Exercise 13.4

(a) If $\lambda, \chi \in \mathrm{Irr}(G)$ and $\lambda(1) = 1$, then $\lambda \cdot \chi \in \mathrm{Irr}(G)$.

(b) The set $\mathrm{Lin}(G) = \{\chi \in \mathrm{Irr}(G) \mid \chi(1) = 1\}$ of linear characters of a finite group G forms a group for the product of characters.

14 Normal Subgroups and Inflation

Whenever a group homomorphism $G \longrightarrow H$ and a representation of H are given, we obtain a representation of G by composition. In particular, we want to apply this principle to normal subgroups $N \trianglelefteq G$ and the corresponding quotient homomorphism, which we always denote by $\pi : G \longrightarrow G/N, g \mapsto gN$.

We will see that by this means, copies of the character tables of quotient groups of G all appear in the character table of G. This observation, although straightforward, will allow us to fill out the character table of a group very rapidly, provided it possesses normal subgroups.

Definition 14.1 (*Inflation*)

Let $N \trianglelefteq G$ and let $\pi : G \longrightarrow G/N, g \mapsto gN$ be the quotient homomorphism. Given a $\mathbb{C}$-representation $\rho : G/N \longrightarrow \mathrm{GL}(V)$, we set

$$\mathrm{Inf}_{G/N}^{G}(\rho) := \rho \circ \pi : G \longrightarrow \mathrm{GL}(V)\,.$$

This is a $\mathbb{C}$-representation of G (see Exercise 9.10), called the **inflation of ρ from G/N to G**.

Note that some texts also call $\mathrm{Inf}_{G/N}^{G}(\rho)$ the *lift* or the *restriction* of ρ along π.

Remark 14.2

(a) If the character afforded by ρ is χ, then by Exercise 9.10(i), the character afforded by $\mathrm{Inf}_{G/N}^{G}(\rho)$ is $\mathrm{Inf}_{G/N}^{G}(\chi) := \chi \circ \pi$. We also call it the **inflation of χ from G/N to G**. Clearly, the values of $\mathrm{Inf}_{G/N}^{G}(\chi)$ are given by the formula

$$\mathrm{Inf}_{G/N}^{G}(\chi)(g) = \chi(gN) \qquad \forall\, g \in G\,.$$

(b) By Exercise 9.10(iii), if ρ (resp. χ) is irreducible, then so is $\mathrm{Inf}_{G/N}^{G}(\rho)$ (resp. $\mathrm{Inf}_{G/N}^{G}(\chi)$)

Exercise 14.3

Let $N \trianglelefteq G$ and let $\rho : G/N \longrightarrow \mathrm{GL}(V)$ be a $\mathbb{C}$-representation of G/N. Compute the kernel of $\mathrm{Inf}_{G/N}^{G}(\rho)$ provided that ρ is faithful.

Definition 14.4 (*Kernel of a character*)

The **kernel of a character** χ of G is $\ker(\chi) := \{g \in G \mid \chi(g) = \chi(1)\}$.

Example 6

(a) $\chi = \mathbf{1}_G$ the trivial character $\Rightarrow \ker(\chi) = G$.

(b) $G = \mathfrak{S}_3$, $\chi = \chi_2$ the sign character $\Rightarrow \ker(\chi) = C_1 \cup C_3 = \langle(123)\rangle$; whereas $\ker(\chi_3) = \{1\}$. (See Example 5.)

Lemma 14.5

Let $\rho : G \longrightarrow \mathrm{GL}(V)$ be a $\mathbb{C}$-representation of G affording the character ψ. Then $\ker(\psi) = \ker(\rho)$, thus it is a normal subgroup of G.

Proof: [Exercise] ∎

Theorem 14.6

Let $N \trianglelefteq G$. Then

$$\begin{array}{rccc} \mathrm{Inf}^G_{G/N}: & \{\text{characters of } G/N\} & \longrightarrow & \{\text{characters } \psi \text{ of } G \mid N \leqslant \ker(\psi)\} \\ & \chi & \mapsto & \mathrm{Inf}^G_{G/N}(\chi) \end{array}$$

is a bijection and so is its restriction to the irreducible characters

$$\begin{array}{rccc} \mathrm{Inf}^G_{G/N}: & \mathrm{Irr}(G/N) & \longrightarrow & \{\psi \in \mathrm{Irr}(G) \mid N \leqslant \ker(\psi)\} \\ & \chi & \mapsto & \mathrm{Inf}^G_{G/N}(\chi)\,. \end{array}$$

Proof: First we prove that the first map is well-defined and bijective.

- Let χ be a character of G/N. By Remark 14.2, N is in the kernel of $\mathrm{Inf}^G_{G/N}(\chi)$, hence the first map is well-defined.
- Now let ψ be a character of G with $N \leqslant \ker(\psi)$ and assume ψ is afforded by the $\mathbb{C}$-representation $\rho : G \longrightarrow \mathrm{GL}(V)$.

$$\begin{array}{ccc} G & \xrightarrow{\ \rho\ } & \mathrm{GL}(V) \\ {\scriptstyle\pi}\downarrow & \circlearrowleft & \nearrow {\scriptstyle\exists!\,\tilde{\rho}} \\ G/N & & \end{array}$$

By Lemma 14.5 we have $\ker(\psi) = \ker(\rho) \geqslant N$. Therefore, by the universal property of the quotient, ρ induces a unique $\mathbb{C}$-representation $\tilde{\rho} : G/N \longrightarrow \mathrm{GL}(V)$ with the property that $\tilde{\rho} \circ \pi = \rho$.

Letting χ be the character afforded by $\tilde{\rho}$, it follows that $\rho = \mathrm{Inf}^G_{G/N}(\tilde{\rho})$ and $\psi = \mathrm{Inf}^G_{G/N}(\chi)$. Thus the 1st map is surjective. Its injectivity is clear (e.g. by Remark 14.2).

The second map is well-defined by the above and Exercise 14.3(a). It is injective because it is just the restriction of the 1st map to the $\mathrm{Irr}(G/N)$, whereas it is surjective by the same argument as above as the constructed representation $\tilde{\rho}$ is clearly irreducible if ρ is, as $\tilde{\rho} \circ \pi = \rho$. ∎

Exercise 14.7

Let G be a finite group. Prove that if $N \trianglelefteq G$, then

$$N = \bigcap_{\substack{\chi \in \mathrm{Irr}(G) \\ N \subseteq \ker(\chi)}} \ker(\chi)\,.$$

It follows immediately from the above exercise that the lattice of normal subgroups of G can be read off from its character table. The theorem also implies that it can be read off from the character table, whether the group is abelian or simple.

Corollary 14.8

(a) Inflation from the abelianization induces a bijection

$$\mathrm{Inf}_{G/G'}^{G} : \mathrm{Irr}(G/G') \xrightarrow{\sim} \mathrm{Lin}(G) \ .$$

In particular, G has precisely $|G : G'|$ linear characters.

(b) The group G is abelian if and only if all its irreducible characters are linear.

Proof: (a) First, we claim that if $\psi \in (G)$, then G' is in its kernel. Indeed, if $\psi(1) = 1$, then $\psi : G \longrightarrow \mathbb{C}^{\times}$ is a group homomorphism. Therefore, as $\mathbb{C}^{\times}$ is abelian,

$$\psi([g,h]) = \psi(ghg^{-1}h^{-1}) = \psi(g)\psi(h)\psi(g)^{-1}\psi(h)^{-1} = \psi(g)\psi(g)^{-1}\psi(h)\psi(h)^{-1} = 1$$

for all $g, h \in G$, and hence $G' = \langle [g,h] \mid g,h \in G \rangle \leqslant \ker(\chi)$. In addition, any irreducible character of G/G' is linear by Proposition 6.1 because G/G' is abelian. Thus Theorem 14.6 yields a bijection

$$\mathrm{Irr}(G/G') = \mathrm{Lin}(G/G') \xrightarrow[\mathrm{Inf}_{G/G'}^{G}]{\sim} \{\psi \in \mathrm{Irr}(G) \mid G' \leqslant \ker(\psi)\} = \mathrm{Lin}(G),$$

as required.

(b) The group G is abelian if and only if $G/G' = G$, which happens if and only if $\mathrm{Inf}_{G/G'}^{G} = \mathrm{Id}$. Hence, the claim follows from (a). ■

Corollary 14.9

A finite group G is simple $\iff$ $\chi(g) \neq \chi(1)\ \forall\, g \in G \setminus \{1\}$ and $\forall\, \chi \in \mathrm{Irr}(G) \setminus \{\mathbf{1}_G\}$.

Proof: [Exercise] ■

Exercise 14.10

Compute the complex character table of the alternating group A_4 through the following steps:

1. Determine the conjugacy classes of A_4 (there are 4 of them) and the corresponding centraliser orders.
2. Determine the degrees of the 4 irreducible characters of A_4.
3. Determine the linear characters of A_4.
4. Determine the non-linear character of A_4 using the 2nd Orthogonality Relations.

To finish this section we show how to compute the character table of the symmetric group S_4 combining several of the techniques we have developed in this chapter.

Example 7 (*The character table of S_4*)

Again the conjugacy classes of S_4 are given by the cycle types. We fix

$$C_1 = \{\mathrm{Id}\},\ C_2 = [(1\ 2)],\ C_3 = [(1\ 2\ 3)],\ C_4 = [(1\ 2)(3\ 4)],\ C_5 = [(1234)]$$

$$\Rightarrow \quad r = 5,\ |C_1| = 1,\ |C_2| = 6,\ |C_3| = 8,\ |C_4| = 3,\ |C_5| = 6\,.$$

Hence $|\operatorname{Irr}(G)| = |C(G)| = 5$ and as always we may assume that $\chi_1 = \mathbf{1}_G$ is the trivial character.

Recall that $V_4 = \{\mathrm{Id}, (1\ 2)(3\ 4), (1\ 3)(2\ 4), (1\ 4)(2\ 3)\} \trianglelefteq S_4$ with $S_4/V_4 \cong S_3$ (AGS or Einführung in die Algebra!). Therefore, by Theorem 14.6 we can "inflate" the character table of $S_4/V_4 \cong S_3$ to S_4 (see Example 5 for the character table of S_3). This provides us with three irreducible characters χ_1, χ_2 and χ_3 of S_4:

	Id	(1 2)	(1 2 3)	(1 2)(3 4)	(1 2 3 4)
$\lvert C_G(g_i)\rvert$	24	4	3	8	4
χ_1	1	1	1	1	1
χ_2	1	–1	1	1	–1
χ_3	2	0	–1	2	0
χ_4	·	·	·	·	·
χ_5	·	·	·	·	·

Here we have computed the values of χ_2 and χ_3 using Remark 14.2 as follows:

- Inflation preserves degrees, hence it follows from Example 5 that $\chi_2(\mathrm{Id}) = 1$ and $\chi_3(\mathrm{Id}) = 2$. (Up to relabelling!)
- As $C_4 = [(1\ 2)(3\ 4)] \subseteq V_4$, $(1\ 2)(3\ 4) \in \ker(\chi_i)$ for $i = 2, 3$ and hence $\chi_2((1\ 2)(3\ 4)) = 1$ and $\chi_3((1\ 2)(3\ 4)) = 2$.
- By Remark 14.2 the values of χ_2 and χ_3 at (1 2) and (1 2 3) are given by the corresponding values in the character table of S_3. (Here it is enough to argue that the isomorphism between S_4/V_4 and S_3 must preserve orders of elements, hence also the cycle type in this case.)
- Finally, we compute that $\overline{(1\ 2\ 3\ 4)} = \overline{(1\ 2)} \in S_4/V_4$, hence $\chi_i((1\ 2\ 3\ 4)) = \chi_i((1\ 2))$ for $i = 2, 3$.

Therefore, it remains to compute χ_4 and χ_5. To begin with the degree formula yields

$$\sum_{i=1}^{5} \chi_i(\mathrm{Id})^2 = 24 \quad\Longrightarrow\quad \chi_4(\mathrm{Id})^2 + \chi_5(\mathrm{Id})^2 = 18 \quad\Longrightarrow\quad \chi_4(\mathrm{Id}) = \chi_5(\mathrm{Id}) = 3\,.$$

Next, the 2nd Orthogonality Relations applied to the 3rd column with itself read

$$\sum_{i=1}^{5} \chi_i((1\ 2\ 3))\overline{\chi_i((1\ 2\ 3))} = \sum_{i=1}^{5} \chi_i((1\ 2\ 3))\chi_i((1\ 2\ 3)^{-1}) = |C_G((1\ 2\ 3))| = 3\,,$$

hence $1 + 1 + 1 + \chi_4((1\ 2\ 3))^2 + \chi_5(1\ 2\ 3))^2 = 3$ and it follows that $\chi_4((1\ 2\ 3)) = \chi_5((1\ 2\ 3)) = 0$. Similarly, the 2nd Orthogonality Relations applied to the 2nd column with itself / the 4th column with itself and the 5th column with itself yield that all other entries squared are equal to 1, hence

all other entries are ± 1.
The 2nd Orthogonality Relations applied to the 1st and 2nd columns give the 2nd column, i.e. $\chi_4((1\ 2)) = 1$ and $\chi_5((1\ 2)) = -1$ (up to swapping χ_4 and χ_5).
Then the 1st Orthogonality Relations applied to the 3rd and the 4th row yield

$$0 = \sum_{k=1}^{5} \frac{1}{|C_G(g_k)|}\chi_3(g_k)\overline{\chi_4(g_k)} = \frac{6}{24} + \frac{1}{4}\chi_4((1\ 2)(3\ 4)) \;\Rightarrow\; \chi_4((1\ 2)(3\ 4)) = -1\,.$$

Similar with the 3rd row and the 5th row: $\chi_5((1\ 2)(3\ 4)) = -1$. Finally the 1st Orthogonality Relations applied to the 1st and the 4th (resp. 5th) row yield $\chi_4((1\ 2\ 3\ 4)) = -1$ (resp. $\chi_5((1\ 2\ 3\ 4)) = 1$). Thus the character table of S_4 is:

	Id	(1 2)	(1 2 3)	(1 2)(3 4)	(1 2 3 4)
$\lvert C_G(g_i)\rvert$	24	4	3	8	4
χ_1	1	1	1	1	1
χ_2	1	-1	1	1	-1
χ_3	2	0	-1	2	0
χ_4	3	1	0	-1	-1
χ_5	3	-1	0	-1	1

Remark 14.11

Two non-isomorphic groups can have the same character table. E.g.: Q_8 and D_8, but $Q_8 \not\cong D_8$. Thus, the character table does not determine:

- the group up to isomorphism;
- the full lattice of subgroups;
- the orders of elements.

Exercise 14.12

Compute the character tables of D_8 and Q_8.
[Hint: In each case, determine the commutator subgroup and deduce that there are 4 linear characters.]

Exercise 14.13 (*The determinant of a representation*)

If $\rho : G \longrightarrow \mathrm{GL}(V)$ is a $\mathbb{C}$-representation of G and $\det : \mathrm{GL}(V) \longrightarrow \mathbb{C}^*$ denotes the determinant homomorphism, then we define a linear character of G via

$$\det_\rho := \det \circ \rho : G \longrightarrow \mathbb{C}^*\,,$$

called the **determinant** of ρ. Prove that, although the finite groups D_8 and Q_8 have the same character table, they can be distinguished by considering the determinants of their irreducible $\mathbb{C}$-representations.

Exercise 14.14

Prove the follwing assertions:

(a) If G is a non-abelian simple group (or more generally if G is perfect, i.e. $G = [G, G]$), then the image $\rho(G)$ of any $\mathbb{C}$-representation $\rho : G \longrightarrow \mathrm{GL}(V)$ is a subgroup of $\mathrm{SL}(V)$.

(b) No simple group G has an irreducible character of degree 2.

Assume that G is simple and $\rho : G \longrightarrow \mathrm{GL}_2(\mathbb{C})$ is an irreducible matrix representation of G with character χ and proceed as follows:

1. Prove that ρ is faithful and G is non-abelian.
3. Determine the determinant $\det_\rho$ of ρ.
4. Prove that $|G|$ is even and G admits an element x of order 2.
5. Prove that $\langle x \rangle \lhd G$ and conclude that assertion (b) holds.

Chapter 5. Integrality and Theorems of Burnside's

The main aim of this chapter is to prove *Burnside's $p^a q^b$ theorem*, which provides us with a solubility criterion for finite groups of order $p^a q^b$ with p, q prime numbers, which is extremely hard to prove by purely group theoretic methods. To reach this aim, we need to develop techniques involving the integrality of character values and further results of Burnside's on the vanishing of character values.

Notation: throughout this chapter, unless otherwise specified, we let:

- G denote a finite group;
- $K := \mathbb{C}$ be the field of complex numbers;
- $\text{Irr}(G) := \{\chi_1, \dots, \chi_r\}$ denote the set of pairwise distinct irreducible characters of G;
- $C_1 = [g_1], \dots, C_r = [g_r]$ denote the conjugacy classes of G, where $g_1, \dots, g_r$ is a fixed set of representatives; and
- we use the convention that $\chi_1 = \mathbf{1}_G$ and $g_1 = 1 \in G$.

In general, unless otherwise stated, all groups considered are assumed to be finite and all $\mathbb{C}$-vector spaces / modules over the group algebra considered are assumed to be finite-dimensional.

15 Algebraic Integers and Character Values

First we investigate the algebraic nature of character values.

Recall: (See Appendix D for details.)
An element $b \in \mathbb{C}$ which is integral over $\mathbb{Z}$ is called an *algebraic integer*. In other words, $b \in \mathbb{C}$ is an algebraic integer if b is a root of monic polynomial $f \in \mathbb{Z}[X]$.
Algebraic integers have the following properties:

- The integers are clearly algebraic integers.
- Roots of unity are algebraic integers, as they are roots of polynomials of the form $X^m - 1 \in \mathbb{Z}[X]$.
- The algebraic integers form a subring of $\mathbb{C}$. In particular, sums and products of algebraic integers are again algebraic integers.
- If $b \in \mathbb{Q}$ is an algebraic integer, then $b \in \mathbb{Z}$. In other words $\{b \in \mathbb{Q} \mid b \text{ algebraic integer}\} = \mathbb{Z}$.

Corollary 15.1

Character values are algebraic integers.

Proof: By the above, roots of unity are algebraic integers. Since the algebraic integers form a ring, so are sums of roots of unity. Hence the claim follows from Property 7.5(b). ∎

16 Central Characters

We now extend representations/characters of finite groups to "representations/characters" of the centre of the group algebra $\mathbb{C}G$ in order to obtain further results on character values, which we will use in the next sections in order to prove Burnside's $p^a q^b$ theorem.

Definition 16.1 (*Class sums*)

The elements $\widehat{C}_j := \sum_{g\in C_j} g \in \mathbb{C}G$ $(1 \leqslant j \leqslant r)$ are called the **class sums** of G.

Lemma 16.2

The class sums $\{\widehat{C}_j \mid 1 \leqslant j \leqslant r\}$ form a $\mathbb{C}$-basis of $Z(\mathbb{C}G)$. In other words, $Z(\mathbb{C}G) = \bigoplus_{j=1}^r \mathbb{C}\widehat{C}_j$.

Proof: Notice that the class sums are clearly $\mathbb{C}$-linearly independent because the group elements are.

'$\supseteq$': $\forall\ 1 \leqslant j \leqslant r$ and $\forall\ g \in G$, we have

$$g \cdot \widehat{C}_j = g(g^{-1}\widehat{C}_j g) = \widehat{C}_j \cdot g\,.$$

Extending by $\mathbb{C}$-linearity, we get $a\cdot\widehat{C}_j = \widehat{C}_j\cdot a\ \ \forall\, 1 \leqslant j \leqslant r$ and $\forall\, a \in \mathbb{C}G$. Thus $\bigoplus_{j=1}^r \mathbb{C}\widehat{C}_j \subseteq Z(\mathbb{C}G)$.

'$\subseteq$': Let $a \in Z(\mathbb{C}G)$ and write $a = \sum_{g\in G} \lambda_g g$ with $\{\lambda_g\}_{g\in G} \in \mathbb{C}$. Since a is central, for every $h \in G$, we have

$$\sum_{g\in G} \lambda_g g = a = hah^{-1} = \sum_{g\in G} \lambda_g (hgh^{-1})\,.$$

Comparing coefficients yield $\lambda_g = \lambda_{hgh^{-1}}\ \forall\ g, h \in G$. Namely, the coefficients λ_g are constant on the conjugacy classes of G, and hence

$$a = \sum_{j=1}^r \lambda_{g_j}\widehat{C}_j \in \bigoplus_{j=1}^r \mathbb{C}\widehat{C}_j\,.$$

∎

Now, notice that by definition the class sums $\widehat{C}_j$ $(1 \leqslant j \leqslant r)$ are elements of the subring $\mathbb{Z}G$ of $\mathbb{C}G$, hence of the centre of $\mathbb{Z}G$.

Corollary 16.3

(a) $Z(\mathbb{Z}G)$ is finitely generated as a $\mathbb{Z}$-module.

(b) The centre $Z(\mathbb{Z}G)$ of the group ring $\mathbb{Z}G$ is integral over $\mathbb{Z}$; in particular the class sums $\widehat{C}_j$ $(1 \leqslant j \leqslant r)$ are integral over $\mathbb{Z}$.