Representation theory of finite groups is originally concerned with the ways of writing a finite group $G$ as a group of matrices, that is using group homomorphisms from $G$ to the general linear group $GL_n(K)$ of invertible $n \times n$-matrices with coefficients in a field $K$ for some positive integer $n$. Thus, we shall first define representations of groups using this approach. Our aim is then to translate such homomorphisms $G \to GL_n(K)$ into the language of module theory in order to be able to apply the theory we have developed so far.

**Notation:** throughout this chapter, unless otherwise specified, we let $G$ denote a finite group and $K$ be a commutative ring. Moreover, all modules considered are assumed to be **finitely generated**, hence of **finite rank** if they are free.

**References:**


### 14 Linear representations of finite groups

**Definition 14.1 (K-representation, matrix representation)**

(a) A **$K$-representation** of $G$ is a group homomorphism $\rho : G \to GL(V)$, where $V \cong K^n$ $(n \in \mathbb{Z}_{>0})$ is a free $K$-module of finite rank.
(b) A **matrix representation** of $G$ is a group homomorphism $X : G \rightarrow \text{GL}_n(K)$ ($n \in \mathbb{Z}_{>0}$).

In both cases the integer $n$ is called the **degree** of the representation.

A representation is called an **ordinary representation** if $K$ is a field of characteristic zero (or more generally of characteristic not dividing $|G|$), and it is called a **modular representation** if $K$ is a field of characteristic $p$ dividing $|G|$.

**Remark 14.2**

Recall that every choice of a basis $B$ of $V$ yields a group homomorphism $\alpha_B : \text{GL}(V) \rightarrow \text{GL}_n(K)$, $\varphi \mapsto (\varphi)_B$ (where $(\varphi)_B$ denotes the matrix of $\varphi$ in the basis $B$). Therefore, a $K$-representation $\rho : G \rightarrow \text{GL}(V)$ together with the choice of a basis $B$ of $V$ gives rise to a matrix representation of $G$:

$$G \overset{\rho}{\longrightarrow} \text{GL}(V) \overset{\alpha_B}{\longrightarrow} \text{GL}_n(K)$$

Conversely, any matrix representation $X : G \rightarrow \text{GL}_n(K)$ gives rise to a $K$-representation

$$\rho : G \rightarrow \text{GL}(K^n)$$

$$g \mapsto \rho(g) : K^n \rightarrow K^n, v \mapsto X(g)v$$

where $X(g)v$ is the standard matrix multiplication (namely we set $V = K^n$).

**Example 8**

(a) If $G$ is an arbitrary finite group, then

$$\rho : G \rightarrow \text{GL}(K) \cong K^\times$$

$$g \mapsto \rho(g) := 1_K$$

is a $K$-representation of $G$, called the **trivial representation** of $G$.

(b) Let $G = S_n$ ($n \geq 1$) be the symmetric group on $n$ letters. Let $\{e_1, \ldots, e_n\}$ be the standard basis of $V := K^n$. Then

$$\rho : S_n \rightarrow \text{GL}(K^n)$$

$$\sigma \mapsto \rho(\sigma) : K^n \rightarrow K^n, e_i \mapsto e_{\sigma(i)}$$

is a $K$-representation, called the **natural representation** of $S_n$.

(c) More generally, if $X$ is a finite $G$-set, i.e. a finite set endowed with a left action $\cdot : G \times X \rightarrow X$, and $V$ is a free $K$-module with basis $\{e_x \mid x \in X\}$, then

$$\rho_X : G \rightarrow \text{GL}(V)$$

$$g \mapsto \rho_X(g) : V \rightarrow V, e_x \mapsto e_{g \cdot x}$$

is a $K$-representation of $G$, called the **permutation representation**.

Clearly (b) is a special case of (c) with $G = S_n$ and $X = \{1, 2, \ldots, n\}$.

If $X = \hat{G}$ and the left action $\cdot : G \times X \rightarrow X$ is just the multiplication in $G$, then $\rho_X = : \rho_{\text{reg}}$ is called the **regular representation** of $G$. 
Definition 14.3 (Equivalent representations)

Let \( \rho_1 : G \to GL(V_1) \) and \( \rho_2 : G \to GL(V_2) \) be two representations of \( G \), where \( V_1, V_2 \) are two free \( K \)-modules of finite rank. Then \( \rho_1 \) and \( \rho_2 \) are called equivalent (or similar, or isomorphic) if there exists a \( K \)-isomorphism \( \alpha : V_1 \to V_2 \) such that \( \rho_2(g) = \alpha \circ \rho_1(g) \circ \alpha^{-1} \) for each \( g \in G \).

\[
\begin{array}{c}
V_1 \xrightarrow{\rho_1(g)} V_1 \\
\alpha^{-1} \circ \quad \circ \quad \circ \downarrow \alpha \\
V_2 \xrightarrow{\rho_2(g)} V_2
\end{array}
\]

In this case, we write \( \rho_1 \sim \rho_2 \).

Clearly \( \sim \) is an equivalence relation.

15 The group algebra and its modules

We now want to be able to see \( K \)-representations of a group \( G \) as modules, and more precisely as modules over a \( K \)-algebra depending on the group \( G \), which is called the group algebra:

Lemma-Definition 15.1 (Group algebra)

The group ring \( KG \) is the ring whose elements are the linear combinations \( \sum_{g \in G} \lambda_g g \) with \( \lambda_g \in K \), and addition and multiplication are given by

\[
\sum_{g \in G} \lambda_g g + \sum_{g \in G} \mu_g g = \sum_{g \in G} (\lambda_g + \mu_g) g \quad \text{and} \quad (\sum_{g \in G} \lambda_g g) \cdot (\sum_{h \in G} \mu_h h) = \sum_{g, h \in G} (\lambda_g \mu_h) gh
\]

respectively. Thus \( KG \) is a \( K \)-algebra, which as a \( K \)-module is free with basis \( G \). Hence we usually call \( KG \) the group algebra of \( G \) over \( K \) rather than simply group ring.

Proof: By definition \( KG \) is a free \( K \)-module with basis \( G \), and the multiplication in \( G \) is extended by \( K \)-bilinearity to the given multiplication \( \cdot : KG \times KG \to KG \). It is then straightforward that \( KG \) bears both the structures of a ring and of a \( K \)-module. Finally, axiom (A3) of \( K \)-algebras follows directly from the definition of the multiplication and the commutativity of \( K \).

Remark 15.2

Clearly the \( K \)-rank of \( KG \) is \( |G| \) and \( G \subseteq (KG)^{\times} \). Moreover, \( KG \) is commutative if and only if \( G \) is an abelian group. Also note that if \( K = \mathbb{F} \) is a field, then it is clear that \( KG \) a left Artinian ring because we may consider \( K \)-dimensions, so that by Hopkin's Theorem a \( KG \)-module is finitely generated if and only if it admits a composition series.

Proposition 15.3

(a) Any \( K \)-representation \( \rho : G \to GL(V) \) of \( G \) gives rise to a \( KG \)-module structure on \( V \), where the external composition law is defined by the map

\[
\cdot : G \times V \to V \quad \quad \quad (g, v) \mapsto g \cdot v := \rho(g)(v)
\]

extended by \( K \)-linearity to the whole of \( KG \).
(b) Conversely, every KG-module \((V, +, \cdot)\) defines a \(K\)-representation
\[
\rho_V: \quad G \longrightarrow \text{GL}(V) \\
g \mapsto \rho_V(g): V \longrightarrow V, \, v \mapsto \rho_V(g) := g \cdot v
\]
of the group \(G\).

**Proof:**

(a) Since \(V\) is a \(K\)-module, it is equipped with an internal addition \(+\) such that \((V, +)\) is an abelian group. It is then straightforward to check that the given external composition law makes \((V, +)\) into a KG-module.

(b) Clearly, it follows from the KG-module axioms that \(\rho_V(g) \in \text{GL}(V)\) and also that \(\rho_V(g_1g_2) = \rho_V(g_1) \circ \rho_V(g_2)\) for all \(g_1, g_2 \in G\), hence \(\rho_V\) is a group homomorphism.

Notice that, since \(G\) is a group, the map \(KG \longrightarrow KG\) such that \(\rho: \quad G \longrightarrow G\) is an anti-automorphism. It follows that any left KG-module \(M\) may be regarded as a right KG-module via the right \(G\)-action \(m \cdot g := g^{-1} \cdot m\). Thus the sidedness of KG-modules is not usually an issue.

**Example 9**

The trivial representation of Example 8(b) corresponds to the so-called **trivial KG-module**, that is the commutative ring \(K\) itself seen as a KG-module via the \(G\)-action
\[
\cdot: G \times K \longrightarrow K \\
(g, \lambda) \longmapsto g \cdot \lambda := \lambda
\]
extended by \(K\)-linearity to the whole of \(KG\).

**Exercise 15.4**

Let \(G\) be a finite group and let \(K\) be a commutative ring. Prove that the regular representation \(\rho_{\text{reg}}\) of \(G\) defined in Example 8(c) corresponds to the regular KG-module \(KG\) via Proposition 15.3.

**Remark 15.5**

More generally, through Proposition 15.3, we may transport terminology and properties from KG-modules to representations and conversely.

For instance, we say that a representation is **irreducible** (or **simple**) if the corresponding KG-module is irreducible (= simple). (Notice that it is tradition to use the term simple for modules, and the term irreducible for representations.)

**Lemma 15.6**

Two representations \(\rho_1: G \longrightarrow \text{GL}(V_1)\) and \(\rho_2: G \longrightarrow \text{GL}(V_2)\) are equivalent if and only if \(V_1 \cong V_2\) as KG-modules.

**Proof:** If \(\rho_1 \sim \rho_2\) and \(\alpha: V_1 \longrightarrow V_2\) is a \(K\)-isomorphism such that \(\rho_2(g) = \alpha \circ \rho_1(g) \circ \alpha^{-1}\) for each \(g \in G\), then by Proposition 15.3 for every \(v \in V_1\) and every \(g \in G\) we have
\[
g \cdot \alpha(v) = \rho_2(g) (\alpha(v)) = \alpha (\rho_1(g)(v)) = \alpha (g \cdot v),
\]
Exercise 15.9 (Lemma 15.8)

Proof: Let \( \rho : V_1 \rightarrow V_2 \) be a \( KG \)-isomorphism and for each \( g \in G \) we have
\[
\alpha(g) \circ \rho_1(g) \circ \alpha^{-1}(v) = \alpha(g) \cdot \rho_1(g) \cdot \alpha^{-1}(v) = \alpha(g \cdot \alpha^{-1}(v)) = g \cdot \alpha^{-1}(v) = g \cdot \rho_2(g)(v),
\]
hence \( \rho_2(g) = \alpha(g) \circ \rho_1(g) \circ \alpha^{-1} \) for each \( g \in G \).

Finally we introduce an ideal of \( KG \) which encodes a lot of information about \( KG \)-modules.

Proposition-Definition 15.7 (The augmentation ideal)

The map \( \varepsilon : KG \rightarrow K, \sum_{g \in G} \lambda_g g \mapsto \sum_{g \in G} \lambda_g \) is an algebra homomorphism, called the augmentation homomorphism (or map). Its kernel \( \ker(\varepsilon) = I(KG) \) is an ideal and it is called the augmentation ideal of \( KG \). The following statements hold:

(a) \( I(KG) = \{ \sum_{g \in G} \lambda_g g \in KG \mid \sum_{g \in G} \lambda_g = 0 \} = \text{ann}_{KG}(K) \) and if \( K \) is a field \( I(KG) \supseteq J(KG) \);

(b) \( KG/\ker(\varepsilon) \cong K \) as \( K \)-algebras;

(c) \( I(KG) \) is a free \( K \)-module of rank \(|G| - 1\) with \( K \)-basis \( \{ g - 1 \mid g \in G \setminus \{1\} \} \);

Proof: Clearly, the map \( \varepsilon : KG \rightarrow K \) is the unique extension by \( K \)-linearity of the trivial representation \( G \rightarrow K^\times \subseteq K, g \mapsto 1_K \) to \( KG \), hence is an algebra homomorphism and its kernel is an ideal of the algebra \( KG \).

(a) \( I(KG) = \ker(\varepsilon) = \{ \sum_{g \in G} \lambda_g g \in KG \mid \sum_{g \in G} \lambda_g = 0 \} \) by definition of \( \varepsilon \). The second equality is obvious by definition of \( \text{ann}_{KG}(K) \), and the last inclusion follows from the definition of the Jacobson radical.

(b) follows from the 1st isomorphism theorem.

(c) Let \( \sum_{g \in G} \lambda_g g \in I(KG) \). Then \( \sum_{g \in G} \lambda_g = 0 \) and hence
\[
\sum_{g \in G} \lambda_g g = \sum_{g \in G} \lambda_g g - \sum_{g \in G} \lambda_g = \sum_{g \in G} \lambda_g (g - 1) = \sum_{g \in G \setminus \{1\}} \lambda_g (g - 1),
\]
which proves that the set \( \{ g - 1 \mid g \in G \setminus \{1\} \} \) generates \( I(KG) \) as a \( K \)-module. The above computations also shows that
\[
\sum_{g \in G \setminus \{1\}} \lambda_g (g - 1) = 0 \quad \implies \quad \sum_{g \in G} \lambda_g g = 0
\]
Hence \( \lambda_g = 0 \ \forall \ g \in G \), which proves that the set \( \{ g - 1 \mid g \in G \setminus \{1\} \} \) is also \( K \)-linearly independent, hence a \( K \)-basis of \( I(KG) \).

Lemma 15.8

If \( K \) is a field of positive characteristic \( p \) and \( G \) is \( p \)-group, then \( I(KG) = J(KG) \).

Exercise 15.9 (Proof of Lemma 15.8. Proceed as indicated.)

(a) (Facultative: you can accept this result and treat (b), (c) and (d) only.) Recall that an ideal \( I \) of a ring \( R \) is called a nil ideal if each element of \( I \) is nilpotent. Prove that if \( I \) is a nil left ideal in a left Artinian ring \( R \) then \( I \) is nilpotent.
(b) Prove that \( g - 1 \) is a nilpotent element for each \( g \in G \setminus \{1\} \) and deduce that \( I(KG) \) is a nil ideal of \( KG \).

(c) Deduce from (a) and (b) that \( I(KG) \subseteq J(KG) \) using Exercise 10 on Exercise Sheet 2.

(d) Conclude that \( I(KG) = J(KG) \) using Proposition-Definition 15.7.

16 Semisimplicity and Maschke’s Theorem

Throughout this section, we assume that \( K \) is a field.

Our first aim is to prove that the semisimplicity of the group algebra depends on both the characteristic of the field and the order of the group.

Theorem 16.1 (Maschke)

If \( \text{char}(K) \nmid |G| \), then \( KG \) is a semisimple \( K \)-algebra.

Proof: By Proposition-Definition 11.2, we need to prove that every s.e.s. \( 0 \rightarrow L \xrightarrow{\theta} M \xrightarrow{\psi} N \rightarrow 0 \) of \( KG \)-modules splits. However, the field \( K \) is clearly semisimple (again by Proposition-Definition 11.2). Hence any such sequence regarded as a s.e.s. of \( K \)-vector spaces and \( K \)-linear maps splits. So let \( \sigma : N \rightarrow M \) be a \( K \)-linear section for \( \psi \) and set

\[
\bar{\sigma} := \frac{1}{|G|} \sum_{g \in G} g^{-1} \sigma g : N \rightarrow M \quad n \mapsto \frac{1}{|G|} \sum_{g \in G} g^{-1} \sigma (gn).
\]

We may divide by \( |G| \), since \( \text{char}(K) \nmid |G| \) implies that \( |G| \in K^\times \). Now, if \( h \in G \) and \( n \in N \), then

\[
\bar{\sigma}(hn) = \frac{1}{|G|} \sum_{g \in G} g^{-1} \sigma (ghn) = h \frac{1}{|G|} \sum_{g \in G} (gh)^{-1} \sigma (ghn) = h \bar{\sigma}(n)
\]

and

\[
\psi \bar{\sigma}(n) = \frac{1}{|G|} \sum_{g \in G} \psi (g^{-1} \sigma (gn)) \overset{\psi\text{-lin}}{=} \frac{1}{|G|} \sum_{g \in G} g^{-1} \psi \sigma (gn) = \frac{1}{|G|} \sum_{g \in G} g^{-1} gn = n
\]

where the last-but-one equality holds because \( \psi \sigma = \text{Id}_N \). Thus \( \bar{\sigma} \) is a \( KG \)-linear section for \( \psi \).

Example 10

If \( K = \mathbb{C} \) is the field of complex numbers, then \( \mathbb{C}G \) is a semisimple \( \mathbb{C} \)-algebra, since \( \text{char}(\mathbb{C}) = 0 \).

It turns out that the converse to Maschke’s theorem also holds. We obtain it using the properties of the augmentation ideal.

Theorem 16.2 (Converse of Maschke’s Theorem)

If \( KG \) is a semisimple \( K \)-algebra, then \( \text{char}(K) \nmid |G| \).
Proof: Set char($K$) = $p$ and let us assume that $p \mid |G|$. In particular $p$ must be a prime number. We have to prove that then $KG$ is not semisimple.

Claim: If $0 \neq V \subseteq KG$ is a $KG$-submodule of $KG^o$, then $V \cap I(KG) \neq 0$.

Indeed: Let $v = \sum_{g \in G} \lambda_g g \in V \setminus \{0\}$. If $\varepsilon(v) = 0$ we are done. Else, set $t := \sum_{h \in G} h$. Then

$$\varepsilon(t) = \sum_{h \in G} 1 = |G| = 0$$

as char($K$) $\mid |G|$. Hence $t \in I(KG)$. Now consider the element $tv$. On the one hand $tv \in V$ since $V$ is a submodule of $KG^o$, and on the other hand $tv \in I(KG) \setminus \{0\}$ since

$$tv = \left( \sum_{h \in G} h \right) \left( \sum_{g \in G} \lambda_g g \right) = \sum_{h, g \in G} (1_K \cdot \lambda_g) h g = \sum_{g \in G} \left( \sum_{x \in G} \lambda_g \right) x = \sum_{x \in G} \varepsilon(v)x \Rightarrow \varepsilon(tv) = \sum_{x \in G} \varepsilon(v) = |G| \varepsilon(v) = 0.$$ 

The Claim implies that $I(KG)$, which is a $KG$-submodule by definition, cannot have a complement in $KG^o$. Therefore, by Proposition-Definition 11.1 $KG^o$ is not semisimple and hence $KG$ is not semisimple by Theorem-Definition 11.2.

In the case the field $K$ is algebraically closed, the following Exercise offers a second proof exploiting Artin-Wedderburn.

**Exercise 16.3 (Proof of the Converse of Maschke’s Theorem for $K = \overline{K}$)**

Assume $K = \overline{K}$ is an algebraically closed field of characteristic $p$ with $p \mid |G|$. Set $T := \langle \sum_{g \in G} g \rangle_K$.

(a) Prove that we have a series of $KG$-submodules given by $KG^o \supseteq I(KG) \supseteq T \supseteq 0$.

(b) Deduce that $KG^o$ has at least two composition factors isomorphic to the trivial module $K$.

(c) Deduce that $KG$ is not a semisimple $K$-algebra using Theorem 13.2.

### 17 Simple modules over algebraically closed fields

Throughout this section, we assume that $K = \overline{K}$ is an algebraically closed field.

As mentioned in Chapter 2, §13 this hypothesis may always be replaced by the weaker assumption that the field $K$ is a splitting field for the group algebra $KG$, which we simply call a splitting field for $G$.

We state here some elementary facts about simple $KG$-modules, which we obtain as consequences of the Artin-Wedderburn structure theorem.

**Corollary 17.1**

There are only finitely many isomorphism classes of simple $KG$-modules, or equivalently, there are only finitely many irreducible $K$-representations of $G$, up to similarity.

Proof: Since $K = \overline{K}$, the first claim follows from Corollary 13.4 and the equivalent characterisation from Proposition 15.3.
Corollary 17.2

If \( G \) is an abelian group, then any simple \( KG \)-module is one-dimensional, or equivalently, all irreducible \( K \)-representations of \( G \) have degree one.

Proof: Since \( K = \overline{K} \) and \( KG \) is commutative the first claim follows from Corollary 13.5 and the equivalent characterisation from Proposition 15.3.

Corollary 17.3

Let \( p \) be a prime number. If \( G \) is a \( p \)-group and \( \text{char}(K) = p \), then the trivial module is the unique simple \( KG \)-module, up to isomorphism.

Proof: By Lemma 15.8 we have \( J(KG) = I(KG) \). Thus \( KG/J(KG) \cong K \) as \( K \)-algebras by Proposition-Definition 15.7. Now, as \( K \) is commutative, \( Z(K) = K \), and it follows from Corollary 13.4 that

\[ |M(KG)| = \dim_K Z(KG/J(KG)) = \dim_K K = 1. \]

Remark 17.4

Another standard proof for Corollary 17.3 consists in using a result of Brauer’s stating that \( |M(KG)| \) equals the number of conjugacy classes of \( G \) of order not divisible by the characteristic of the field \( K \).

Corollary 17.5

If \( \text{char}(K) \nmid |G| \), then \( |G| = \sum_{S \in M(KG)} \dim_K(S)^2 \).

Proof: Since \( \text{char}(K) \nmid |G| \), the group algebra \( KG \) is semisimple by Maschke’s Theorem. Thus it follows from Theorem 13.2 that

\[ \sum_{S \in M(KG)} \dim_K(S)^2 = \dim_K(KG) = |G|. \]