Ex 49: Let $\alpha, c\alpha \in \Phi$, where $c \in \mathbb{R}$.

Then $\langle \alpha, c\alpha \rangle = \frac{2c(\alpha,\alpha)}{c^2(\alpha,\alpha)} = \frac{2}{c}$, but (R4) forces $\frac{2}{c} \in \mathbb{Z}$, so that

$$c \in \frac{2}{\mathbb{Z}}$$

Set $c = \frac{2}{d}$, where $d \in \mathbb{Z}$. Then $\langle c\alpha, \alpha \rangle = c\langle \alpha, \alpha \rangle = 2c = \frac{4}{d}$

Again, (R4) yields $\frac{4}{d} \in \mathbb{Z}$, so that $d \in \{\pm 1, \pm 2, \pm 4\}$.

$\Rightarrow$ $c = \frac{2}{d} \in \left\{\pm\frac{1}{2}, \pm 1, \pm 2\right\}$ as desired. (*)

Let $\Phi' := \left\{\alpha \in \Phi : 2\alpha \notin \Phi\right\}$. By assumption, it is clear that (R1) and (R4) hold.

Also for $\alpha, \beta \in \Phi' \subset \Phi$, we have $s_\alpha(\beta) \in \Phi$, since the latter satisfies (R2).

Now if $2s_\alpha(\beta) \in \Phi' \subset \Phi$, we would have $s_\alpha(2\beta) \in \Phi$, which by (R2) for Φ, implies $2\beta \in \Phi$, a contradiction

$\Rightarrow$ (R2) holds for Φ'.

Finally, let $\alpha \in \Phi'$, $c \in \mathbb{R}$ be s.t. $c\alpha \in \Phi'$.

Clearly $c \in \{\pm\frac{1}{2}, \pm 1, \pm 2\}$ by (*), and in fact $c \neq \pm 2$ by assumption.

$\rightsquigarrow$ $c \in \left\{\pm\frac{1}{2}, \pm 1\right\}$.

Now if $c = \pm\frac{1}{2}$, then $\pm\beta := \pm\frac{1}{2}\alpha \in \Phi$ satisfies $2(\pm\beta) = \pm\alpha \in \Phi$, a contradiction

$\rightsquigarrow$ $c \in \{\pm 1\}$.

$\Rightarrow$ (R3) holds as well //

Ex. 50: Recall: An L-module V belongs to the category $\mathcal{O}$ iff the following cond. hold:

1. V is H-semisimple, ie. $V = \bigoplus_{\lambda \in H^*} V_\lambda$, where $V_\lambda = \{v \in V : hv = \lambda(h)v \ \forall h \in H\}$

2. For every $\lambda \in H^*$, we have $\dim V_\lambda < \infty$.

3. $\exists$ finitely many $\mu_1, \dots, \mu_s \in H^*$ s.t. for every $\lambda \in H^*$ with $V_\lambda \neq 0$, $\exists\, 1 \le i \le s$ s.t. $\lambda \le \mu_i$.

Claim 1: Let $V \in \mathcal{O}$ and consider a submodule W of V for L. Then $W \in \mathcal{O}$ as well.

Proof: Let $V = \bigoplus_{\lambda \in H^*} V_\lambda$ be the dec. of V into a direct sum of wt. spaces, as in 1. above, and define $W_\lambda := W \cap V_\lambda$ for all λ. Then one checks that $W_\lambda = \{w \in W : hw = \lambda(h)w \ \forall h \in H\}$, and that

$$W = \bigoplus_{\lambda \in H^*} W_\lambda,$$

showing that 1. holds.

Also $W_\lambda \subset V_\lambda \ \forall \lambda \in H^*$ and hence 2. holds as well.

Finally, 3 is obvious as well, since $W_\lambda \neq 0 \Rightarrow V_\lambda \neq 0$.

// Claim 1

Claim 2: Let $V \in \mathcal{O}$ and consider a submodule W of V for L. Then $V/W \in \mathcal{O}$.

Proof: Let $V = \bigoplus_{\lambda \in H^*} V_\lambda$, $W = \bigoplus_{\lambda \in H^*} V_\lambda$ be as in the pf. of Claim 1.

Set $U_\lambda := V_\lambda / W_\lambda$, and define

$$\varphi : V \longrightarrow \bigoplus_{\lambda \in H^*} U_\lambda, \qquad (v_\lambda)_{\lambda \in H^*} \longmapsto (\pi_\lambda(v_\lambda))_{\lambda \in H^*}$$

, where $\pi_\lambda : V_\lambda \twoheadrightarrow U_\lambda$

Then $\ker \varphi = \bigoplus_{\lambda \in H^*} W_\lambda$, hence $V/W \cong \bigoplus_{\lambda \in H^*} U_\lambda$.

$\Rightarrow$ 1 holds

Also, $\dim U_\lambda \leq \dim V_\lambda < \infty$, hence 2. holds

Finally, 3 holds as well, since $\dim U_\lambda \neq 0 \Rightarrow V_\lambda \neq 0$ //

Ex. 52: $\sigma := \exp(\operatorname{ad}x)\exp(\operatorname{ad}(-y))\exp(\operatorname{ad}x) \in \operatorname{Aut}(L)$.

Preliminary computations: $\cdot\ \exp(\operatorname{ad}x)(x) = (id + \operatorname{ad}x + \frac{1}{2}\operatorname{ad}x^2 + \ldots)(x) = x + 0 + \ldots + 0 = x$

$\cdot\ \exp(\operatorname{ad}x)(y) = y + [x,y] + \frac{1}{2}[x,[x,y]] + \frac{1}{4}[x,[x,[x,y]]] + \ldots$

$= y + h + \frac{1}{2}[x,h] + \frac{1}{4}[x,[x,h]] + \ldots$

$= y + h - x + \frac{1}{2}\underbrace{[x,x]}_{0} + \ldots$

$= y + h - x$

$\cdot\ \exp(\operatorname{ad}x)(h) = h + [x,h] + \frac{1}{2}[x,[x,h]] + \ldots$

$= h - 2x$

$\cdot\ \exp(\operatorname{ad}(-y))(x) = x - [y,x] + \frac{1}{2}[y,[y,x]] - \frac{1}{4}[y,[y,[y,x]]] + \ldots$

$= x + h - \frac{1}{2}[y,h] + \frac{1}{4}[y,[y,h]] + \ldots$

$= x + h - y$

$\cdot\ \exp(\operatorname{ad}(-y))(y) = y$

$\cdot\ \exp(\operatorname{ad}(-y))(h) = h - [y,h] + \frac{1}{2}[y,[y,h]] + \ldots$

$= h - 2y$

Hence: $\sigma(x) = \exp(\operatorname{ad}x)\exp(\operatorname{ad}(-y))\exp(\operatorname{ad}x)(x)$

$= \exp(\operatorname{ad}x)\exp(\operatorname{ad}(-y))(x)$

$= \exp(\operatorname{ad}x)(x + h - y)$

$= x + h - 2x - y - h + x$

$= -y$

$\sigma(y) = \exp(\operatorname{ad}x)\exp(\operatorname{ad}(-y))\exp(\operatorname{ad}x)(y)$

$= \exp(\operatorname{ad}x)\exp(\operatorname{ad}(-y))(y + h - x)$

$= \exp(\operatorname{ad}x)(y + h - 2y - x - h + y)$

$= \exp(\operatorname{ad}x)(-x)$

$= -x$

$\sigma(h) = \exp(\operatorname{ad}x)\exp(\operatorname{ad}(-y))\exp(\operatorname{ad}x)(h)$

$= \exp(\operatorname{ad}x)\exp(\operatorname{ad}(-y))(h - 2x)$

$= \exp(\operatorname{ad}x)(h - \cancel{2y} - 2x - 2h + \cancel{2y})$

$= \exp(\operatorname{ad}x)(-2x - h)$

$= -2x - h + 2x$

$= -h$

Therefore $\sigma^2(x) = \sigma(-y) = x$, $\sigma^2(y) = \sigma(-x) = y$, $\sigma^2(h) = \sigma(-h) = h$, so that $\sigma^2 = id$, as $\{x, y, h\}$ is a basis for L.

Finally, we show that σ acts as conj. by $x-y$

Preliminary computations : $\bullet\ (x-y)(y-x) = xy - x^2 - y^2 + yx$

$$x^2 = 0,\ y^2 = 0 \qquad = xy + yx \qquad xy = \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix},\ yx = \begin{pmatrix} 0 & 0 \\ 0 & 1 \end{pmatrix}$$

$$= 1$$

$$\Rightarrow \boxed{(x-y)^{-1} = y-x}$$

$$\bullet\ (x-y)x(y-x) = (x-y)(xy-x^2) = (x^2 - x^3 - yxy + yx^2) = -yxy = -\begin{pmatrix} & 0 \\ 1 & \end{pmatrix}\overbrace{\begin{pmatrix} & 1 \\ & \end{pmatrix}\begin{pmatrix} & 0 \\ 1 & \end{pmatrix}}^{\begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix}} = -y$$

$$\bullet\ (x-y)y(y-x) = (x-y)(y^2-yx) = xy^2 - xyx - y^3 + y^2x = -xyx = -\begin{pmatrix} & 1 \\ & \end{pmatrix}\underbrace{\begin{pmatrix} & 0 \\ 1 & \end{pmatrix}\begin{pmatrix} & 1 \\ & \end{pmatrix}}_{\begin{pmatrix} 0 & 0 \\ 0 & 1 \end{pmatrix}} = -x$$

$\bullet\ (x-y)h(y-x) = (x-y)(hy - hx) = xhy - xhx - yhy + yhx$

$$= \begin{pmatrix} & 1 \\ & \end{pmatrix}\begin{pmatrix} 1 & \\ & -1 \end{pmatrix}\begin{pmatrix} & \\ 1 & \end{pmatrix} - \begin{pmatrix} & 1 \\ & \end{pmatrix}\begin{pmatrix} 1 & \\ & -1 \end{pmatrix}\begin{pmatrix} & 1 \\ & \end{pmatrix} - yhy + yhx$$

$$= \begin{pmatrix} & -1 \\ & \end{pmatrix}\begin{pmatrix} & 0 \\ 1 & \end{pmatrix} - \begin{pmatrix} & -1 \\ & \end{pmatrix}\begin{pmatrix} & 1 \\ & \end{pmatrix} - \begin{pmatrix} & \\ 1 & \end{pmatrix}\begin{pmatrix} 1 & \\ & -1 \end{pmatrix}\begin{pmatrix} & \\ 1 & \end{pmatrix} + yhx$$

$$= \begin{pmatrix} -1 & \\ & \end{pmatrix} - \begin{pmatrix} & \\ 1 & \end{pmatrix}\begin{pmatrix} & \\ 1 & \end{pmatrix} + \begin{pmatrix} & \\ 1 & \end{pmatrix}\begin{pmatrix} 1 & \\ & -1 \end{pmatrix}\begin{pmatrix} & 1 \\ & \end{pmatrix}$$

$$= \begin{pmatrix} -1 & \\ & \end{pmatrix} + \begin{pmatrix} & \\ 1 & \end{pmatrix}\begin{pmatrix} & 1 \\ & \end{pmatrix}$$

$$= -h$$

//