EXAMPLE

THE CHARACTER TABLE OF A_5

USING INDUCTION FROM A_4

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And $|C_1| = 1$, $|C_2| = 15$, $|C_3| = 20$, $|C_4| = |C_5| = 12$.

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Next: $\mathrm{Irr}(A_5) = ?$ and $X(A_5) = ?$

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In Exercise Sheet 5 (Ex. 1) we computed $X(A_4)$, hence we can now induce the irreducible of A_4 to A_5 in order to obtain elements of $\mathrm{Irr}(A_5) \setminus \{1_{A_5}\} =: \{\chi_2, \chi_3, \chi_4, \chi_5\}$:

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Set $H := A_4 \leq A_5 \implies |H| = 12$

Recall:

$\lvert[g_i]\rvert$	1	3	4	4
g_i	Id	(12)(34)	(123)	(132)
1_H	1	1	1	1
χ_2^H	1	1	ω	ω^2
χ_3^H	1	1	ω^2	ω
χ_4^H	3	-1	0	0

with $\omega :=$ primitive 3rd root of $1_{\mathbb{C}}$.

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$1_H\uparrow_H^G((12)(34)) = \frac{1}{12}\cdot 12 = 1$

$1_H\uparrow_H^G((123)) = \frac{1}{12}\cdot 24 = 2$

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conj. by the elements $x\in A_4$ yield an element in V_4 $\rightsquigarrow \varphi^\circ(x^{-1}gx)=1$, else $\varphi^\circ(x^{-1}gx)=0$

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Recall: $\mathrm{Ind}_H^G(\varphi)(g) = \frac{1}{|H|}\sum_{x\in G} \varphi^0(x^{-1}gx)$ where $\varphi^0(y) = \begin{cases} \varphi(y) & , y\in H \\ 0 & , y\notin H \end{cases}$

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"$1_H\uparrow_H^G = (5,1,2,0,0)$"

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$$<1_H\uparrow_H^G, 1_G>_G = <1_H, \underbrace{1_G\downarrow_H^G}_{=1_H}>_H = 1$$

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$$= \langle 1_H\uparrow_H^G, 1_H\uparrow_H^G\rangle_G - 2 + 1$$

$$= \frac{1}{|G|}\cdot\sum_{g\in G} 1_H\uparrow_H^G(g)\cdot 1_H\uparrow_H^G(g^{-1}) - 1$$

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$$= \frac{1}{60} \cdot (1 \cdot 5 \cdot 5 + 15 \cdot 1 \cdot 1 + 20 \cdot 2 \cdot 2) - 1 = 1$$

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$$= \frac{1}{60} \cdot (1\cdot 5\cdot 5 + 15\cdot 1\cdot 1 + 20\cdot 2\cdot 2) - 1 = 1$$

$\Rightarrow 1_H\uparrow_H^G - 1_G =: \chi_4$ is irreducible

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As $\chi_2(1), \chi_3(1), \chi_5(1) \mid |A_5| = 60$ $\Rightarrow$ $\chi_2(1), \chi_3(1), \chi_5(1) \in \{3,4,5,6\}$

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Degree formula: $\chi_2(1)^2 + \chi_3(1)^2 + \chi_5(1)^2 = |A_5| - \chi_1(1)^2 - \chi_4(1)^2$

$= 60 - 1 - 16 = 43$

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$$= 60 - 1 - 16 = 43$$

$\Rightarrow$ $\boxed{\chi_2(1) = 3, \ \chi_3(1) = 3, \ \chi_5(1) = 5}$ (! Possibility)

Hence

$$X(A_5) = \begin{array}{c|ccccc}
 & C_1 & C_2 & C_3 & C_4 & C_5 \\
\hline
|C_k| & 1 & 15 & 20 & 12 & 12 \\
|C_G(g_k)| & 60 & 4 & 3 & 5 & 5 \\
\hline
\chi_1 & 1 & 1 & 1 & 1 & 1 \\
\chi_2 & 3 & & & & \\
\chi_3 & 3 & & & & \\
\chi_4 & 4 & 0 & 1 & -1 & -1 \\
\chi_5 & 5 & & & &
\end{array}$$

Hence

$X(A_5) =$

	C_1	C_2	C_3	C_4	C_5
$\lvert C_k \rvert$	1	15	20	12	12
$\lvert C_G(g_k) \rvert$	60	4	3	5	5
χ_1	1	1	1	1	1
χ_2	3		0		
χ_3	3		0		
χ_4	4	0	1	-1	-1
χ_5	5			0	0

Zeros follow from Cor. 17.7 as

$$\gcd(\chi_2(1), |C_3|) = \gcd(\chi_3(1), |C_3|) = \gcd(\chi_5(1), |C_4|) = \gcd(\chi_5(1), |C_5|) = 1$$

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	C_1	C_2	C_3	C_4	C_5
$\lvert C_k \rvert$	1	15	20	12	12
$\lvert C_G(g_k) \rvert$	60	4	3	5	5
χ_1	1	1	1	1	1
χ_2	3		0		
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χ_5	5	1	-1	0	0

- 2[nd] Orth. Rel. 1st & 3rd col's: $0 = 1\cdot 1 + 4\cdot 1 + 5\cdot \chi_5(g_3) \Rightarrow \chi_5(g_3) = -1$
- 1[st] Orth. Rel: 1st & 5[th] rows: $0 = 1\cdot 5 + 1\cdot \chi_5(g_2) + 1\cdot(-1) + 1\cdot 0 + 1\cdot 0 \Rightarrow \chi_5(g_2) = 1$

Hence

$X(A_5) =$

	C_1	C_2	C_3	C_4	C_5
$\lvert C_k \rvert$	1	15	20	12	12
$\lvert C_G(g_k) \rvert$	60	4	3	5	5
χ_1	1	1	1	1	1
χ_2	3	-1	0	$-\varphi-\varphi^4$	$-\varphi^2-\varphi^3$
χ_3	3	-1	0	$-\varphi^2-\varphi^3$	$-\varphi-\varphi^4$
χ_4	4	0	1	-1	-1
χ_5	5	1	-1	0	0

$\left(\varphi := e^{2\pi i/5}\right)$

- Same method as for χ_4 yields: $\chi_2 = \psi\uparrow^{A_5}_{\langle(12345)\rangle} - \chi_4 - \chi_5$ with $\psi =$ "χ_3" in Ex. 4.
- Finally $\chi_3 := \overline{\chi_2}$