Exercise 9.9

Let V be a $\mathbb{C}G$-module affording the character χ_V. Consider the $\mathbb{C}$-subspace of fixed points under the action of G, that is, $V^G := \{v \in V \mid g \cdot v = v \ \forall\, g \in G\}$. Prove that

$$\dim_{\mathbb{C}} V^G = \frac{1}{|G|} \sum_{g \in G} \chi_V(g)$$

in two different ways:

1. considering the scalar product of χ_V with the trivial character $\mathbf{1}_G$;
2. seeing V^G as the image of the projector $\pi : V \longrightarrow V, v \mapsto \frac{1}{|G|}\sum_{g\in G} g \cdot v$.

10 The Regular Character

Recall from Example 1(d) that a finite G-set X induces a *permutation representation*

$$\begin{array}{rccl} \rho_X : & G & \longrightarrow & \mathrm{GL}(V) \\ & g & \mapsto & \rho_X(g) : V \longrightarrow V, e_x \mapsto e_{g\cdot x} \end{array}$$

where V is a $\mathbb{C}$-vector space with basis $\{e_x \mid x \in X\}$ (i.e. indexed by the set X). Given $g \in G$ write $\mathrm{Fix}_X(g) := \{x \in X \mid g \cdot x = x\}$ for the set of fixed points of g on X.

Proposition 10.1 (***Character of a permutation representation***)

Let X be a G-set and let χ_X denote the character of the associated permutation representation ρ_X. Then

$$\chi_X(g) = |\,\mathrm{Fix}_X(g)| \qquad \forall\, g \in G\,.$$

Proof: Let $g \in G$. The diagonal entries of the matrix of $\rho_X(g)$ expressed in the basis $B := \{e_x \mid x \in X\}$ are:

$$\Big(\big(\rho_X(g)\big)_B\Big)_{xx} = \begin{cases} 1 & \text{if } g \cdot x = x \\ 0 & \text{if } g \cdot x \neq x \end{cases} \qquad \forall\, x \in X\,.$$

Hence taking traces, we get $\chi_X(g) = \sum_{x\in X} \Big(\big(\rho_X(g)\big)_B\Big)_{xx} = |\,\mathrm{Fix}_X(g)|$. ■

For the action of G on itself by left multiplication, by Example 1(d), $\rho_X = \rho_{\mathrm{reg}}$ is the regular representation of G. In this case, we obtain the values of the *regular character*.

Corollary 10.2 (***The regular character***)

Let χ_{reg} denote the character of the regular representation ρ_{reg} of G. Then

$$\chi_{\mathrm{reg}}(g) = \begin{cases} |G| & \text{if } g = 1_G, \\ 0 & \text{otherwise.} \end{cases}$$

Proof: This follows immediately from Proposition 10.1 since $\mathrm{Fix}_G(1_G) = G$ and $\mathrm{Fix}_G(g) = \varnothing$ for every $g \in G \setminus \{1_G\}$. ∎

Theorem 10.3 (*Decomposition of the regular representation*)

The multiplicity of an irreducible $\mathbb{C}$-representation of G as a constituent of ρ_{reg} equals its degree. In other words,

$$\chi_{\mathrm{reg}} = \sum_{\chi \in \mathrm{Irr}(G)} \chi(1)\chi \,.$$

Proof: By Corollary 9.3 we have $\chi_{\mathrm{reg}} = \sum_{\chi \in \mathrm{Irr}(G)} \langle \chi_{\mathrm{reg}}, \chi \rangle_G \chi$, where for each $\chi \in \mathrm{Irr}(G)$,

$$\langle \chi_{\mathrm{reg}}, \chi \rangle_G = \frac{1}{|G|} \sum_{g \in G} \underbrace{\chi_{\mathrm{reg}}(g)}_{\substack{=\delta_{1g}|G| \\ \text{by Cor. 10.2}}} \overline{\chi(g)} = \frac{|G|}{|G|}\chi(1) = \chi(1) \,.$$

The claim follows. ∎

Remark 10.4

In particular, the theorem tells us that each irreducible $\mathbb{C}$-representation (considered up to equivalence) occurs with multiplicity at least one in a decomposition of the regular representation into irreducible subrepresentations.

Corollary 10.5 (*Degree formula*)

The order of the group G is given in terms of its irreducible character by the formula

$$|G| = \sum_{\chi \in \mathrm{Irr}(G)} \chi(1)^2 \,.$$

Proof: Evaluating the regular character at $1 \in G$ yields

$$|G| = \chi_{\mathrm{reg}}(1) = \sum_{\chi \in \mathrm{Irr}(G)} \chi(1)\chi(1) = \sum_{\chi \in \mathrm{Irr}(G)} \chi(1)^2 \,.$$

∎

Exercise 10.6

Use the degree formula to give a second proof of Proposition 6.1 when $K = \mathbb{C}$. In other words, prove that if G is a finite abelian group, then

$$\mathrm{Irr}(G) = \mathrm{Lin}(G) \,,$$

the set of linear characters of G.

Chapter 4. The Character Table

In Chapter 3 we have proved that for any finite group G the equality $|\operatorname{Irr}(G)| = |C(G)| =: r$ holds. Thus the values of the irreducible characters of G can be recorded in an $r \times r$-matrix, called the *character table* of G. The entries of this matrix are related to each other in subtle manners, many of which are encapsulated in the 1st Orthogonality Relations and their consequences, as for example the degree formula. Our aim in this chapter is to develop further tools and methods to compute character tables.

Notation: throughout this chapter, unless otherwise specified, we let:

- G denote a finite group;
- $K := \mathbb{C}$ be the field of complex numbers;
- $|\operatorname{Irr}(G)| = |C(G)| =: r$;
- $\operatorname{Irr}(G) = \{\chi_1, \ldots, \chi_r\}$ denote the set of pairwise distinct irreducible characters of G;
- $C_1 = [g_1], \ldots, C_r = [g_r]$ denote the conjugacy classes of G, where $g_1, \ldots, g_r$ is a fixed set of representatives; and
- we use the convention that $\chi_1 = \mathbf{1}_G$ and $g_1 = 1 \in G$.

In general, unless otherwise stated, all groups considered are assumed to be finite and all $\mathbb{C}$-vector spaces / modules over the group algebra considered are assumed to be finite-dimensional.

11 The Character Table of a Finite Group

Definition 11.1 (*Character table*)

The **character table of** G is the matrix $X(G) := \Big(\chi_i(g_j)\Big)_{ij} \in M_r(\mathbb{C})$.

Example 4 (*The character table of a cyclic group*)

Let $G = \langle g \mid g^n = 1\rangle$ be cyclic of order $n \in \mathbb{Z}_{>0}$. Since G is abelian,

$$\operatorname{Irr}(G) = \{\text{linear characters of } G\}$$

by Proposition 6.1 and $|\operatorname{Irr}(G)| = |G| = n$. Moreover, each conjugacy class is a singleton:

$$\forall\, 1 \leqslant j \leqslant r = n : \quad C_j = \{g_j\} \text{ and we set } g_j := g^{j-1}.$$

Let ζ be a primitive n-th root of unity in $\mathbb{C}$, so that $\{\zeta^i \mid 1 \leqslant i \leqslant n\}$ are all the n-th roots of unity. Now, each $\chi_i : G \to \mathbb{C}^\times$ is a group homomorphism and is determined by $\chi_i(g)$, which has to be an n-th root of $1_\mathbb{C}$. Therefore, we have n possibilities for $\chi_i(g)$. We set

$$\chi_i(g) := \zeta^{i-1} \quad \forall\, 1 \leqslant i \leqslant n \quad \Rightarrow \quad \chi_i(g^j) = \zeta^{(i-1)j} \quad \forall\, 1 \leqslant i \leqslant n, 0 \leqslant j \leqslant n-1$$

Thus the character table of G is

$$X(G) = \Big(\chi_i(g_j)\Big)_{\substack{1\leqslant i\leqslant n\\ 1\leqslant j\leqslant n}} = \Big(\chi_i(g^{j-1})\Big)_{\substack{1\leqslant i\leqslant n\\ 1\leqslant j\leqslant n}} = \Big(\zeta^{(i-1)(j-1)}\Big)_{\substack{1\leqslant i\leqslant n\\ 1\leqslant j\leqslant n}},$$

which we visualise as follows:

	1	g	g^2	$\dots$	g^{n-1}
$\chi_1 = \mathbf{1}_G$	1	1	1	$\dots$	1
χ_2	1	ζ	ζ^2	$\dots$	ζ^{n-1}
χ_3	1	ζ^2	ζ^4	$\dots$	$\zeta^{2(n-1)}$
$\dots$	$\dots$	$\dots$	$\dots$	$\dots$	$\dots$
χ_n	1	ζ^{n-1}	$\zeta^{2(n-1)}$	$\dots$	$\zeta^{(n-1)^2}$

Example 5 (*The character table of S_3*)

Let now $G := S_3$ be the symmetric group on 3 letters. Recall from the AGS/*Einführung in die Algebra* that the conjugacy classes of S_3 are

$$C_1 = \{\mathrm{Id}\},\ C_2 = \{(1\ 2), (1\ 3), (2\ 3)\},\ C_3 = \{(1\ 2\ 3), (1\ 3\ 2)\}$$

$$\Rightarrow \quad r = 3, |C_1| = 1, |C_2| = 3, |C_3| = 2\,.$$

In Example 2(d) we have exhibited three non-equivalent irreducible matrix representations of S_3, which we denoted ρ_1, ρ_2, ρ_3. For each $1 \leqslant i \leqslant 3$ let χ_i be the character of ρ_i and n_i be its degree, so that $n_1 = n_2 = 1$ and $n_3 = 2$. Hence

$$n_1^2 + n_2^2 + n_3^2 = 6 = |G|\,.$$

Therefore, the degree formula tells us that ρ_1, ρ_2, ρ_3 are all the irreducible matrix representations of S_3, up to equivalence. We note that $n_1 = n_2 = 1$, $n_3 = 2$ is in fact the unique solution (up to relabelling) to the equation given by the degree formula! Taking traces of the matrices in Example 2(d) yields the character table of S_3.

	Id	(1 2)	(1 2 3)
χ_1	1	1	1
χ_2	1	-1	1
χ_3	2	0	-1

In the next sections we want to develop further techniques to compute character tables of finite groups, before we come back to further examples of such tables for larger groups.

Exercise 11.2

Compute the character table of the Klein-four group $C_2 \times C_2$ and of $C_2 \times C_2 \times C_2$.

12 The 2nd Orthogonality Relations

The 1st Orthogonality Relations provide us with orthogonality relations between the rows of the character table. They can be rewritten as follows in terms of matrices.

Exercise 12.1

Let G be a finite group. Set $X := X(G)$ and

$$C := \begin{bmatrix} |C_G(g_1)| & 0 & \cdots & 0 \\ 0 & |C_G(g_2)| & \ddots & \vdots \\ \vdots & \ddots & \ddots & 0 \\ 0 & \cdots & 0 & |C_G(g_r)| \end{bmatrix} \in M_r(\mathbb{C})\,.$$

Use the Orbit-Stabiliser Theorem in order to prove that the 1st Orthogonality Relations can be rewritten under the form

$$X C^{-1} \overline{X}^{\mathrm{Tr}} = I_r\,,$$

where $\overline{X}^{\mathrm{Tr}}$ denotes the transpose of the complex-conjugate $\overline{X}$ of the character table X of G. Deduce that the character table is invertible.

There are also some orthogonality relations between the columns of the character table. These can easily be deduced from the 1st Orthogonality Relations given above in terms of matrices.

Theorem 12.2 (*2nd Orthogonality Relations*)

With the notation of Exercise 12.1 we have

$$X^{\mathrm{Tr}}\,\overline{X} = C\,.$$

In other words,

$$\sum_{\chi \in \mathrm{Irr}(G)} \chi(g_i)\overline{\chi(g_j)} = \delta_{ij}\frac{|G|}{|[g_i]|} = \delta_{ij}|C_G(g_i)| \qquad \forall\, 1 \leqslant i, j \leqslant r\,.$$

Proof: Taking complex conjugation of the formula given by the 1st Orthogonality Relations (Exercise 12.1) yields:

$$X C^{-1} \overline{X}^{\mathrm{Tr}} = I_r \qquad \Longrightarrow \qquad \overline{X} C^{-1} X^{\mathrm{Tr}} = I_r$$

Now, since X is invertible, so are all the matrices in the above equations and hence $X^{\mathrm{Tr}} = \left(\overline{X}C^{-1}\right)^{-1}$. It follows that

$$X^{\mathrm{Tr}}\,\overline{X} = \left(\overline{X}C^{-1}\right)^{-1}\overline{X} = C\overline{X}^{-1}\overline{X} = C\,.$$

The second formula is now obtained by considering the entry (i, j) in the above matrix equation for all $1 \leqslant i, j \leqslant r$. ■

Exercise 12.3

Prove that the degree formula can be read off from the 2nd Orthogonality Relations.

13 Tensor Products of Representations and Characters

Tensor products of vector spaces and matrices are recalled/introduced in Appendix C. We are now going to use this construction to build *products* of characters.

Proposition 13.1

Let G and H be finite groups, and let $\rho_V : G \longrightarrow \mathrm{GL}(V)$ and $\rho_W : H \longrightarrow \mathrm{GL}(W)$ be $\mathbb{C}$-representations with characters χ_V and χ_W respectively. Then

$$\begin{array}{rccl} \rho_V \otimes \rho_W : & G \times H & \longrightarrow & \mathrm{GL}(V \otimes_{\mathbb{C}} W) \\ & (g,h) & \mapsto & (\rho_V \otimes \rho_W)(g,h) := \rho_V(g) \otimes \rho_W(h) \end{array}$$

(where $\rho_V(g) \otimes \rho_W(h)$ is the tensor product of the $\mathbb{C}$-endomorphisms $\rho_V(g) : V \longrightarrow V$ and $\rho_W(h) : W \longrightarrow W$ as defined in Lemma-Definition C.4) is a $\mathbb{C}$-representation of $G \times H$, called the **tensor product** of ρ_V and ρ_W, and the corresponding character, which we denote by $\chi_{V \otimes_{\mathbb{C}} W}$, is

$$\chi_{V \otimes_{\mathbb{C}} W} = \chi_V \cdot \chi_W ,$$

where $\chi_V \cdot \chi_W(g,h) := \chi_V(g) \cdot \chi_W(h) \ \forall \ (g,h) \in G \times H$.

Proof: First note that $\rho_V \otimes \rho_W$ is well-defined by Lemma-Definition C.4 and it is a group homomorphism because

$$\begin{aligned} (\rho_V \otimes \rho_W)(g_1g_2, h_1h_2)[v \otimes w] &= \big(\rho_V(g_1g_2) \otimes \rho_W(h_1h_2)\big)[v \otimes w] \\ &= \rho_V(g_1g_2)[v] \otimes \rho_W(h_1h_2)[w] \\ &= \rho_V(g_1) \circ \rho_V(g_2)[v] \otimes \rho_W(h_1) \circ \rho_W(h_2)[w] \\ &= \rho_V(g_1) \otimes \rho_W(h_1)\big[\rho_V(g_2)[v] \otimes \rho_W(h_2)[w]\big] \\ &= \big(\rho_V(g_1) \otimes \rho_W(h_1)\big) \circ \big(\rho_V(g_2) \otimes \rho_W(h_2)\big)[v \otimes w] \\ &= \big(\rho_V \otimes \rho_W\big)(g_1,h_1) \circ \big(\rho_V \otimes \rho_W\big)(g_2,h_2)[v \otimes w] \end{aligned}$$

$\forall \ g_1, g_2 \in G,\ h_1, h_2 \in H,\ v \in V,\ w \in W.$ Furthermore, for each $g \in G$ and each $h \in H$,

$$\chi_{V \otimes_{\mathbb{C}} W}(g,h) = \mathrm{Tr}\left((\rho_V \otimes \rho_W)(g,h)\right) = \mathrm{Tr}\left(\rho_V(g) \otimes \rho_W(h)\right) = \mathrm{Tr}\left(\rho_V(g)\right) \cdot \mathrm{Tr}\left(\rho_W(h)\right) = \chi_V(g) \cdot \chi_W(h)$$

by Lemma-Definition C.4, hence $\chi_{V \otimes_{\mathbb{C}} W} = \chi_V \cdot \chi_W$. ■

Remark 13.2

The diagonal inclusion $\iota : G \longrightarrow G \times G, g \mapsto (g,g)$ of G in the product $G \times G$ is a group homomorphism with $\iota(G) \cong G$. Therefore, if $G = H$, then

$$G \xrightarrow{\iota} G \times G \xrightarrow{\chi_V \cdot \chi_W} \mathbb{C}, g \mapsto (g,g) \mapsto \chi_V(g) \cdot \chi_W(g)$$

becomes a character of G, which we also denote by $\chi_V \cdot \chi_W$.

Corollary 13.3

If G and H are finite groups, then $\mathrm{Irr}(G \times H) = \{\chi \cdot \psi \mid \chi \in \mathrm{Irr}(G), \psi \in \mathrm{Irr}(H)\}$.

Proof: [Exercise]. Hint: Use Corollary 9.8(d) and the degree formula. ■