Exercise 17.5

Prove that, if $\chi \in \mathrm{Irr}(G)$, then $\chi(1) \mid |G : Z(\chi)|$. Deduce that $\chi(1) \mid |G : Z(G)|$.

This allows us to prove an important criterion, due to Burnside, for character values to be zero.

Theorem 17.6 (*Burnside*)

Let $\chi \in \mathrm{Irr}(G)$ and let $C = [g]$ be a conjugacy class of G such that $\gcd(\chi(1), |C|) = 1$. Then $\chi(g) = 0$ or $g \in Z(\chi)$.

Proof: As $\gcd(\chi(1), |C|) = 1$, there exist $u, v \in \mathbb{Z}$ such that $u\chi(1) + v|C| = 1$ Set $\alpha := \frac{\chi(g)}{\chi(1)}$. Then

$$\alpha = \frac{\chi(g)}{\chi(1)} \cdot 1 = \frac{\chi(g)}{\chi(1)}\big(u\chi(1) + v|C|\big) = u\chi(g) + v\frac{|C|\chi(g)}{\chi(1)} = u\chi(g) + v\omega_\chi(C)$$

is an algebraic integer because both $\chi(g)$ and $\omega_\chi(C)$ are. Now, set $m := |\langle g \rangle|$ and let $\zeta_m := e^{\frac{2\pi i}{m}}$. As $\chi(g)$ is a sum of m-th roots of unity, certainly $\chi(g) \in \mathbb{Q}(\zeta_m)$. Let $\mathcal{G}$ be the Galois group of the Galois extension $\mathbb{Q} \subseteq \mathbb{Q}(\zeta_m)$. Then for each field automorphism $\sigma \in \mathcal{G}$, $\sigma(\alpha)$ is also an algebraic integer because α and $\sigma(\alpha)$ are roots of the same monic integral polynomial. Hence $\beta := \prod_{\sigma\in\mathcal{G}} \sigma(\alpha)$ is also an algebraic integer and because $\sigma(\beta) = \beta$ for every $\sigma \in \mathcal{G}$, β is an element of the fixed field of $\mathcal{G}$, namely $\beta \in \mathbb{Q}$ (Galois theory). Therefore $\beta \in \mathbb{Z}$.
If $g \in Z(\chi)$, then there is nothing to do. Thus we may assume that $g \notin Z(\chi)$. Then $|\chi(g)| \neq \chi(1)$, so that by Property 7.5(c) we must have $|\chi(g)| < \chi(1)$ and hence $|\alpha| < 1$. Now, again by Property 7.5(b), $\chi(g) = \varepsilon_1 + \ldots + \varepsilon_n$ with $n = \chi(1)$ and $\varepsilon_1, \ldots, \varepsilon_n$ m-th roots of unity. Therefore, for each $\sigma \in \mathcal{G}\setminus\{\mathrm{Id}\}$, we have $\sigma(\chi(g)) = \sigma(\varepsilon_1) + \ldots + \sigma(\varepsilon_n)$ with $\sigma(\varepsilon_1), \ldots, \sigma(\varepsilon_n)$ m-th roots of unity, because $\varepsilon_1, \ldots, \varepsilon_n$ are. It follows that

$$|\sigma(\chi(g))| = |\sigma(\varepsilon_1) + \ldots + \sigma(\varepsilon_n)| \leqslant |\sigma(\varepsilon_1)| + \ldots + |\sigma(\varepsilon_n)| = n = \chi(1)$$

and hence

$$|\sigma(\alpha)| = \frac{1}{\chi(1)}|\sigma(\chi(g))| \leqslant \frac{\chi(1)}{\chi(1)} = 1\,.$$

Thus

$$|\beta| = |\prod_{\sigma\in\mathcal{G}} \sigma(\alpha)| = \underbrace{|\alpha|}_{<1} \cdot \prod_{\sigma\in\mathcal{G}\setminus\{\mathrm{Id}\}} \underbrace{|\sigma(\alpha)|}_{\leqslant 1} < 1\,.$$

The only way an integer satisfies this inequality is $\beta = 0$. Thus $\alpha = 0$ as well, which implies that $\chi(g) = 0$. ∎

Corollary 17.7

Assume now that G is a non-abelian simple group. In the situation of Theorem 17.6 if we assume moreover that $\chi(1) > 1$ and $C \neq \{1\}$, then it is always the case that $\chi(g) = 0$.

Proof: Assume $\chi(g) \neq 0$, then Theorem 17.6 implies that $g \in Z(\chi)$, so $Z(\chi) \neq 1$. As G is simple and $Z(\chi) \trianglelefteq G$ by Proposition 17.3(a), we have $Z(\chi) = G$. Moreover, the fact that G is simple also implies that $\ker(\chi) = 1$, as if it were G, then χ would be reducible. Thus, it follows from Proposition 17.3 that

$$G = Z(\chi)/\ker(\chi) = Z(G/\ker(\chi)) = Z(G) = 1\,,$$

where the last equality holds because G is simple non-abelian. This is contradiction. ∎

18 Burnside's $p^a q^b$-Theorem

Character theory has many possible applications to the to the structure of finite groups. We consider in this section on of the most famous of these: the proof of Burnside's $p^a q^b$ theorem.

Example 10

To begin with we consider two possible minor applications of character theory to finite groups. Both are results of the *Einfürung in die Algebra*, for which you have already seen purely group-theoretic proofs.

(a) G finite group such that $|G| = p^2$ for some prime number $p \implies G$ is abelian.

- **Proof using character theory**. By Corollary 16.7 we have $\chi(1) \mid |G|$ for each $\chi \in \mathrm{Irr}(G)$. Thus
$$\chi(1) \in \{1, p, p^2\}\,.$$
Therefore the degree formula reads
$$p^2 = |G| = \sum_{\chi \in \mathrm{Irr}(G)} \chi(1)^2 = \underbrace{\mathbf{1}_G(1)^2}_{=1} + \sum_{\substack{\chi \in \mathrm{Irr}(G) \\ \chi \neq \mathbf{1}_G}} \chi(1)^2\,,$$
which implies that it is not possible that the degree of an irreducible character of G is p or p^2. In other words, all the irreducible characters of G are linear, and thus G is abelian by Corollary 14.8.

(b) G is a non-trivial p-group $\implies G$ is soluble.

[Recall from the *Einfürung in die Algebra* that a finite group G is **soluble** if it admits a chain of subgroups
$$1 = G_0 < G_1 < \ldots < G_s = G$$
such that for $1 \leqslant i \leqslant s$, $G_{i-1} \lhd G_i$ and G_i/G_{i-1} is cyclic of prime order. Moreover, we have the following very useful *solubility criterion*, sometimes coined "the sandwich principle": if $H \trianglelefteq G$ is a normal subgroup, then the group G is soluble if and only if both G and G/H are soluble.]

- **Proof using character theory**. By induction on $|G| =: p^a$ ($a \in \mathbb{Z}_{>0}$). If $|G| = p$ or $|G| = p^2$, then G is abelian (cyclic in the former case). Finite abelian groups are clearly soluble because they are products of cyclic groups of prime power order.
Therefore, we may assume that $|G| \geqslant p^3$. As in (a) Corollary 16.7 implies that
$$\chi(1) \in \{1, p, p^2, \ldots, p^a\} \quad \text{for each } \chi \in \mathrm{Irr}(G)\,.$$
Now, again the degree formula yields
$$p^a = |G| = 1 + \sum_{\substack{\chi \in \mathrm{Irr}(G) \\ \chi \neq \mathbf{1}_G}} \chi(1)^2\,.$$
and for this equality to hold, there must be at least p linear characters of G (including the trivial character). Thus it follows from Corollary 14.8 that $G' \lneq G$. Hence both G' and G/G' are soluble by the induction hypothesis $\Rightarrow$ G is soluble by the *sandwich principle*.

Theorem 18.1 (*Burnside*)

Let G be a finite non-abelian simple group. If C is a conjugacy class of G such that $|C| = p^a$ with p prime and $a \in \mathbb{Z}_{\geqslant 0}$, then $C = \{1\}$.

Proof: Assume ab absurdo that $C \neq \{1\}$ and choose $g \in C$. In particular $g \neq 1$. Since G is non-abelian simple $G = G'$ and it follows from Corollary 14.8 that the unique linear character of G is the trivial character. Hence, for each $\chi \in \mathrm{Irr}(G) \setminus \{\mathbf{1}_G\}$, either $p \mid \chi(1)$ or $1 = \gcd(\chi(1), p) = \gcd(\chi(1), |C|)$, and in the case in which $p \nmid \chi(1)$, then $\chi(g) = 0$ by Corollary 17.7. Therefore, the Second Orthogonality Relations read

$$0 = 1 + \sum_{\substack{\chi \in \mathrm{Irr}(G) \\ \chi \neq \mathbf{1}_G}} \underbrace{\chi(g)}_{\substack{=0 \text{ if} \\ p \nmid \chi(1)}} \underbrace{\overline{\chi(1)}}_{=\chi(1)} = 1 + \sum_{\substack{\chi \in \mathrm{Irr}(G) \\ p \mid \chi(1)}} \chi(g)\chi(1)$$

and dividing by p yields

$$\underbrace{\sum_{\substack{\chi \in \mathrm{Irr}(G) \\ p \mid \chi(1)}} \underbrace{\frac{\chi(1)}{p}}_{\in \mathbb{Z}} \underbrace{\chi(g)}_{\substack{\text{algebraic} \\ \text{integer}}}}_{\text{algebraic integer}} = -\frac{1}{p} \in \mathbb{Q} \setminus \mathbb{Z}\,.$$

This contradicts the fact that rational numbers which are algebraic integers are integers. It follows that $g = 1$ is the only possibility and hence $C = \{1\}$. ∎

As a consequence, we obtain Burnside's $p^a q^b$ theorem, which can be found in the literature under two different forms. The first version provides us with a "non-simplicity" criterion and the second version with a solubility criterion, which is extremely hard to prove by purely group theoretic methods.

Theorem 18.2 (*Burnside's $p^a q^b$ Theorem, "simple" version*)

Let p, q be prime numbers and let $a, b \in \mathbb{Z}_{\geqslant 0}$ be integers such that $a + b \geqslant 2$. If G is a finite group of order $p^a q^b$, then G is not simple.

Proof: First assume that $a = 0$ or $b = 0$. Then G is a q-group with $q^2 \mid |G|$, resp. a p-group with $p^2 \mid |G|$. Therefore the centre of G is non-trivial (*Einfürung in die Algebra*), thus of non-trivial prime power order. Therefore there exists an element $g \in Z(G)$ of order q (resp. p) and $1 \neq \langle g \rangle \lhd G$ is a proper non-trivial normal subgroup. Hence G is not simple.

We may now assume that $a \neq 0 \neq b$. Let $Q \in \mathrm{Syl}_q(G)$ be a Sylow q-subgroup of G (i.e. $|Q| = q^b$). Again, as Q is a q-group, we have $Z(Q) \neq \{1\}$ and we can choose $g \in Z(Q) \setminus \{1\}$. Then

$$Q \leqslant C_G(g)$$

and therefore the Orbit-Stabiliser Theorem yields

$$|[g]| = |G : C_G(g)| = p^r$$

for some non-negative integer $r \leqslant a$. If $r = 0$, then $p^r = 1$ and $G = C_G(g)$, so that $g \in Z(G)$. Hence $Z(G) \neq \{1\}$ and G is not simple by the same argument as above. If $p^r > 1$, then G cannot be simple by Theorem 18.1. ∎

Theorem 18.3 (*Burnside's $p^a q^b$ Theorem, "soluble" version*)

Let p, q be prime numbers and $a, b \in \mathbb{Z}_{\geqslant 0}$. Then any finite group of order $p^a q^b$ is soluble.

Proof: Let G be a finite group of order $p^a q^b$. We proceed by induction on $a + b$.

- $a + b \in \{0, 1\} \implies G$ is either trivial or cyclic of prime order, hence clearly soluble.

- $a + b \geqslant 2 \implies G$ is not simple by the "simple" version of Burnside's $p^a q^b$ theorem. Hence there exists a proper non-trivial normal subgroup H in G and both $|H|, |G/H| < p^a q^b$. Therefore both H and G/H are soluble by the induction hypothesis. Thus G is soluble by the sandwich principle. ■

Chapter 6. Induction and Restriction of Characters

In this chapter we present important methods to construct / relate characters of a group, given characters of subgroups or overgroups. The main idea is that we would like to be able to use the character tables of groups we know already in order to compute the character tables of subgroups or overgroups of these groups.

Notation: throughout this chapter, unless otherwise specified, we let:

- G denote a finite group, $H \leqslant G$ and $N \trianglelefteq G$, $i_H : H \longrightarrow G, h \mapsto h$ is the canonical inclusion of H in G and $\pi_N : G \longrightarrow G/N, g \mapsto gN$ is the quotient morphism;
- $K := \mathbb{C}$ be the field of complex numbers;
- $\mathrm{Irr}(G) := \{\chi_1, \ldots, \chi_r\}$ denote the set of pairwise distinct irreducible characters of G;
- $C_1 = [g_1], \ldots, C_r = [g_r]$ denote the conjugacy classes of G, where $g_1, \ldots, g_r$ is a fixed set of representatives; and
- we use the convention that $\chi_1 = \mathbf{1}_G$ and $g_1 = 1 \in G$.

In general, unless otherwise stated, all groups considered are assumed to be finite and all $\mathbb{C}$-vector spaces / modules over the group algebra considered are assumed to be finite-dimensional.

19 Induction and Restriction

We aim at *inducing* and *restricting* characters from subgroups, resp. overgroups. We start with the operation of induction, which is a subtle operation to construct a class function on G from a given class function on a subgroup $H \leqslant G$. We will focus on characters in a second step.

Definition 19.1 (*Induced class function*)

Let $H \leqslant G$ and $\varphi \in \mathcal{C}l(H)$ be a class function on H. Then the **induction of φ from H to G** is

$$\mathrm{Ind}_H^G(\varphi) =: \varphi\uparrow_H^G \; : \quad \begin{array}{ccl} G & \longrightarrow & \mathbb{C} \\ g & \mapsto & \varphi\uparrow_H^G(g) := \frac{1}{|H|}\sum_{x \in G} \varphi^\circ(xgx^{-1}) \,, \end{array}$$

where for $y \in G$, $\varphi^\circ(y) := \begin{cases} \varphi(y) & \text{if } y \in H, \\ 0 & \text{if } y \notin H. \end{cases}$

Remark 19.2

With the notation of Definition 19.1 the following holds:

(a)
$$\varphi\uparrow_H^G(g) = \frac{1}{|H|}\sum_{x\in G}\varphi^\circ(xgx^{-1}) = \frac{1}{|H|}\sum_{\substack{x\in G\\ xgx^{-1}\in H}}\varphi(xgx^{-1})\,;$$

(b) the function $\varphi\uparrow_H^G$ is a class function on G, because for every $g, y \in G$,
$$\varphi\uparrow_H^G(ygy^{-1}) = \frac{1}{|H|}\sum_{x\in G}\varphi^\circ(xygy^{-1}x^{-1}) \overset{s:=yx}{=} \frac{1}{|H|}\sum_{s\in G}\varphi^\circ(sgs^{-1}) = \varphi\uparrow_H^G(g)\,.$$

In contrast, the operation of restriction is based on the more elementary idea that any map can be restricted to a subset of its domain. For class functions / representations / characters we are essentially interested in restricting these (seen as maps) to subgroups.

Definition 19.3 (*Restricted class function*)

Let $H \leqslant G$ and $\psi \in \mathcal{C}l(G)$ be a class function on G. Then the **restriction of** ψ **from** G **to** H is
$$\mathrm{Res}_H^G(\psi) := \psi\downarrow_H^G := \psi|_H = \psi\circ i_H\,.$$
This is obviously again a class function on H.

Remark 19.4

If ψ is a character of G afforded by the $\mathbb{C}$-representation $\rho : G \longrightarrow \mathrm{GL}(V)$, then clearly $\psi\downarrow_H^G$ is the character afforded by the $\mathbb{C}$-representation $\mathrm{Res}_H^G(\rho) := \rho\downarrow_H^G := \rho|_H = \rho\circ i_H : H \longrightarrow \mathrm{GL}(V)$. See Exercise 9.10(i).

Exercise 19.5

Let $H \leqslant J \leqslant G$and let $g \leqslant G$. Prove the following assertions:

(a) $\varphi \in \mathcal{C}l(H) \implies \varphi\uparrow_H^G(g) = \sum_{\substack{Hx\in H\backslash G\\ Hx=Hxg}}\varphi(xgx^{-1})\,;$

(b) $\varphi \in \mathcal{C}l(H) \implies (\varphi\uparrow_H^J)\uparrow_J^G = \varphi\uparrow_H^G$ (transitivity of induction);

(c) $\psi \in \mathcal{C}l(G) \implies (\psi\downarrow_J^G)\downarrow_H^J = \psi\downarrow_H^G$ (transitivity of restriction);

(d) the maps
$$\mathrm{Ind}_H^G : \mathcal{C}l(H) \longrightarrow \mathcal{C}l(G), \varphi \mapsto \varphi\uparrow_H^G \quad \text{and} \quad \mathrm{Res}_H^G : \mathcal{C}l(G) \longrightarrow \mathcal{C}l(H), \psi \mapsto \psi\downarrow_H^G$$
are $\mathbb{C}$-linear;

(e) $\varphi \in \mathcal{C}l(H)$ and $\psi \in \mathcal{C}l(G) \implies \psi\cdot\varphi\uparrow_H^G = \left(\psi\downarrow_H^G\cdot\varphi\right)\uparrow_H^G$ (Frobenius formula).