GL}(V) \longrightarrow \mathbb{C}^*$ denotes the determinant homomorphism, then we define a linear character of G through

$$\det_\rho := \det \circ \rho : G \longrightarrow \mathbb{C}^*\,,$$

called the **determinant of** ρ. Prove that, although the finite groups D_8 and Q_8 have the "same" character table, they can be distinguished by considering the determinants of their irreducible $\mathbb{C}$-representations.