Chapter 7. Frobenius Groups

In this chapter we show how to understand the irreducible characters of an important class of finite groups: the *Frobenius groups*. After Burnside's $p^a q^b$-Theorem this provides us with a second fundamental application of character theory to the structure theory of finite groups.

Notation: throughout this chapter, unless otherwise specified, we let:

- G denote a finite group in multiplicative notation with neutral element $1 := 1_G$;
- $K := \mathbb{C}$ be the field of complex numbers.

In general, unless otherwise stated, all groups considered are assumed to be finite and all $\mathbb{C}$-vector spaces / modules over the group algebra considered are assumed to be finite-dimensional.

22 Frobenius group / Frobenius complement / Frobenius kernel

Definition 22.1 (*Frobenius group / Frobenius complement*)

A finite group G admitting a non-trivial proper subgroup H such that

$$H \cap {}^{g}H = \{1\} \qquad \forall\, g \in G \backslash H$$

is called a **Frobenius group** with **Frobenius complement** H or a **Frobenius group with respect to** H.

Note: The definition implies immediately that $N_G(H) = H$. Also, a Frobenius complement need not be unique.

Example 15

Assume $P \in \mathrm{Syl}_p(G)$ is such that $|P| = p$ and $N_G(P) = P < G$. (In words: P is cyclic of order p and self-normalising!) Then, clearly, $P \cap {}^{g}P = \{1\}$ for any $g \in G \backslash P = G \backslash N_G(P)$, and so G is a Frobenius group with Frobenius complement P.

This yields immediately that the following well-known groups are Frobenius groups:

(1) the symmetric group $\mathfrak{S}_3$ is a Frobenius group with Frobenius complement $\langle(1\ 2)\rangle$;

(2) the dihedral group

$$D_{2(2m+1)} = \langle a, b \mid a^{2m+1} = b^2 = (ab)^2 = 1\rangle$$

of order $2(2m+1)$ with $m \in \mathbb{Z}_{\geqslant 1}$ is a Frobenius group with Frobenius complement $\langle b\rangle$;

(3) the alternating group $\mathfrak{A}_4$ is a Frobenius group with Frobenius complement $\langle(1\ 2\ 3)\rangle$.

Theorem 22.2 (Frobenius)

If G is a Frobenius group with Frobenius complement H, then there exists a normal subgroup $N \trianglelefteq G$ such that $G = HN$ and $H \cap N = \{1\}$. Moreover, such an N is uniquely determined, and it is called the **Frobenius kernel**.

We see below that the normal subgroup N is easily defined as a set and proved to be unique with the required properties; the crux of the difficulty lies in proving that it is a subgroup of G. This requires character theoretical arguments!

Proof: Define $N := \big(G \setminus \bigcup_{g\in G} {}^gH\big) \cup \{1\}$.

Claim 1: $H \cap N = \{1\}$ and $|N| = |G : H|$.

Indeed, from the definition of N we have $H \cap N = \{1\}$ and from the definition of Frobenius complement, $H = N_G(H)$, so there are exactly $|G : N_G(H)| = |G : H|$ distinct conjugates ${}^gH$ of H because if $g, x \in G$ then we have:

$${}^gH = {}^xH \Leftrightarrow x^{-1}g \in N_G(H) = H \Leftrightarrow gH = xH\,.$$

Moreover, these have only the identity element in common, because if $g, x \in G$ are such that ${}^gH \neq {}^xH$, then $x^{-1}g \notin N_G(H) = H$, so by the definition of the Frobenius complement,

$$\{1\} = {}^{x^{-1}g}H \cap H = {}^{x^{-1}}({}^gH \cap {}^xH)\,,$$

proving that ${}^gH \cap {}^xH = \{1_G\}$. It now follows that

$$\Big|\bigcup_{g\in G} {}^gH\Big| = |G : H| \cdot (|H| - 1) + 1\,.$$

Claim 2: if G contains a normal subgroup $\widetilde{N}$ such that $\widetilde{N}H = G$ and $\widetilde{N} \cap H = \{1_G\}$, then $\widetilde{N} = N$.

(*Be careful! At this stage, this does not mean that such an* $\widetilde{N}$ *exists!*)

Indeed, since $\widetilde{N} \cap H = \{1_G\}$ and $\widetilde{N} \trianglelefteq G$, certainly

$$\widetilde{N} \cap {}^gH = {}^g\widetilde{N} \cap {}^gH = {}^g(\widetilde{N} \cap H) = {}^g\{1\} = \{1\}$$

for any $g \in G$, whence $\widetilde{N} \subseteq N$. Moreover, $\widetilde{N}$ is such that $|\widetilde{N}| = |G : H| = |N|$, where the 2nd equality holds by Claim 1, proving that $\widetilde{N} = N$.

Claim 3: if $\theta \in \mathcal{C}l(H)$ is such that $\theta(1_H) = 0$, then $\theta\uparrow_H^G\downarrow_H^G = \theta$.

To begin with, the values of $\theta\uparrow_H^G\downarrow_H^G$ and θ at 1 coincide since by Corollary 19.8 we have $\theta\uparrow_H^G\downarrow_H^G(1_H) = |G : H| \cdot \theta(1) = 0$. Now, let $h \in H\setminus\{1\}$. Then, given $x \in G$, $\theta^\circ(xhx^{-1}) \neq 0$ only if ${}^xh \neq 1$ and ${}^xh \in H \cap {}^xH$, so $x \in H$. Moreover, as θ is a class function, we get

$$\theta\uparrow_H^G(h) = \frac{1}{|H|}\sum_{x\in G}\theta^\circ(xhx^{-1}) = \frac{1}{|H|}\sum_{x\in H}\theta(h) = \theta(h)\,,$$

as required.

Claim 4: $\mathrm{Ind}_H^G : \{\theta \in \mathcal{C}l(H) \mid \theta(1) = 0\} \longrightarrow \mathcal{C}l(G), \theta \mapsto \theta\uparrow_H^G$ is an isometry with respect to the scalar products $\langle -, - \rangle_H$ and $\langle -, - \rangle_G$.

Indeed, let $\theta, \eta \in \mathcal{C}l(H)$ be such that $\theta(1_H) = 0 = \eta(1)$. Then Frobenius Reciprocity and Claim 3 yield

$$\langle \theta\uparrow_H^G, \eta\uparrow_H^G \rangle_G = \langle \theta\uparrow_H^G\downarrow_H^G, \eta \rangle_H = \langle \theta, \eta \rangle_H\,,$$

as desired.

Claim 5: If $\eta \in \mathrm{Irr}(H)\backslash\{\mathbf{1}_H\}$ (exists as $H \neq \{1\}$) and $\theta := \eta - \eta(1)\mathbf{1}_H$, then $\eta^* := \theta\uparrow_H^G + \eta(1)\mathbf{1}_G$ is an irreducible character of G.

Clearly, $\theta \in \mathbb{Z}\mathrm{Irr}(H) \subseteq \mathcal{C}l(H)$, $\theta(1) = 0$, and $\eta^* \in \mathbb{Z}\mathrm{Irr}(G) \subseteq \mathcal{C}l(G)$ (see Remark 19.6). Now, on the one hand by Claim 4, we have

$$\langle \theta\uparrow_H^G, \theta\uparrow_H^G \rangle_G = \langle \theta, \theta \rangle_H = \langle \eta, \eta \rangle_H + \eta(1)^2\,.$$

On the other hand, by Frobenius reciprocity, $\langle \theta\uparrow_H^G, \mathbf{1}_G \rangle_G = \langle \theta, \mathbf{1}_H \rangle_H = -\eta(1)$, hence the above together with the fact that $\theta\uparrow_H^G$ is virtual character (by Remark 19.6) implies that

$$\begin{aligned}
\langle \eta^*, \eta^* \rangle_G &= \langle \theta\uparrow_H^G + \eta(1)\mathbf{1}_G, \theta\uparrow_H^G + \eta(1)\mathbf{1}_G \rangle_G \\
&= \langle \theta\uparrow_H^G, \theta\uparrow_H^G \rangle_G + 2\eta(1)\langle \theta\uparrow_H^G, \mathbf{1}_G \rangle_G + \eta(1)^2\langle \mathbf{1}_G, \mathbf{1}_G \rangle_G \\
&= \langle \eta, \eta \rangle_H + \eta(1)^2 + 2\eta(1) \cdot (-\eta(1)) + \eta(1)^2 \\
&= \langle \eta, \eta \rangle_H = 1
\end{aligned}$$

As η^* is a virtual character, it now follows that $\pm\eta^* \in \mathrm{Irr}(G)$. However,

$$\eta^*(1) = \theta\uparrow_H^G(1) + \eta(1)\mathbf{1}_G(1) = 0 + \eta(1) \cdot 1 = \eta(1) > 1\,,$$

whence $\eta^* \in \mathrm{Irr}(G)$.

The next claim eventually proves that N is a normal subgroup of G.

Claim 6: $N = \bigcap_{\eta \in \mathrm{Irr}(H)} \ker(\eta^*)$.

By Claim 5 ,

$$M := \bigcap_{\eta \in \mathrm{Irr}(H)} \ker(\eta^*)$$

defines a normal subgroup of G. First we claim that $M \leqslant N$. Observe that Claim 3 implies that for any $\eta \in \mathrm{Irr}(H)$,

$$\eta^*\downarrow_H^G = \theta\uparrow_H^G\downarrow_H^G + \eta(1)\mathbf{1}_G\downarrow_H^G = \theta + \eta(1)\mathbf{1}_H = \eta\,.$$

Thus, if $h \in M \cap H$, then for all $\eta \in \mathrm{Irr}(H)$, we have

$$\eta^*(1) = \eta^*(h) = \eta(h).$$

It follows that $\eta(h) = \eta(1)$ for all $\eta \in \mathrm{Irr}(H)$, and so

$$M \cap H \leqslant \bigcap_{\eta \in \mathrm{Irr}(H)} \ker(\eta) = \{1\}$$

where the last equality holds by Exercise 14.7. This proves that $M \cap H = \{1\}$, whence also $M \cap {}^xH = \{1\}$ for each $x \in G$ since $M \trianglelefteq H$. Therefore $M \leqslant N$, and it remains to prove that $N \leqslant M$. So, let $g \in N\backslash\{1\}$. Then, by the definition of N, for every $x \in G$, we have $g \notin {}^xH$. Hence, by definition of induced characters, $\theta\uparrow_H^G(g) = 0$ for each θ as defined in Claim 5, and so $\eta^*(g) = \eta^*(1)$ for each $\eta \in \mathrm{Irr}(H)\backslash\{\mathbf{1}_H\}$. It follows that $g \in \ker(\eta^*)$ for each $\eta \in \mathrm{Irr}(H)$, proving that $g \in M$. This proves Claim 6.

The statement of the theorem now follows from Claim 6, Claim 1 and Claim 2. ∎

Remark 22.3

(a) To use standard group theory terminology, the theorem says that the Frobenius kernel is a *normal complement of H in G* and that G is an *internal semi-direct product of N by H*.

(b) There is no known proof of Frobenius' theorem which does not make use of character theory.

(c) Thompson proved that the Frobenius kernel N of a Frobenius group is always a nilpotent group (i.e. N is the direct product of its Sylow subgroups).

Exercise 22.4

(a) Find two non-isomorphic finite groups which are Frobenius groups and not isomorphic to any of the Frobenius groups given in Example 15.

(b) Find two infinite families of non-abelian finite groups which are not Frobenius groups.

Justify your answers with proofs.

23 Characters of Frobenius groups

We now construct the whole character table of an arbitrary Frobenius group.

Theorem 23.1 (*Brauer's Permutation Lemma*)

Let A, B be finite groups, and assume that A acts on both $\mathrm{Irr}(B)$ and $C(B)$ via left actions

$$A \times \mathrm{Irr}(B) \to \mathrm{Irr}(B), \quad (u, \chi) \mapsto u.\chi,$$
$$A \times C(B) \to C(B), \quad (u, C) \mapsto u.C,$$

such that $(a.\chi)(a.c) = \chi(c)$ for each $a \in A$, each $c \in C$ and each $C \in C(B)$. Then

$$|\,\mathrm{Fix}_{\mathrm{Irr}(B)}(a)| = |\{\chi \in \mathrm{Irr}(B) \mid a.\chi = \chi\}| = |\{C \in C(B) \mid a.C = C\}| = |\,\mathrm{Fix}_{C(B)}(a)|$$

for every $u \in A$. (In other words, the permutation representations induced by the two actions afford the same character.) Moreover, the number of A-orbits on $\mathrm{Irr}(B)$ and on $C(B)$ coincide.

Proof: Set $h := |\,\mathrm{Irr}(B)| = |C(B)|$ and write $\mathrm{Irr}(B) = \{\chi_1, \dots, \chi_h\} =: X_1$ and $C(B) = \{C_1, \dots, C_h\} =: X_2$. By Example 1(d), the A-actions on $\mathrm{Irr}(B) =: X_1$ and $C(B) =: X_2$ define permutation representations

$$\rho_{X_1} : A \to \mathrm{GL}(\mathbb{C}^h) \cong \mathrm{GL}_h(\mathbb{C}) \text{ and } \rho_{X_2} : A \to \mathrm{GL}(\mathbb{C}^h) \cong \mathrm{GL}_h(\mathbb{C})$$

respectively, which we see as matrix representations w.r.t. to the ordered $\mathbb{C}$-basis $(\chi_1, \dots, \chi_h)$ and $(C_1, \dots, C_h)$, respectively. Moreover, we denote by χ_{X_1} and χ_{X_2}, respectively, the characters afforded by these representations. Now, by Proposition 10.1, for each $a \in A$ we have

$$\chi_{X_1}(a) = |\mathrm{Fix}_{X_1}(a)| \qquad \text{and} \qquad \chi_{X_2}(a) = |\mathrm{Fix}_{X_2}(a)|\,.$$

Hence, in order to complete the proof of the first claim, it is enough to prove that $\chi_{X_1} = \chi_{X_2}$. Fix $a \in A$ and observe that the action of a on X_1 and X_2 permutes the rows and the columns of $X(N)$, sending the row indexed by χ_i to the row indexed by $a.\chi_i$, and the column indexed by C_j to the column indexed by $a^{-1}.C_j$. (The reason for this choice will become clear in the next lines.) Then, the permutation of the rows is given by left multiplication with $\rho_{X_1}(a)$, i.e. $\rho_{X_1}(a)X(N)$, and the permutation of the columns is given by right multiplication with $\rho_{X_2}(a)$, i.e. $X(N)\rho_{X_2}(a)$. Moreover, the hypothesis of the theorem implies that

$$(a.\chi)(c) = \chi(a^{-1}.c) \qquad \forall\, a \in A,\ \forall\, c \in C,\ \forall\, C \in C(B)\,.$$

It follows that

$$\rho_{X_1}(a)X(N) = X(N)\rho_{X_2}(a)$$

and hence, since $X(N)$ is an invertible matrix, we get $\rho_{X_1}(a) = X(N)\rho_{X_2}(a)X(N)^{-1}$, proving that $\rho_{X_1} \sim \rho_{X_2}$ and the claim follows.
For the last claim, remember that the number of A-orbits on $\mathrm{Irr}(B)$ is given by $\langle \chi_{X_1}, 1_B\rangle_B$ and the number of A-orbits on $\mathrm{Irr}(B)$ is given by $\langle \chi_{X_2}, 1_B\rangle_B$. Now, both numbers are equal by the first claim. ■

We want to apply Brauer's Permutation Lemma in order to obtain information on the character table of Frobenius groups.

Remark 23.2

If $N \trianglelefteq G$, then G acts by conjugation on the sets $\mathrm{Irr}(N)$ and $C(N)$. In other words, there are left G-actions

$$\begin{array}{ccc} G \times \mathrm{Irr}(N) & \longrightarrow & \mathrm{Irr}(N) \\ (g,\chi) & \mapsto & g.\chi := {}^{g}\chi \end{array}$$

and

$$\begin{array}{ccc} G \times C(N) & \longrightarrow & C(N) \\ (g,C) & \mapsto & g.C := {}^{g}C = \{gcg^{-1} \mid c \in C\}\,. \end{array}$$

Moreover, it follows from the definition of a conjugate character that these actions satisfy the condition

$$(g.\chi)(g.C) = \chi(C) \qquad \forall\, C \in C(N)\,.$$

It follows that we may apply Brauer's Permutation Lemma to this setting.

Theorem 23.3

Let G be an arbitrary finite group. Assume that $N \trianglelefteq G$ and assume that $C_G(n) \leqslant N$ for all $n \in N\backslash\{1\}$. Then, the following assertions hold:

(a) if $\psi \in \mathrm{Irr}(N)\backslash\{\mathbf{1}_N\}$, then $\psi\uparrow_N^G \in \mathrm{Irr}(G)$;

(b) if $\chi \in \mathrm{Irr}(G)$ is such that $N \nleqslant \ker(\chi)$, then there exists $\psi \in \mathrm{Irr}(N)$ such that $\chi = \psi\uparrow_N^G$.

Proof:

(a) First, it follows from the Mackey formula that $\langle \psi\uparrow_N^G, \psi\uparrow_N^G\rangle_G = \sum_{xN\in G/N}\langle {}^{x}\psi, \psi\rangle_N$ (See Exercise 20.5.) Thus, to prove that $\psi\uparrow_N^G$ is irreducible, it suffices to prove that $\psi \neq {}^{x}\psi$ for each $x \notin N$, since then the latter sum is equal to 1. Now, by Brauer's Permutation Lemma and Remark 23, it is enough to prove that for each conjugacy class $[1] \neq C \in C(N)$ and each $x \in G$ that the equality $xCx^{-1} = C$ implies that $x \in N$. So, let $n \in C$. Then, $xCx^{-1} = C$ implies that $xnx^{-1} = yny^{-1}$ for some $y \in N$ and hence $y^{-1}x \in C_G(n) \leqslant N$ by the hypothesis, proving that $x \in N$, as required.

(b) Since $N \nleqslant \ker(\chi)$, certainly $\chi\downarrow_N^G$ has at least one non-trivial constituent, say $\psi \in \mathrm{Irr}(N)\backslash\{\mathbf{1}_N\}$. Moreover, Frobenius reciprocity yields

$$\langle \chi, \psi\uparrow_N^G\rangle_G = \langle \chi\downarrow_N^G, \psi\rangle_H \neq 0\,.$$

Thus χ is a constituent of $\psi\uparrow_N^G$, but then this $\chi = \psi\uparrow_N^G$ since $\psi\uparrow_N^G \in \mathrm{Irr}(G)$ by (a). ■

This leads to the following characterisation of the irreducible characters of Frobenius groups.

Theorem 23.4

Let G be a Frobenius group with Frobenius complement H and Frobenius kernel N. Then,

$$\mathrm{Irr}(G) = \mathrm{Inf}_{G/N}^G\left(\mathrm{Irr}(G/N)\right) \sqcup \left\{\psi\uparrow_N^G \mid \psi \in \mathrm{Irr}(N)\backslash\{\mathbf{1}_N\}\right\}.$$

Note. Notice that the Frobenius complement H does not occur in the description of $\mathrm{Irr}(G)$. Thus, choosing a different Frobenius complement would not change the result. Also notice that the second set $\{\psi\uparrow_N^G \mid \psi \in \mathrm{Irr}(N)\backslash\{\mathbf{1}_N\}\}$ may contain repetitions! In order to describe all characters of G which do not have N in their kernel, it suffices to consider a set of representatives of the G-orbits instead of $\mathrm{Irr}(N)\backslash\{\mathbf{1}_N\}$.

Proof: It follows from Theorem 23.3 that it suffices to prove that $C_G(n) \leqslant N$ for all $n \in N\backslash\{1\}$. So, let $n \in N\backslash\{1\}$ and suppose that $C_G(n) \nleqslant N$. Then, by the definition of N, there exists $x \in G$ such that $C_G(n) \cap {}^xH \neq \{1\}$. Now, conjugating by x^{-1} and replacing n with $x^{-1}nx$, we may assume that $C_G(n) \cap H \neq \{1\}$. Thus, given $1 \neq h \in C_G(n) \cap H$, we have $h \in H \cap {}^nH$, which contradicts the fact that H is a Frobenius complement. ■

Exercise 23.5

Compute that character table of the dihedral group $D_{2(2m+1)}$ for any $m \in \mathbb{Z}_{\geqslant 1}$.