Exercise 45: Write $L = sp_4(k)$, $\tilde{L} = so_5(k)$, and recall that

$$L = \left\{ \begin{pmatrix} x & a & b & c \\ d & y & c & e \\ f & g & -x & -d \\ g & i & -a & -y \end{pmatrix} : x,y,a,b,c,d,e,f,g,i \in k \right\},$$

$$\tilde{L} = \left\{ \begin{pmatrix} x & a & b & 0 & c \\ d & y & e & -c & 0 \\ f & g & 0 & -b & -e \\ 0 & i & -f & -x & -d \\ -i & 0 & -g & -a & -y \end{pmatrix} : x,y,a,b,c,d,e,f,g,i \in k \right\}$$

Also set $h := a(e_{11}-e_{33}) + b(e_{22}-e_{44}) \in L$, $\tilde{h} := a(e_{11}-e_{44}) + b(e_{22}-e_{55}) \in \tilde{L}$.

One checks (using the identity $[e_{ij}, e_{kl}] = \delta_{jk}e_{il} - \delta_{il}e_{kj}$) that

$$h(e_{12}-e_{43}) = (a-b)(e_{12}-e_{43})$$
$$h(e_{13}) = 2a\,e_{13}$$
$$h(e_{14}+e_{23}) = (a+b)(e_{14}+e_{23})$$
$$h(e_{24}) = 2b\,e_{24}$$

Set $\alpha : H \to K$, $h \mapsto a-b$, $\quad \beta : H \to K$, $h \mapsto 2b$, $\quad e_\alpha = e_{12}-e_{43}$, $e_\beta = e_{24}$, $e_{\alpha+\beta} = e_{14}+e_{23}$, $e_{2\alpha+\beta} = e_{13}$

$$\tilde{h}(e_{12}-e_{54}) = (a-b)(e_{12}-e_{54})$$
$$\tilde{h}(e_{13}-e_{34}) = a(e_{13}-e_{34})$$
$$\tilde{h}(e_{15}-e_{24}) = (a+b)(e_{15}-e_{24})$$
$$\tilde{h}(e_{23}-e_{35}) = b(e_{23}-e_{35})$$

Set $\tilde{\alpha} : H \to K$, $h \mapsto a-b$, $\quad \tilde{\beta} : H \to K$, $h \mapsto b$, $\quad e_{\tilde{\alpha}} = e_{12}-e_{54}$, $e_{\tilde{\beta}} = e_{23}-e_{35}$, $e_{\tilde{\alpha}+\tilde{\beta}} = e_{13}-e_{34}$, $e_{\tilde{\alpha}+2\tilde{\beta}} = e_{15}-e_{24}$

$h_\alpha = [e_\alpha, e_{-\alpha}] = [e_{12}-e_{43}, e_{21}-e_{34}] = (e_{11}-e_{33}) - (e_{22}-e_{44})$ $\quad$ $h_{\tilde{\alpha}} = [e_{12}-e_{54}, e_{21}-e_{45}] = (e_{11}-e_{44}) - (e_{22}-e_{55})$

$h_\beta = [e_{24}, e_{42}] = e_{22}-e_{44}$ $\quad$ $h_{\tilde{\beta}} = [e_{23}-e_{35}, e_{32}-e_{53}] = e_{22}-e_{55}$

Setting $h_\alpha := h_1 - h_2$, $x_\alpha = e_{12} - e_{43}$, $x_{-\alpha} = x_\alpha^t$, one checks that $[x_\alpha, x_{-\alpha}] = h_\alpha$, $[h_\alpha, x_\alpha] = 2x_\alpha$,
$h_\beta := 2h_2$, $x_\beta = e_{24}$, $x_{-\beta} = x_\beta^t$ $[h_\alpha, x_{-\alpha}] = -2x_{-\alpha}$.

Similarly, $h_{\tilde\alpha} := \tilde h_1 - \tilde h_2$, $x_{\tilde\alpha} = e_{12} - e_{54}$, $x_{-\tilde\alpha} = x_{\tilde\alpha}^t$, $[x_{\tilde\alpha}, x_{-\tilde\alpha}] = h_{\tilde\alpha}$, $[h_{\tilde\alpha}, x_{\tilde\alpha}] = 2x_{\tilde\alpha}$, $[h_{\tilde\alpha}, x_{-\tilde\alpha}] = -2x_{-\tilde\alpha}$
$h_{\tilde\beta} := \tilde h_2$, $x_{\tilde\beta} = e_{23} - e_{35}$, $x_{-\tilde\beta} = x_{\tilde\beta}^t$

Goal: To define a linear map $\varphi : L \longrightarrow \tilde L$ that is an iso. of Lie algebras.

Idea: φ should send $x_{\pm\alpha}$ to $x_{\pm\tilde\beta}$, and $x_{\pm\beta}$ to $x_{\pm\tilde\alpha}$.

Q: What does it imply on other basis elements?

Observe that we should have $\underset{①}{\varphi([h_\alpha, x_\alpha])} = \underset{②}{[\varphi(h_\alpha), \varphi(x_\alpha)]}$.

$$\left.\begin{array}{l} ① = \varphi(2x_\alpha) = 2\varphi(x_\alpha) = 2x_{\tilde\beta} \\ ② = [\varphi(h_\alpha), x_{\tilde\beta}] \end{array}\right\} \Rightarrow \text{might as well set } \varphi(h_\alpha) = h_{\tilde\beta}$$

Similarly $\varphi(h_\beta) = h_{\tilde\alpha}$.

Go on like this... $\underset{①}{\varphi([x_\alpha, x_\beta])} = \underset{②}{[\varphi(x_\alpha), \varphi(x_\beta)]}$

$$\left.\begin{array}{l} ① = \varphi([e_{12}-e_{43}, e_{24}]) = \varphi(e_{14}+e_{23}) \\ ② = [x_{\tilde\beta}, x_{\tilde\alpha}] = [e_{23}-e_{35}, e_{12}-e_{54}] = -e_{13}+e_{34} \end{array}\right\} \varphi(\underbrace{e_{14}+e_{23}}_{x_{\alpha+\beta}}) = \underbrace{-e_{13}+e_{34}}_{-x_{\tilde\alpha+\tilde\beta}}$$

Etc: Get
$$\begin{array}{rcl} \varphi : L & \longrightarrow & \tilde L \\ h_\alpha & \longmapsto & h_{\tilde\beta} \\ h_\beta & \longmapsto & h_{\tilde\alpha} \\ x_{\pm\alpha} & \longmapsto & x_{\pm\tilde\beta} \\ x_{\pm\beta} & \longmapsto & x_{\pm\tilde\alpha} \\ x_{\pm(\alpha+\beta)} & \longmapsto & -x_{\pm(\tilde\alpha+\tilde\beta)} \\ x_{\pm(2\alpha+\beta)} & \longmapsto & \frac{1}{2}x_{\pm(\tilde\alpha+2\tilde\beta)} \end{array}$$
, where $x_{-\alpha-\beta} = x_{\alpha+\beta}^t$

Can check that $\varphi([v,w]) = [\varphi(v), \varphi(w)]$ $\forall v, w \in L$.

Exercise 46. 1. Let $1 \leq j \leq n$, $\alpha \in R^+ - \{\alpha_j\}$.

As $\alpha \neq \alpha_j$, we get that $\alpha = \sum_{r=1}^{n} a_r \alpha_r$, with $a_s > 0$ for some $s \neq j$.

Now $s_{\alpha_j}(\alpha) = \alpha - \langle \alpha, \alpha_j \rangle \alpha_j$

$$= \sum_{\substack{r=1 \\ r \neq j}}^{n} a_r \alpha_r + (a_j - \langle \alpha, \alpha_j \rangle) \alpha_j$$

$\Rightarrow$ the coeff. before α_s is $a_s > 0$, forcing $s_{\alpha_j}(\alpha) \in R^+$ // 1.

2. Let $\alpha \in R^+$, $\alpha = \sum a_r \alpha_r$, with $a_r \geq 0$ $\forall r$.

As $(-,-)$ is a scalar product, one has

$$0 < (\alpha, \alpha) = \sum \underbrace{a_r}_{\geq 0} (\alpha, \alpha_r)$$

$\Rightarrow$ $(\alpha, \alpha_r) > 0$ for at least one r // 2.

3. Let $\alpha \in \Phi^+ - \Delta$.

By 2. $\exists$ $1 \leq j \leq n$ s.t. $\langle \alpha, \alpha_j \rangle > 0$.

Let then $B := s_{\alpha_j}(\alpha) = \alpha - \langle \alpha, \alpha_j \rangle \alpha_j$.

We have

- $s_{\alpha_j}(B) = s_{\alpha_j}(s_{\alpha_j}(\alpha)) = \alpha$ ✓
- $B = s_{\alpha_j}(\alpha) \in R^+$ by 1. ✓
- $ht(B) = ht(\alpha) - \langle \alpha, \alpha_j \rangle < ht(\alpha)$ ✓

// 3.

Algorithm to find R^+: Set $R_0^+ := \Delta = \{\alpha_1, \dots, \alpha_n\}$

$$R_1^+ := R_0^+ \cup \left(\bigcup_{j=1}^{n} s_{\alpha_j}(R_0^+) \cap R^+ \right)$$

$$\vdots$$

$$R_k^+ := R_{k-1}^+ \cup \left(\bigcup_{j=1}^{n} s_{\alpha_j}(R_{k-1}^+ \setminus R_{k-2}^+) \cap R^+ \right)$$

4. Illustration of the algorithm:

$\boxed{4.1\ R = A_2,}$ $\Delta = \{\alpha_1, \alpha_2\}$, $C(\Delta) = \begin{pmatrix} 2 & -1 \\ -1 & 2 \end{pmatrix}$

- $R_0^+ = \Delta$ ✓

- $R_1^+ = ?$

Need to compute $s_{\alpha_j}(\alpha_r)$ $\forall\ 1 \leq r \neq j \leq 2$ (indeed, $s_{\alpha_j}(\alpha_j) = -\alpha_j \notin R^+$)

$s_{\alpha_1}(\alpha_2) = \alpha_2 - \langle \alpha_2, \alpha_1 \rangle \alpha_1 = \alpha_2 + \alpha_1$

$s_{\alpha_2}(\alpha_1) = \alpha_1 - \langle \alpha_1, \alpha_2 \rangle \alpha_2 = \alpha_1 + \alpha_2$

$\rightsquigarrow$ $R_1^+ = R_0^+ \cup \{\alpha_1 + \alpha_2\} = \{\alpha_1, \alpha_2, \alpha_1 + \alpha_2\}$

- $R_2^+ = ?$

Need to compute $s_{\alpha_j}(\underbrace{\alpha_1 + \alpha_2}_{2-1=1})$ $\forall j = 1, 2$

$s_{\alpha_1}(\alpha_1 + \alpha_2) = \alpha_1 + \alpha_2 - \widetilde{\langle \alpha_1 + \alpha_2, \alpha_1 \rangle} \alpha_1 = \alpha_1 + \alpha_2 - \alpha_1 = \alpha_2$

$s_{\alpha_2}(\alpha_1 + \alpha_2) = \dots = \alpha_1$

Conclusion: $R^+ = R_0^+ \cup R_1^+ = \{\alpha_1, \alpha_2, \alpha_1 + \alpha_2\}$

$\boxed{4.2\ R = G_2}$ $\Delta = \{\alpha_1, \alpha_2\}$, $C(\Delta) = \begin{pmatrix} 2 & -1 \\ -3 & 2 \end{pmatrix}$

- $R_0^+ = \Delta$ ✓

- $R_1^+ = ?$

Again, need to compute $s_{\alpha_j}(\alpha_r)$ $\forall\ 1 \leq r \neq j \leq 2$

$s_{\alpha_1}(\alpha_2) = \alpha_2 - \overbrace{\langle \alpha_2, \alpha_1 \rangle}^{= -3} \alpha_1 = \alpha_2 + 3\alpha_1$

$s_{\alpha_2}(\alpha_1) = \alpha_1 - \langle \alpha_1, \alpha_2 \rangle \alpha_2 = \alpha_1 + \alpha_2$

$\rightsquigarrow$ $R_1^+ = R_0^+ \cup \{\alpha_1 + \alpha_2, 3\alpha_1 + \alpha_2\} = \{\alpha_1, \alpha_2, \alpha_1 + \alpha_2, 3\alpha_1 + \alpha_2\}$

$R_2^+ = ?$

Need to compute $s_{\alpha_j}(\alpha_1 + \alpha_2)$, $s_{\alpha_j}(3\alpha_1 + \alpha_2)$, $j = 1, 2$.

Rmk: already know that $s_{\alpha_2}(\alpha_1 + \alpha_2) = \alpha_1$, $s_{\alpha_1}(3\alpha_1 + \alpha_2) = \alpha_2 \dots$

$$s_{\alpha_1}(\alpha_1+\alpha_2) = \alpha_1+\alpha_2 - \overset{2-3=-1}{\langle \alpha_1+\alpha_2,\alpha_1\rangle}\alpha_1 = \alpha_1+\alpha_2+\alpha_1 = 2\alpha_1+\alpha_2$$

$$s_{\alpha_2}(3\alpha_1+\alpha_2) = 3\alpha_1+\alpha_2 - \overset{-3+2=-1}{\langle 3\alpha_1+\alpha_2,\alpha_2\rangle}\alpha_2 = 3\alpha_1+2\alpha_2$$

$$\leadsto R_2^+ = \{\alpha_1,\alpha_2,\alpha_1+\alpha_2,\ 2\alpha_1+\alpha_2,\ 3\alpha_1+\alpha_2,\ 3\alpha_1+2\alpha_2\}$$

Finally, one checks that $s_{\alpha_j}(R_2^+)\subset R_2^+$, hence $\Delta = R_2^+$

$\boxed{4.3 : R = B_3}$ $\Delta = \{\alpha_1,\alpha_2,\alpha_3\}$, $C(\Delta) = \begin{pmatrix} 2 & -1 & 0 \\ -1 & 2 & -2 \\ 0 & -1 & 2 \end{pmatrix}$.

- $R_0^+ = \Delta$ ✓
- $R_1^+ = ?$

$$s_{\alpha_1}(\alpha_2) = \alpha_2 - \overset{-1}{\langle \alpha_2,\alpha_1\rangle}\alpha_1 = \alpha_1+\alpha_2, \qquad s_{\alpha_2}(\alpha_1) = \alpha_1 - \overset{-1}{\langle \alpha_1,\alpha_2\rangle}\alpha_2 = \alpha_1+\alpha_2.$$

$$s_{\alpha_1}(\alpha_3) = 0 = s_{\alpha_3}(\alpha_1)$$

$$s_{\alpha_2}(\alpha_3) = \alpha_3 - \overset{-1}{\langle \alpha_3,\alpha_2\rangle}\alpha_2 = \alpha_2+\alpha_3$$

$$s_{\alpha_3}(\alpha_2) = \alpha_2 - \overset{-2}{\langle \alpha_2,\alpha_3\rangle}\alpha_3 = \alpha_2+2\alpha_3$$

$$\leadsto R_1^+ = \{\alpha_1,\alpha_2,\alpha_3,\ \alpha_1+\alpha_2,\ \alpha_2+\alpha_3,\ \alpha_2+2\alpha_3\}$$

- $R_2^+ = ?$

Rmk: already know that $s_{\alpha_1}(\alpha_1+\alpha_2)=\alpha_2$, $s_{\alpha_2}(\alpha_1+\alpha_2)=\alpha_1$, $s_{\alpha_2}(\alpha_2+\alpha_3)=\alpha_3$, $s_{\alpha_3}(\alpha_2+2\alpha_3)=\alpha_2$

$$s_{\alpha_1}(\alpha_2+\alpha_3) = \alpha_2+\alpha_3 - \overset{-1}{\langle \alpha_2+\alpha_3,\alpha_1\rangle}\alpha_1 = \alpha_1+\alpha_2+\alpha_3$$

$$s_{\alpha_1}(\alpha_2+2\alpha_3) = \alpha_2+2\alpha_3 - \overset{-1}{\langle \alpha_2+2\alpha_3,\alpha_1\rangle}\alpha_1 = \alpha_1+\alpha_2+2\alpha_3$$

$$s_{\alpha_2}(\alpha_2+2\alpha_3) = \alpha_2+2\alpha_3 - \overset{2-2=0}{\langle \alpha_2+2\alpha_3,\alpha_2\rangle}\alpha_2 = \alpha_2+2\alpha_3$$

$$s_{\alpha_3}(\alpha_1+\alpha_2+\alpha_3) = \alpha_1+\alpha_2+\alpha_3 - \overset{0-2+2=0}{\langle \alpha_1+\alpha_2+\alpha_3,\alpha_3\rangle}\alpha_3 = \alpha_1+\alpha_2+\alpha_3$$

$$\leadsto R_2^+ = \{\alpha_1,\alpha_2,\alpha_3,\ \alpha_1+\alpha_2,\ \alpha_2+\alpha_3,\ \alpha_2+2\alpha_3,\ \alpha_1+\alpha_2+\alpha_3,\ \alpha_1+\alpha_2+2\alpha_3\}$$

- $R_3^+ = ?$ Again, know the values of $s_{\alpha_1}(\alpha_1+\alpha_2+\alpha_3)$, $s_{\alpha_3}(\alpha_1+\alpha_2+\alpha_3)$, $s_{\alpha_1}(\alpha_1+\alpha_2+2\alpha_3)$

$$s_{\alpha_2}(\alpha_1+\alpha_2+\alpha_3) = \alpha_1+\alpha_2+\alpha_3 - \overset{-1+2-1=0}{\langle \alpha_1+\alpha_2+\alpha_3,\alpha_2\rangle}\alpha_2 = \alpha_1+\alpha_2+\alpha_3$$

$$s_{\alpha_2}(\alpha_1+\alpha_2+2\alpha_3) = \alpha_1+\alpha_2+2\alpha_3 - \overset{-1+2-2=-1}{\langle \alpha_1+\alpha_2+2\alpha_3,\alpha_2\rangle}\alpha_2 = \alpha_1+2\alpha_2+2\alpha_3$$

$$s_{\alpha_3}(\alpha_1+\alpha_2+2\alpha_3) \in R_2^+$$

$$\leadsto R_3^+ = R_2^+ \cup \{\alpha_1+2\alpha_2+2\alpha_3\} = \{\alpha_1,\alpha_2,\alpha_3,\alpha_1+\alpha_2,\alpha_2+\alpha_3,\alpha_1+\alpha_2+\alpha_3,\alpha_2+2\alpha_3,\alpha_1+\alpha_2+2\alpha_3,\alpha_1+2\alpha_2+2\alpha_3\}$$

- Finally, one checks that $s_{\alpha_j}(\alpha_1+2\alpha_2+2\alpha_3) \in R_2^+ \ \forall j \leadsto R^+ = R_3^+$

$\boxed{4.4.\ R = D_4}$ $\Delta = \{a_1, a_2, a_3, a_4\}$ $C(\Delta) = \begin{pmatrix} 2 & -1 & & \\ -1 & 2 & -1 & -1 \\ & -1 & 2 & 0 \\ & -1 & 0 & 2 \end{pmatrix}$

- $R_0^+ = \Delta$ ✓

- $R_1^+ = ?$

$s_{a_1}(a_2) = s_{a_2}(a_1) = a_1 + a_2$

$s_{a_1}(a_3) = s_{a_1}(a_4) = s_{a_3}(a_1) = s_{a_4}(a_1) = 0$

$s_{a_2}(a_3) = a_2 + a_3 = s_{a_3}(a_2)$

$s_{a_2}(a_4) = a_2 + a_4 = s_{a_4}(a_2)$

$s_{a_3}(a_4) = s_{a_4}(a_3) = 0$

$\rightsquigarrow R_1^+ = \{a_1, a_2, a_3, a_4, a_1 + a_2, a_2 + a_3, a_2 + a_4\}$

- $R_2^+ = ?$

$s_{a_3}(a_1 + a_2) = a_1 + a_2 + a_3 = s_{a_1}(a_2 + a_3)$

$s_{a_4}(a_1 + a_2) = a_1 + a_2 + a_4 = s_{a_1}(a_2 + a_4)$

$s_{a_3}(a_2 + a_4) = a_2 + a_3 + a_4 = s_{a_4}(a_2 + a_3)$

$\rightsquigarrow R_2^+ = R_1^+ \cup \{a_1 + a_2 + a_3, a_1 + a_2 + a_4, a_2 + a_3 + a_4\}$

- $R_3^+ = ?$

$s_{a_1}(a_2 + a_3 + a_4) = s_{a_3}(a_1 + a_2 + a_4) = s_{a_4}(a_1 + a_2 + a_3) = a_1 + a_2 + a_3 + a_4$

$\rightsquigarrow R_3^+ = R_2^+ \cup \{a_1 + a_2 + a_3 + a_4\}$

$R_4^+ = ?$

$s_{a_2}(a_1 + a_2 + a_3 + a_4) = a_1 + 2a_2 + a_3 + a_4$

$a_2 + a_3 + a_4$

$\rightsquigarrow R_5^+ = \{a_1, a_2, a_3, a_4, a_1 + a_2, a_2 + a_3, a_2 + a_4, a_1 + a_2 + a_3, a_1 + a_2 + a_4, a_1 + a_2 + a_3 + a_4, a_1 + 2a_2 + a_3 + a_4\}$

Notice: $s_{a_j}(a_1 + 2a_2 + a_3 + a_4) \in R_5^+ \quad \forall j$

$$\Rightarrow R^+ = R_5^+$$

Exercise 47: Consider the map

$$\varphi : \{x,y,z\} \longrightarrow \langle w \rangle_K .$$
$$x,z \longmapsto 0$$
$$y \longmapsto w$$

$\Rightarrow$ Extends uniquely to $\varphi : \langle x,y,z \rangle_K \twoheadrightarrow \langle w \rangle_K$ Linear map.

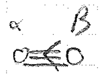
Now $\cdot\ \varphi([xy]-z) = [\varphi(x),\varphi(y)] - \varphi(z) = [0,w]-0 = 0$

$\cdot\ \varphi([yz]-x) = [\varphi(y),\varphi(z)] - \varphi(x) = [w,0]-0 = 0$

$\cdot\ \varphi([zx]-x) = [\varphi(z),\varphi(x)] - \varphi(y) = 0$

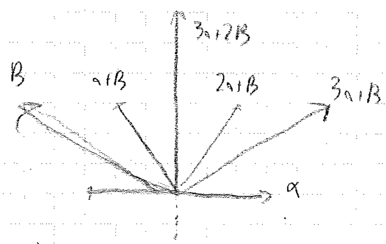
$\Rightarrow$ Get $\bar{\varphi} : L \twoheadrightarrow \langle w \rangle_k$ morph. of Lie algebras

As $x,y \in \ker\varphi$, get $\boxed{L \cong \langle w \rangle_k}$

Exercise 48: 

, pos. long roots : $\{\beta, 3\alpha+\beta, 3\alpha+2\beta\} =: \Phi'^{+}$.

$$\begin{pmatrix} 2 & -1 \\ -3 & 2 \end{pmatrix}$$



(R1) Clear $(0 \notin \Phi', |\Phi'| < \infty, \mathbb{R}^2 = \langle \Phi' \rangle)$ ✓

(R2) Clear as well, as reflections preserve lengths ✓

(R3) Obvious as well

(R4) True, as it was for whole root syst.

Set $\tilde{\alpha} := 3\alpha+\beta$, $\tilde{\beta} := \beta$. Then $\langle \tilde{\alpha}, \tilde{\beta} \rangle = \langle 3\alpha+\beta, \beta \rangle = -3+2 = -1$

$\langle \tilde{\beta}, \tilde{\alpha} \rangle = \langle \beta, 3\alpha+\beta \rangle \overset{*}{=} \langle 3\alpha+\beta, \beta \rangle = -1$

* since length β = length $3\alpha+\beta$

$\Rightarrow$ type Φ' : A_2

Type F_4: $\overset{\alpha_1}{\circ} - \overset{\alpha_2}{\circ} \Rightarrow \overset{\alpha_3}{\circ} - \overset{\alpha_4}{\circ}$ $\quad \begin{pmatrix} 2 & -1 & & \\ -1 & 2 & -2 & \\ & -1 & 2 & -1 \\ & & -1 & 2 \end{pmatrix}$, $\Delta = \{\alpha_1, \alpha_2, \alpha_3, \alpha_4\}$

We start by determining R^+, using the method introduced in Ex. 46.

In red: long roots, in blue: short roots

Notation: We write $\boxed{abcd = a\alpha_1 + b\alpha_2 + c\alpha_3 + d\alpha_4}$ (When convienient)
For example, $\alpha_2 = 0100$.

$R_0^+ = \Delta = \{\alpha_1, \alpha_2, \alpha_3, \alpha_4\}$

$\langle \alpha_1, \alpha_2 \rangle = \langle \alpha_2, \alpha_1 \rangle = -1 \Rightarrow \alpha_1 + \alpha_2 \in R^+$. Also, it is conjugate to α_1 (indeed, $\alpha_1 + \alpha_2 = s_{\alpha_2}(\alpha_1)$), so it is long: $\alpha_1 + \alpha_2 \in R^+$.

$\langle \alpha_2, \alpha_3 \rangle = -2 \Rightarrow \alpha_2 + 2\alpha_3 \in R^+$, $\langle \alpha_3, \alpha_2 \rangle = -1 \Rightarrow \alpha_2 + \alpha_3 \in R^+$, $\langle \alpha_3, \alpha_4 \rangle = \langle \alpha_4, \alpha_3 \rangle = -1 \Rightarrow \alpha_3 + \alpha_4 \in R^+$

$R_1^+ = \Delta \cup \{\alpha_1 + \alpha_2, \alpha_2 + \alpha_3, \alpha_3 + \alpha_4, \alpha_2 + 2\alpha_3\}$

$\langle \alpha_1 + \alpha_2, \alpha_3 \rangle = -2 \Rightarrow \alpha_1 + \alpha_2 + 2\alpha_3 \in R^+$, $\langle \alpha_2 + \alpha_3, \alpha_1 \rangle = -1 \Rightarrow \alpha_1 + \alpha_2 + \alpha_3 \in R^+$, $\langle \alpha_2 + \alpha_3, \alpha_4 \rangle = -1 \Rightarrow \alpha_2 + \alpha_3 + \alpha_4 \in R^+$,
$\langle \alpha_3 + \alpha_4, \alpha_2 \rangle = -1 \Rightarrow \alpha_2 + \alpha_3 + \alpha_4 \in R^+$, $\langle \alpha_2 + 2\alpha_3, \alpha_1 \rangle = -1 \Rightarrow \alpha_1 + \alpha_2 + 2\alpha_3 \in R^+$, $\langle \alpha_2 + 2\alpha_3, \alpha_4 \rangle = -2 \Rightarrow \alpha_2 + 2\alpha_3 + 2\alpha_4 \in R^+$

$R_2^+ = R_1^+ \cup \{1110, 0111, 1120, 0122\}$

$\langle 1110, \alpha_4 \rangle = -1 \Rightarrow 1111 \in R^+$, $\langle 0111, \alpha_3 \rangle = -2+2-1 = -1 \Rightarrow 0121 \in R^+$, $\langle 1120, \alpha_2 \rangle = -1+2-2 = -1 \Rightarrow 1220 \in R^+$,
$\langle 1120, \alpha_4 \rangle = -2 \Rightarrow 1122 \in R^+$

$R_3^+ = R_2^+ \cup \{1111, 0121, 1220, 1122\}$

$\langle 1111, \alpha_3 \rangle = -2+2-1 = -1 \Rightarrow 1121 \in R^+$, $\langle 1220, \alpha_4 \rangle = -2 \Rightarrow 1222 \in R^+$, $\langle 0121, \alpha_1 \rangle = -1 \Rightarrow 1121 \in R^+$,
$\langle 1122, \alpha_2 \rangle = -1+2-2 = -1 \Rightarrow 1222 \in R^+$

$R_4^+ = R_3^+ \cup \{1121, 1222\}$

$\langle 1121, \alpha_2 \rangle = -1+2-2 = -1 \Rightarrow 1221 \in R^+$, $\langle 1222, \alpha_3 \rangle = -4+4-2 \Rightarrow 1242 \in R^+$

$R_5^+ = R_4^+ \cup \{1221, 1242\}$

$\langle 1221, \alpha_3 \rangle = -2+4-1 = -1 \Rightarrow 1231 \in R^+$, $\langle 1242, \alpha_2 \rangle = -1+2-4 = -1 \Rightarrow 1342 \in R^+$

$\Rightarrow R_6^+ = R_5^+ \cup \{1231, 1342\}$

$\langle 1231, \alpha_4 \rangle = -3+2 = -1 \Rightarrow 1232 \in R^+$, $\langle 1342, \alpha_1 \rangle = -3+2 = -1 \Rightarrow 2342 \in R^+$

$\Rightarrow R_7^+ = R_6^+ \cup \{1232, 2342\}$

Finally, $\langle 1232, \alpha_1 \rangle = 0$, $\langle 1232, \alpha_2 \rangle = -1+4-3 = 0$, $\langle 1232, \alpha_3 \rangle = -4+6-2 = 0$, $\langle 1232, \alpha_4 \rangle = -3+4 = 1$,
$\langle 2342, \alpha_1 \rangle = 1$, $\langle 2342, \alpha_2 \rangle = 0$, $\langle 2342, \alpha_3 \rangle = 0$, $\langle 2342, \alpha_4 \rangle = 0$

$\Rightarrow$ Done!

$R^+ = \{1000, 0100, 0010, 0001\} \cup \{1100, 0110, 0011\} \cup \{1110, 0120, 0111\} \cup \{1120, 1111, 0121\}$
$\cup \{1220, 1121, 0122\} \cup \{1221, 1122\} \cup \{1222, 1231\}$
$\cup \{1232\} \cup \{1242\} \cup \{1342\} \cup \{2342\}$

(Here Grouped by height)

$\Rightarrow R^+_{long} = \{\alpha_1, \alpha_2, \alpha_1+\alpha_2, \alpha_2+2\alpha_3, \alpha_1+\alpha_2+2\alpha_3, \alpha_1+2\alpha_2+2\alpha_3, \alpha_2+2\alpha_3+2\alpha_4, \alpha_1+\alpha_2+2\alpha_3+2\alpha_4, \alpha_1+2\alpha_2+2\alpha_3+2\alpha_4,$
$\alpha_1+2\alpha_2+4\alpha_3+2\alpha_4, \alpha_1+3\alpha_2+4\alpha_3+2\alpha_4, 2\alpha_1+3\alpha_2+4\alpha_3+2\alpha_4\}$

What could a basis be? ($\Delta_{long} = ?$)

We must have $\alpha_1, \alpha_2 \in \Delta_{long}$, but it is not enough...

$\Rightarrow$ add $\alpha_2+2\alpha_3$, but also $\alpha_2+2\alpha_3+2\alpha_4$.

$\Delta_{long} = \{\alpha, \beta, \gamma, \delta\} = \{\alpha_1, \alpha_2, \alpha_2+2\alpha_3, \alpha_2+2\alpha_3+2\alpha_4\}$.

Then
- $\alpha_1 = \alpha$
- $\alpha_2 = \beta$
- $1100 = \alpha+\beta$
- $1120 = \gamma$
- $1220 = \alpha+\beta+\gamma$
- $0122 = \delta$
- $1122 = \alpha+\delta$
- $1222 = \alpha+\beta+\delta$
- $1242 = \alpha+\gamma+\delta$
- $1342 = \alpha+\beta+\gamma+\delta$
- $2342 = 2\alpha+\beta+\gamma+\delta$.

$\Rightarrow$ Have a basis!

All roots have same length.

Finally: $\langle \alpha, \beta \rangle = \langle a_1, a_2 \rangle = -1$

$\langle \alpha, \gamma \rangle = \langle a_1, a_2 + 2a_3 \rangle = \langle a_2 + 2a_3, a_1 \rangle = -1$

$\langle \alpha, \delta \rangle = \langle a_1, a_2 + 2a_3 + 2a_4 \rangle = -1$

$\langle \beta, \gamma \rangle = \langle a_2, a_2 + 2a_3 \rangle = \langle a_2 + 2a_3, a_2 \rangle = 2 - 2 = 0$

$\langle \beta, \delta \rangle = \langle a_2, a_2 + 2a_3 + 2a_4 \rangle = \langle a_2 + 2a_3 + 2a_4, a_2 \rangle = 0$

$$\begin{aligned}\langle \gamma, \delta \rangle = \langle a_2 + 2a_3, a_2 + 2a_3 + 2a_4 \rangle &= \langle a_2, a_2 + 2a_3 + 2a_4 \rangle + 2\langle a_3, a_2 + 2a_3 + 2a_4 \rangle \\ &= \langle a_2 + 2a_3 + 2a_4, a_2 \rangle + 2\langle a_3, a_2 + 2a_3 + 2a_4 \rangle \\ &= 0 + 2\langle a_3, a_2 + 2a_3 + 2a_4 \rangle \\ &= 0 + \langle a_2 + 2a_3 + 2a_4, a_3 \rangle \\ &= 0 + (-2 + 4 - 2) \\ &= 0\end{aligned}$$

$$\Rightarrow \begin{pmatrix} 2 & -1 & -1 & -1 \\ -1 & 2 & 0 & 0 \\ -1 & 0 & 2 & 0 \\ -1 & 0 & 0 & 2 \end{pmatrix}$$

$$\langle a_2 + 2a_3 + 2a_4, a_3 \rangle = \frac{2(a_2 + 2a_3 + 2a_4, a_3)}{(a_3, a_3)} = \underbrace{\frac{2(a_3, a_2 + 2a_3 + 2a_4)}{|a_2 + 2a_3 + 2a_4|^2}}_{= \langle a_3, a_2 + 2a_3 + 2a_4 \rangle} \cdot \overbrace{\frac{|a_2 + 2a_3 + 2a_4|^2}{|a_3|^2}}^{2}$$

$$\Rightarrow \langle a_3, a_2 + 2a_3 + 2a_4 \rangle = \frac{1}{2} \langle a_2 + 2a_3 + 2a_4, a_3 \rangle$$

Conclusion: Relabel $\alpha' = \beta$, $\beta' = \alpha$, $\gamma' = \gamma$, $\delta' = \delta$

$$\rightsquigarrow C = \begin{pmatrix} 2 & -1 & 0 & 0 \\ -1 & 2 & -1 & -1 \\ 0 & -1 & 2 & 0 \\ 0 & -1 & 0 & 2 \end{pmatrix}, \text{ type } D_4$$

Long roots in F_4 form a root system of type D_4