- Applying the Orthogonality Relations yields:
  1st, 3rd column $\Rightarrow \chi_5(g_3) = -1$ and the scalar product $\langle \chi_1, \chi_5 \rangle_G = 0 \Rightarrow \chi_5(g_2) = 1$.
- Finally, to fill out the remaining gaps, we can induce from the cyclic subgroup $Z_5 := \langle (1\ 2\ 3\ 4\ 5) \rangle \leqslant A_5$. Indeed, choosing the non-trivial irreducible character $\psi$ of $Z_5$ which was denoted "$\chi_3$" in Example 4 gives

  $$\psi\uparrow_{Z_5}^{G} = (12, 0, 0, \zeta^2 + \zeta^3, \zeta + \zeta^4)$$

  where $\zeta = \exp(2\pi \mathbf{i}/5)$ is a primitive 5-th root of unity. Then we compute that

  $$\langle \psi\uparrow_{Z_5}^{G}, \chi_4 \rangle_G = 1 = \langle \psi\uparrow_{Z_5}^{G}, \chi_5 \rangle_G \quad \Longrightarrow \quad \psi\uparrow_{Z_5}^{G} - \chi_4 - \chi_5 = (3, -1, 0, -\zeta - \zeta^4, -\zeta^2 - \zeta^3)$$

  and this character must be irreducible, because it is not the sum of 3 copies of the trivial character. Hence we set $\chi_2 := \psi\uparrow_{Z_5}^{G} - \chi_4 - \chi_5$ and the values of $\chi_3$ then easily follow from the 2nd Othogonality Relations:

| | $C_1$ | $C_2$ | $C_3$ | $C_4$ | $C_5$ |
|---|---|---|---|---|---|
| $\lvert C_k \rvert$ | 1 | 15 | 20 | 12 | 12 |
| $\lvert C_G(g_k) \rvert$ | 60 | 4 | 3 | 5 | 5 |
| $\chi_1$ | 1 | 1 | 1 | 1 | 1 |
| $\chi_2$ | 3 | $-1$ | 0 | $-\zeta - \zeta^4$ | $-\zeta^2 - \zeta^3$ |
| $\chi_3$ | 3 | $-1$ | 0 | $-\zeta^2 - \zeta^3$ | $-\zeta - \zeta^4$ |
| $\chi_4$ | 4 | 0 | 1 | $-1$ | $-1$ |
| $\chi_5$ | 5 | 1 | $-1$ | 0 | 0 |

**Remark 19.9 (*Induction of $\mathbb{C}H$-modules*)**

If you have attended the lecture *Commutative Algebra* you have studied the *tensor product of modules*. In the M.Sc. lecture *Representation Theory* you will see that induction of modules is defined through a tensor product, extending the scalars from $\mathbb{C}H$ to $\mathbb{C}G$. More precisely, if $M$ is a $\mathbb{C}H$-module, then the induction of $M$ from $H$ to $G$ is defined to be $\mathbb{C}G \otimes_{\mathbb{C}H} M$. Moreover, if $M$ affords the character $\chi$, then $\mathbb{C}G \otimes_{\mathbb{C}H} M$ affords the character $\chi\uparrow_H^G$.

# 20 Clifford Theory

Clifford theory is a generic term for a series of results relating the representation / character theory of a given group $G$ to that of a normal subgroup $N \trianglelefteq G$ through induction and restriction.

**Notation 20.1**

If $H \leqslant G$ and $x \in G$, then we let

$$\begin{array}{rccc} c_x: & H & \longrightarrow & xHx^{-1} \\ & h & \mapsto & xhx^{-1} \end{array}$$

denote the conjugation homomorphism by $x$.

**Definition 20.2 (*Conjugate class function / inertia group*)**

Let $H \leqslant G$, let $\varphi \in \mathcal{C}l(H)$, let $g \in G$ and let $c_{g^{-1}} : gHg^{-1} \longrightarrow H$ denote the conjugation homomorphism by $g^{-1}$. We define:

(a) the **conjugate class function to** $\varphi$ **by** $g$ to be ${}^g\varphi := \varphi \circ c_{g^{-1}} \in \mathcal{C}l(gHg^{-1})$, i.e. the class function on $gHg^{-1}$ given by

$$ {}^g\varphi : gHg^{-1} \longrightarrow \mathbb{C}, x \mapsto \varphi(g^{-1}xg)\,; \textit{and} $$

(b) the **inertia group** of $\varphi$ in $G$ to be $\mathcal{I}_G(\varphi) := \{g \in G \mid {}^g\varphi = \varphi\}$.

**Exercise 20.3**

Let $g, h \in G$. With the notation of Definition 20.2, prove that:

(a) ${}^g\varphi$ is indeed a class function on $gHg^{-1}$;

(b) $\mathcal{I}_G(\varphi) \leqslant G$ and $H \leqslant \mathcal{I}_G(\varphi) \leqslant N_G(H)$;

(c) ${}^g\varphi = {}^h\varphi \;\Leftrightarrow\; h^{-1}g \in \mathcal{I}_G(\varphi) \;\Leftrightarrow\; g\mathcal{I}_G(\varphi) = h\mathcal{I}_G(\varphi)$;

(d) if $\rho : H \longrightarrow \mathrm{GL}(V)$ is a $\mathbb{C}$-representation of $H$ with character $\chi$, then

$$ {}^g\rho := \circ c_{g^{-1}} : gHg^{-1} \longrightarrow \mathrm{GL}(V), x \mapsto \rho(g^{-1}xg) $$

is $\mathbb{C}$-representation of $gHg^{-1}$ with character ${}^g\chi = \chi \circ c_{g^{-1}}$ and ${}^g\chi(1) = \chi(1)$;

(e) if $J \leqslant H$ then ${}^g(\varphi\downarrow^H_J) = ({}^g\varphi)\downarrow^{gHg^{-1}}_{gJg^{-1}}$.

**Exercise 20.4 (*Mackey Formula*)**

Let $H, L \leqslant G$ and let $\varphi \in \mathcal{C}l(H)$. Prove that

$$ (\varphi\uparrow^G_H)\downarrow^G_L = \sum_{LgH \in L\backslash G/H} ({}^g\varphi)\downarrow^{gHg^{-1}}_{gHg^{-1}\cap L}\uparrow^L_{gHg^{-1}\cap L} . $$

**Exercise 20.5**

Deduce from the Mackey formula that if $N \trianglelefteq G$, and $\psi \in \mathrm{Irr}(N)$, then

$$ \langle \psi\uparrow^G_N, \psi\uparrow^G_N \rangle_G = \sum_{xN \in G/N} \langle \psi, {}^x\psi \rangle_N . $$

**Lemma 20.6**

(a) If $H \leqslant G$, $\varphi, \psi \in \mathcal{C}l(H)$ and $g \in G$, then $\langle {}^g\varphi, {}^g\psi \rangle_{gHg^{-1}} = \langle \varphi, \psi \rangle_H$.

(b) If $N \trianglelefteq G$ and $g \in G$, then we have $\psi \in \mathrm{Irr}(N) \;\Leftrightarrow\; {}^g\psi \in \mathrm{Irr}(N)$.

(c) If $N \trianglelefteq G$ and $\psi$ is a character of $N$, then $(\psi\uparrow_N^G)\downarrow_N^G = |\mathcal{I}_G(\psi) : N| \sum_{g\mathcal{I}_G(\psi)\in G/\mathcal{I}_G(\psi)} {}^g\psi$.

**Proof:** (a) Clearly

$$\begin{aligned} \langle {}^g\varphi, {}^g\psi \rangle_{gHg^{-1}} &= \frac{1}{|gHg^{-1}|} \sum_{x\in gHg^{-1}} {}^g\varphi(x)\,\overline{{}^g\psi(x)} \\ &= \frac{1}{|H|} \sum_{x\in gHg^{-1}} \varphi(g^{-1}xg)\overline{\psi(g^{-1}xg)} \\ &\overset{y := g^{-1}xg}{=} \frac{1}{|H|} \sum_{y\in H} \varphi(y)\overline{\psi(y)} = \langle \varphi, \psi \rangle_H . \end{aligned}$$

(b) As $N \trianglelefteq G$, $gNg^{-1} = N$. Thus, if $\psi \in \mathrm{Irr}(N)$, then on the one hand ${}^g\psi$ is also a character of $N$ by Exercise 20.3(d), and on the other hand it follows from (a) that $\langle {}^g\psi, {}^g\psi \rangle_N = \langle \psi, \psi \rangle_N = 1$. Hence ${}^g\psi$ is an irreducible character of $N$. Therefore, if ${}^g\psi \in \mathrm{Irr}(N)$, then $\psi = {}^{g^{-1}}({}^g\psi) \in \mathrm{Irr}(N)$, as required.

(c) If $n \in N$ then so does $g^{-1}ng\ \forall\ g \in G$, hence

$$\psi\uparrow_N^G\downarrow_N^G(n) = \psi\uparrow_N^G(n) = \frac{1}{|N|}\sum_{g\in G}\psi(g^{-1}ng) = \frac{1}{|N|}\sum_{g\in G}{}^g\psi(n) = \frac{|\mathcal{I}_G(\psi)|}{|N|}\sum_{g\in[G/\mathcal{I}_G(\psi)]}{}^g\psi(n).$$

■

## Notation 20.7

Given $N \trianglelefteq G$ and $\psi \in \mathrm{Irr}(N)$, we set $\mathrm{Irr}(G \mid \psi) := \{\chi \in \mathrm{Irr}(G) \mid \langle \chi\downarrow_N^G, \psi \rangle_N \neq 0\}$.

## Theorem 20.8 (Clifford Theory)

Let $N \trianglelefteq G$. Let $\chi \in \mathrm{Irr}(G)$, $\psi \in \mathrm{Irr}(N)$ and set $\mathcal{I} := \mathcal{I}_G(\psi)$. Then the following assertions hold.

(a) If $\psi$ is a constituent of $\chi\downarrow_N^G$, then

$$\chi\downarrow_N^G = e\left(\sum_{g\mathcal{I}_G(\psi)\in G/\mathcal{I}_G(\psi)} {}^g\psi\right),$$

where $e = \langle \chi\downarrow_N^G, \psi \rangle_N = \langle \chi, \psi\uparrow_N^G \rangle_G \in \mathbb{Z}_{>0}$ is called the **ramification index** of $\chi$ in $N$ (or of $\psi$ in $G$). In particular, all the constituents of $\chi\downarrow_N^G$ have the same degree.

(b) Induction from $\mathcal{I} = \mathcal{I}_G(\psi)$ to $G$ induces a bijection

$$\begin{aligned} \mathrm{Ind}_{\mathcal{I}}^G : \quad \mathrm{Irr}(\mathcal{I} \mid \psi) &\longrightarrow \mathrm{Irr}(G \mid \psi) \\ \eta &\mapsto \eta\uparrow_{\mathcal{I}}^G \end{aligned}$$

preserving ramification indices, i.e. $\langle \eta\downarrow_N^{\mathcal{I}}, \psi \rangle_N = \langle \eta\uparrow_{\mathcal{I}}^G\downarrow_N^{\mathcal{I}}, \psi \rangle_N = e$.

**Proof:** (a) By Frobenius reciprocity, $\langle \chi, \psi\uparrow_N^G \rangle_G = \langle \chi\downarrow_N^G, \psi \rangle_N \neq 0$. Thus $\chi$ is a constituent of $\psi\uparrow_N^G$ and therefore $\chi\downarrow_N^G$ is a constituent of $\psi\uparrow_N^G\downarrow_N^G$.
Now, if $\eta \in \mathrm{Irr}(N)$ is an arbitrary constituent of $\chi\downarrow_N^G$ (i.e. $\langle \chi\downarrow_N^G, \eta \rangle_N \neq 0$) then by the above, we have

$$\langle \psi\uparrow_N^G\downarrow_N^G, \eta \rangle_N \geqslant \langle \chi\downarrow_N^G, \eta \rangle_N > 0 .$$

Moroever, by Lemma 20.6(c) the constituents of $\psi\uparrow_N^G\downarrow_N^G$ are preciely $\{{}^g\psi \mid g \in [G/\mathcal{I}_G(\psi)]\}$. Hence $\eta$ is $G$-conjugate to $\psi$. Furthermore, for every $g \in G$ we have

$$\begin{aligned}\langle\chi\downarrow_N^G, {}^g\psi\rangle_N = \frac{1}{|N|}\sum_{h\in N}\chi(h){}^g\psi(h^{-1}) &= \frac{1}{|N|}\sum_{h\in N}\chi(h)\psi(g^{-1}h^{-1}g) \\ &\overset{\chi\in\mathcal{Cl}(G)}{=} \frac{1}{|N|}\sum_{h\in N}\chi(g^{-1}hg)\psi(g^{-1}h^{-1}g) \\ &\overset{s:=g^{-1}hg\in N}{=} \frac{1}{|N|}\sum_{s\in N}\chi(s)\psi(s^{-1}) = \langle\chi\downarrow_N^G, \psi\rangle_N = e\,.\end{aligned}$$

Therefore, every $G$-conjugate ${}^g\psi$ ($g \in [G/\mathcal{I}_G(\psi)]$) of $\psi$ occurs as a constituent of $\chi\downarrow_N^G$ with the same multiplicity $e$. The claim about the degrees is then clear as ${}^g\psi(1) = \psi(1)\ \forall g \in G$.

(b) <u>**Claim 1:** $\eta \in \mathrm{Irr}(\mathcal{I} \mid \psi) \;\Rightarrow\; \eta\uparrow_{\mathcal{I}}^G \in \mathrm{Irr}(G|\psi)$.</u>
Since $\mathcal{I} = \mathcal{I}_{\mathcal{I}}(\psi)$, (a) implies that $\eta\downarrow_N^{\mathcal{I}} = e'\psi$ with $e' = \langle\eta\downarrow_N^{\mathcal{I}}, \psi\rangle_N = \frac{\eta(1)}{\psi(1)} > 0$. Now, let $\chi \in \mathrm{Irr}(G)$ be a constituent of $\eta\uparrow_{\mathcal{I}}^G$. By Frobenius Reciprocity we have

$$0 \neq \langle\chi, \eta\uparrow_{\mathcal{I}}^G\rangle_G = \langle\chi\downarrow_{\mathcal{I}}^G, \eta\rangle_{\mathcal{I}}\,.$$

It follows that $\eta\downarrow_N^{\mathcal{I}}$ is a constituent of $\chi\downarrow_{\mathcal{I}}^G\downarrow_N^{\mathcal{I}}$ and

$$e := \langle\chi\downarrow_N^G, \psi\rangle_N = \langle\chi\downarrow_{\mathcal{I}}^G\downarrow_N^{\mathcal{I}}, \psi\rangle_N \geqslant \langle\eta\downarrow_N^{\mathcal{I}}, \psi\rangle_N = e' > 0\,,$$

hence $\chi \in \mathrm{Irr}(G|\psi)$. Moreover, by (a) we have $e = \langle\chi\downarrow_N^G, {}^g\psi\rangle_N \geqslant e'$ for each $g \in G$. Therefore,

$$\chi(1) = e\sum_{g\in[G/\mathcal{I}]}{}^g\psi(1) \overset{(a)}{=} e|G:\mathcal{I}|\psi(1) \geqslant e'|G:\mathcal{I}|\psi(1) = |G:\mathcal{I}|\eta(1) = \eta\uparrow_{\mathcal{I}}^G(1) \geqslant \chi(1)\,.$$

Thus $e = e'$, $\eta\uparrow_{\mathcal{I}}^G = \chi \in \mathrm{Irr}(G)$, and therefore $\eta\uparrow_{\mathcal{I}}^G \in \mathrm{Irr}(G|\psi)$.

<u>**Claim 2:** $\chi \in \mathrm{Irr}(G \mid \psi) \;\Rightarrow\; \exists!\,\eta \in \mathrm{Irr}(\mathcal{I} \mid \psi)$ such that $\langle\chi\downarrow_{\mathcal{I}}^G, \eta\rangle_{\mathcal{I}} \neq 0$.</u>
Again by (a), as $\chi \in \mathrm{Irr}(G \mid \psi)$, we have $\chi\downarrow_N^G = e\sum_{g\in[G/\mathcal{I}]}{}^g\psi$, where $e = \langle\chi\downarrow_N^G, \psi\rangle_N \in \mathbb{Z}_{>0}$. Therefore, there exists $\eta \in \mathrm{Irr}(\mathcal{I})$ such that

$$\langle\chi\downarrow_{\mathcal{I}}^G, \eta\rangle_{\mathcal{I}} \neq 0 \neq \langle\eta\downarrow_N^{\mathcal{I}}, \psi\rangle_N$$

because $\chi\downarrow_N^G = \chi\downarrow_{\mathcal{I}}^G\downarrow_N^{\mathcal{I}}$, so in particular $\eta \in \mathrm{Irr}(\mathcal{I} \mid \psi)$. Hence existence holds and it remains to see that uniqueness holds. Again by Frobenius reciprocity we have $0 \neq \langle\chi, \eta\uparrow_{\mathcal{I}}^G\rangle_G$. By Claim 1 this forces $\chi = \eta\uparrow_{\mathcal{I}}^G$ and $\eta\downarrow_N^{\mathcal{I}} = e\psi$, so $e$ is also the ramification index of $\psi$ in $\mathcal{I}$.
Now, write $\chi\downarrow_{\mathcal{I}}^G = \sum_{\lambda\in\mathrm{Irr}(\mathcal{I})} a_\lambda\lambda = \sum_{\lambda\neq\eta} a_\lambda\lambda + a_\eta\eta$ with $a_\lambda \geqslant 0$ for each $\lambda \in \mathrm{Irr}(\mathcal{I})$ and $a_\eta > 0$. It follows that

$$(a_\eta - 1)\eta\downarrow_N^{\mathcal{I}} + \sum_{\lambda\neq\eta} a_\lambda\lambda\downarrow_N^{\mathcal{I}} = \underbrace{\chi\downarrow_N^G}_{=e\sum_{g\in[G/\mathcal{I}]}{}^g\psi} - \underbrace{\eta\downarrow_N^{\mathcal{I}}}_{=e\psi} = e\sum_{g\in[G/\mathcal{I}]\setminus[1]}{}^g\psi.$$

Since $\psi$ does not occur in this sum, but occurs in $\eta\downarrow_N^{\mathcal{I}}$, the only possibility is $a_\eta = 1$ and $\lambda \notin \mathrm{Irr}(\mathcal{I}|\psi)$ for $\lambda \neq \eta$. Thus $\eta$ is uniquely determined as the only constituent of $\chi\downarrow_{\mathcal{I}}^G$ in $\mathrm{Irr}(\mathcal{I} \mid \psi)$.

Finally, Claims 1 and 2 prove that $\mathrm{Ind}_{\mathcal{I}}^G : \mathrm{Irr}(\mathcal{I} \mid \psi) \longrightarrow \mathrm{Irr}(G \mid \psi), \eta \mapsto \eta\uparrow_{\mathcal{I}}^G$ is well-defined and bijective, and the proof of Claim 2 shows that the ramification indices are preserved. ■

**Example 13 (*Normal subgroups of index 2*)**

Let $N < G$ be a subgroup of index $|G : N| = 2$ ($\Rightarrow N \lhd G$) and let $\chi \in \mathrm{Irr}(G)$, then either

(1) $\chi\downarrow_N^G \in \mathrm{Irr}(N)$, or

(2) $\chi\downarrow_N^G = \psi + {}^g\psi$ for a $\psi \in \mathrm{Irr}(N)$ and a $g \in G\backslash N$.

Indeed, let $\psi \in \mathrm{Irr}(N)$ be a constituent of $\chi\downarrow_N^G$. Since $|G : N| = 2$, we have $\mathcal{I}_G(\psi) \in \{N, G\}$. Theorem 20.8 yields the following:

- If $\mathcal{I}_G(\psi) = N$ then $\mathrm{Irr}(\mathcal{I}_G(\psi) \mid \psi) = \{\psi\}$ and $\psi\uparrow_N^G = \chi$, so that $e = 1$ and we get $\chi\downarrow_N^G = \psi + {}^g\psi$ for any $g \in G\backslash N$.
- If $\mathcal{I}_G(\psi) = G$ then $G/\mathcal{I}_G(\psi) = \{1\}$, so that

$$\chi\downarrow_N^G = e\psi \qquad \text{with } e = \langle \chi\downarrow_N^G, \psi\rangle_N = \langle \chi, \psi\uparrow_N^G\rangle_G\,.$$

Moroever, by Lemma 20.6(c),

$$\psi\uparrow_N^G\downarrow_N^G = |\mathcal{I}_G(\psi) : N| \sum_{g \in G/\mathcal{I}_G(\psi)} {}^g\psi = 2\psi\,.$$

Hence

$$2\psi(1) = \psi\uparrow_N^G\downarrow_N^G(1) \geqslant \chi\downarrow_N^G(1) = \chi(1) = e\psi(1) \quad \Rightarrow \quad e \leqslant 2\,.$$

Were $e = 2$ then we would have $2\psi(1) = \psi\uparrow_N^G(1)$, hence $\chi = \psi\uparrow_N^G$ and thus

$$1 = \langle \chi, \psi\uparrow_N^G\rangle_G = \langle \chi\downarrow_N^G, \psi\rangle_N = e = 2$$

a contradiction. Whence $e = 1$, which implies that $\chi\downarrow_N^G \in \mathrm{Irr}(N)$. Moreover, $\psi\uparrow_N^G = \chi + \chi'$ for some $\chi' \in \mathrm{Irr}(G)$ such that $\chi' \neq \chi$.

Remember that we have proved that the degree of an irreducible character of a finite group $G$ divides the index of the centre $|G : Z(G)|$. The following consequence of Clifford's theorem due to N. Itô provides us with yet a stronger divisibility criterion.

**Theorem 20.9 (Itô)**

Let $A \leqslant G$ be an abelian subgroup of $G$ and let $\chi \in \mathrm{Irr}(G)$. Then the following assertions hold:

(a) $\chi(1) \leqslant |G : A|$; and

(b) if $A \trianglelefteq G$, then $\chi(1) \,\big|\, |G : A|$.

**Proof:** (a) Exercise!

(b) Let $\psi \in \mathrm{Irr}(A)$ be a constituent of $\chi\downarrow_A^G$, so that in other words $\chi \in \mathrm{Irr}(G \mid \psi)$. By Theorem 20.8(b) there exists $\eta \in \mathrm{Irr}(\mathcal{I}_G(\psi) \mid \psi)$ such that $\chi = \eta\uparrow_{\mathcal{I}_G(\psi)}^G$ and $\eta\downarrow_A^{\mathcal{I}_G(\psi)} = e\psi$ (proof of Claim 2). Now, as $A$ is abelian, all the irreducible characters of $A$ have degree 1 and for each $x \in A$, $\psi(x)$ is an $o(x)$-th root of unity. Hence $\forall\, x \in A$ we have

$$|\eta(x)| = |\eta\downarrow_A^{\mathcal{I}_G(\psi)}(x)| = |e\psi(x)| = e|\psi(x)| = e \cdot 1 = e = \eta(1) \quad \Rightarrow \quad A \subseteq Z(\eta)\,.$$

Therefore, by Remark 17.5, we have

$$\eta(1) \,\Big|\, |\mathcal{I}_G(\psi) : Z(\eta)| \,\Big|\, |\mathcal{I}_G(\psi) : A|$$

and since $\chi = \eta\uparrow_{\mathcal{I}_G(\psi)}^G$ it follows that

$$\chi(1) = |G : \mathcal{I}_G(\psi)|\eta(1) \,\Big|\, |G : \mathcal{I}_G(\psi)| \cdot |\mathcal{I}_G(\psi) : A| = |G : A|\,.$$

■

## 21 The Theorem of Gallagher

In the context of Clifford theory (Theorem 20.8) we understand that irreducibility of characters is preserved by induction from $\mathcal{I}_G(\psi)$ to $G$. Thus we need to understand induction of characters from $N$ to $\mathcal{I}_G(\psi)$, in particular what if $G = \mathcal{I}_G(\psi)$. What can be said about $\mathrm{Irr}(G \mid \psi)$?

**Lemma 21.1**

Let $N \trianglelefteq G$ and let $\psi \in \mathrm{Irr}(N)$ such that $\mathcal{I}_G(\psi) = G$. Then

$$\psi\uparrow_N^G = \sum_{\chi \in \mathrm{Irr}(G)} e_\chi \chi$$

where $e_\chi := \langle \chi\downarrow_N^G, \psi\rangle_N$ is the ramification index of $\chi$ in $N$; in particular

$$\sum_{\chi \in \mathrm{Irr}(G)} e_\chi^2 = |G : N|\,.$$

**Proof:** Write $\psi\uparrow_N^G = \sum_{\chi \in \mathrm{Irr}(G)} a_\chi \chi$ with suitable $a_\chi = \langle \chi, \psi\uparrow_N^G\rangle_G$. By Frobenius reciprocity, $a_\chi \neq 0$ if and only if $\chi \in \mathrm{Irr}(G \mid \psi)$. But by Theorem 20.8: if $\chi \in \mathrm{Irr}(G|\psi)$, then $\chi\downarrow_N^G = e_\chi \psi$, so that

$$e_\chi = \langle \chi\downarrow_N^G, \psi\rangle_N = \langle \chi, \psi\uparrow_N^G\rangle_G = a_\chi\,.$$

Therefore,

$$|G : N|\psi(1) = \psi\uparrow_N^G(1) = \sum_{\chi \in \mathrm{Irr}(G)} a_\chi \chi(1) = \sum_{\chi \in \mathrm{Irr}(G)} e_\chi \chi(1) = \sum_{\chi \in \mathrm{Irr}(G)} e_\chi^2 \psi(1) = \psi(1) \sum_{\chi \in \mathrm{Irr}(G)} e_\chi^2$$

and it follows that $|G : N| = \sum_{\chi \in \mathrm{Irr}(G)} e_\chi^2$. ■

Therefore the multiplicities $\{e_\chi\}_{\chi \in \mathrm{Irr}(G)}$ behave like the irreducible character degrees of the factor group $G/N$. This is not a coincidence in many cases.

**Definition 21.2 (*Extension of a character*)**

Let $N \trianglelefteq G$ and $\chi \in \mathrm{Irr}(G)$ such that $\psi := \chi\downarrow_N^G$ is irreducible. Then we say that $\psi$ **extends to** $G$, and $\chi$ is an **extension of** $\psi$.

### Exercise 21.3

Let $N \trianglelefteq G$ and $\chi \in \mathrm{Irr}(G)$. Prove that

$$\chi\downarrow_N^G\uparrow_N^G = \mathrm{Inf}_{G/N}^G(\chi_{\mathrm{reg}}) \cdot \chi\,,$$

where $\chi_{\mathrm{reg}}$ is the regular character of $G/N$.

### Theorem 21.4 (Gallagher)

Let $N \trianglelefteq G$ and let $\chi \in \mathrm{Irr}(G)$ such that $\psi := \chi\downarrow_N^G \in \mathrm{Irr}(N)$. Then

$$\psi\uparrow_N^G = \sum_{\lambda\in\mathrm{Irr}(G/N)} \lambda(1)\ \mathrm{Inf}_{G/N}^G(\lambda)\cdot\chi,$$

where the characters $\{\mathrm{Inf}_{G/N}^G(\lambda)\cdot\chi \mid \lambda \in \mathrm{Irr}(G/N)\}$ of $G$ are pairwise distinct and irreducible.

**Proof:** By Exercise 21.3 we have $\psi\uparrow_N^G = \mathrm{Inf}_{G/N}^G(\chi_{\mathrm{reg}})\cdot\chi$, where $\chi_{\mathrm{reg}}$ denotes the regular character of $G/N$. Recall that by Theorem 10.3, $\chi_{\mathrm{reg}} = \sum_{\lambda\in\mathrm{Irr}(G/N)} \lambda(1)\,\lambda$, so that we have

$$\psi\uparrow_N^G = \sum_{\lambda\in\mathrm{Irr}(G/N)} \lambda(1)\,\mathrm{Inf}_{G/N}^G(\lambda)\cdot\chi\,.$$

Now, by Lemma 21.1, we have

$$\begin{aligned}|G:N| = \sum_{\chi\in\mathrm{Irr}(G)} e_\chi^2 = \langle\psi\uparrow_N^G, \psi\uparrow_N^G\rangle_G &= \sum_{\lambda,\mu\in\mathrm{Irr}(G/N)} \lambda(1)\mu(1)\langle\mathrm{Inf}_{G/N}^G(\lambda)\cdot\chi, \mathrm{Inf}_{G/N}^G(\mu)\cdot\chi\rangle_G \\ &\geqslant \sum_{\lambda\in\mathrm{Irr}(G/N)} \lambda(1)^2 = |G:N|\,.\end{aligned}$$

Hence equality holds throughout. This proves that

$$\langle\mathrm{Inf}_{G/N}^G(\lambda)\cdot\chi, \mathrm{Inf}_{G/N}^G(\mu)\cdot\chi\rangle = \delta_{\lambda\mu}.$$

By Erercise 13.4, $\mathrm{Inf}_{G/N}^G(\lambda)\cdot\chi$ are characters of $G$ and hence the characters $\{\mathrm{Inf}_{G/N}^G(\lambda)\cdot\chi \mid \lambda\in\mathrm{Irr}(G/N)\}$ are irreducible and pairwise distinct, as claimed. ■

Therefore, given $\psi \in \mathrm{Irr}(N)$ which extends to $\chi \in \mathrm{Irr}(G)$, we get $\mathrm{Inf}_{G/N}^G(\lambda)\cdot\chi$ ($\lambda \in \mathrm{Irr}(G/N)$) as further irreducible characters.

### Example 14

Let $N < G$ with $|G:N| = 2$ ($\Rightarrow N \trianglelefteq G$) and let $\psi \in \mathrm{Irr}(N)$. We saw:

- if $\mathcal{I}_G(\psi) = N$ then $\psi\uparrow_N^G \in \mathrm{Irr}(G)$;
- if $\mathcal{I}_G(\psi) = G$ then $\psi$ extends to some $\chi \in \mathrm{Irr}(G)$ and $\psi^G = \chi + \chi'$ with $\chi' \in \mathrm{Irr}(G)\setminus\{\chi\}$. It follows that $\chi' = \chi\cdot\mathrm{sign}$, where sign is the inflation of the sign character of $G/N \cong S_2$ to $G$.