The Jacobson Radical

Notation. Below R denotes an arbitrary unital and associative ring.

H The Jacobson radical

The Jacobson radical is one of the most important two-sided ideals of a ring. This ideal carries a lot of information about the structure of a ring and that of its modules.

Proposition-Definition H.1 (*Annihilator* / *Jacobson radical*)

(a) Let M be an R-module. Then $\mathrm{ann}_R(M) := \{r \in R \mid rm = 0 \;\; \forall\, m \in M\}$ is a two-sided ideal of R, called **annihilator** of M.

(b) The **Jacobson radical** of R is the two-sided ideal

$$J(R) := \bigcap_{\substack{V \text{ simple} \\ R\text{-module}}} \mathrm{ann}_R(V) = \{x \in R \mid 1 - axb \in R^\times \;\; \forall\, a, b \in R\}\,.$$

(c) If V is a simple R-module, then there exists a maximal left ideal $I \lhd R$ such that $V \cong R^\circ/I$ (as R-modules) and

$$J(R) = \bigcap_{\substack{I \lhd R, \\ I \text{ maximal} \\ \text{left ideal}}} I\,.$$

Remark H.2

(a) Any simple R-module may be seen as a simple $R/J(R)$-module, because $J(R)$ annihilates it.

(b) Conversely, any simple $R/J(R)$-module may be seen as a simple R-module via a change of the base ring through the quotient morphism $R \longrightarrow R/J(R)$.

(c) **Consequence**: R and $R/J(R)$ have the same simple modules.

I Nakayama's Lemma

Theorem I.1 (*Nakayama's Lemma*)

If M is a finitely generated R-module and $J(R)M = M$, then $M = 0$.

Remark I.2

One often needs to apply Nakayama's Lemma to a finitely generated quotient module M/U, where U is an R-submodule of M. In that case the result may be restated as follows:

$$M = U + J(R)M \quad \Longrightarrow \quad U = M$$

J Local rings

Definition J.1

A ring R is said to be **local** $:\Longleftrightarrow$ $R \backslash R^\times$ is a two-sided ideal of R.

Example J.2

(a) Any field K is local because $K \backslash K^\times = \{0\}$ by definition.

(b) Let p be a prime number. Then $R := \{\frac{a}{b} \in \mathbb{Q} \mid p \nmid b\}$ is local as $R \backslash R^\times = \{\frac{a}{b} \in R \mid p|a\}$, which is a 2-sided ideal.

Proposition J.3

Let R be a ring. Then TFAE:

(a) R is local;

(b) $R \backslash R^\times = J(R)$, i.e. $J(R)$ is the unique maximal left ideal of R;

(c) $R/J(R)$ is a skew-field.