BRAUER'S HEIGHT ZERO CONJECTURE FOR QUASI-SIMPLE GROUPS

RADHA KESSAR AND GUNTER MALLE

To the memory of Sandy Green

Abstract. We show that Brauer's height zero conjecture holds for blocks of finite quasi-simple groups. This result is used in Navarro–Späth's reduction of this conjecture for general groups to the inductive Alperin–McKay condition for simple groups.

1. Introduction

In this paper we verify that the open direction of Richard Brauer's 1955 height zero conjecture (BHZ) holds for blocks of finite quasi-simple groups:

Main Theorem. *Let S be a finite quasi-simple group, ℓ a prime and B an ℓ-block of S. Then B has abelian defect groups if and only if all $\chi \in \mathrm{Irr}(B)$ have height zero.*

The proof of one direction of Brauer's height zero conjecture, that blocks with abelian defect groups only contain characters of height zero, was completed in [15]. Subsequently it was shown by Gabriel Navarro and Britta Späth [22] that the other direction of (BHZ) can be reduced to proving the following for all finite quasi-simple groups S:

(1) (BHZ) holds for S, and
(2) the inductive form of the Alperin–McKay conjecture holds for $S/Z(S)$.

Here, we show that the first statement holds. The main case, when S is quasi-simple of Lie type, is treated in Section 2, and then the proof of the Main Theorem is completed in Section 3.

2. Brauer's height zero conjecture for groups of Lie type

In this section we show that (BHZ) holds for quasi-simple groups of Lie type. This constitutes the central part of the proof of our Main Theorem.

Throughout, we work with the following setting. We let $\mathbf{G}$ be a connected reductive linear algebraic group over an algebraic closure of a finite field of characteristic p, and $F : \mathbf{G} \to \mathbf{G}$ a Steinberg endomorphism with finite group of fixed points $\mathbf{G}^F$. It is well-known that apart from finitely many exceptions, all finite quasi-simple groups of Lie type can be obtained as $\mathbf{G}^F/Z$ for some central subgroup $Z \leq \mathbf{G}^F$ by choosing $\mathbf{G}$ simple of simply connected type.

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We let $\mathbf{G}^*$ be dual to $\mathbf{G}$, with compatible Steinberg endomorphism again denoted F. Recall that by the results of Lusztig the set $\mathrm{Irr}(\mathbf{G}^F)$ of complex irreducible characters of $\mathbf{G}^F$ is a disjoint union of rational Lusztig series $\mathcal{E}(\mathbf{G}^F, s)$, where s runs over the semisimple elements of $\mathbf{G}^{*F}$ up to conjugation.

2.1. **Groups of Lie type in their defining characteristic.** We first consider the easier case of groups of Lie type in their defining characteristic, where we need the following:

Lemma 2.1. *Let $\mathbf{G}$ be simple, not of type A_1, with Frobenius endomorphism $F : \mathbf{G} \to \mathbf{G}$. Then every coset of $[\mathbf{G}^F, \mathbf{G}^F]$ in $\mathbf{G}^F$ contains a (semisimple) element centralising a root subgroup of $\mathbf{G}^F$.*

Proof. First note that by inspection any of the rank 2 groups $\mathrm{L}_3(q)$, $\mathrm{U}_3(q)$, and $\mathrm{S}_4(q)$ (and hence also $\mathrm{U}_4(q)$) contains a root subgroup $U \cong \mathbb{F}_q^+$ all of whose non-identity elements are conjugate under a maximally split torus. Now if $\mathbf{G}$ is not of type A_1 with $[\mathbf{G}^F, \mathbf{G}^F] < \mathbf{G}^F$ then it contains an F-stable Levi subgroup $\mathbf{H}$ of type A_2, B_2, or A_3, and thus $\mathbf{G}^F$ contains a root subgroup U all of whose non-trivial elements are conjugate under the maximally split torus of $[\mathbf{H}^F, \mathbf{H}^F] \leq [\mathbf{G}^F, \mathbf{G}^F]$. But $\mathbf{G}^F = [\mathbf{G}^F, \mathbf{G}^F]\mathbf{T}^F$ for any F-stable maximal torus $\mathbf{T}$ of $\mathbf{G}$ (see [21, Ex. 30.13]). Thus any coset of $[\mathbf{G}^F, \mathbf{G}^F]$ in $\mathbf{G}^F$ contains semisimple elements which centralise U. □

Now let $\mathbf{G}$ be simple with dual group $\mathbf{G}^*$, let $\pi : \mathbf{G} \to \mathbf{G}_{\mathrm{ad}}$ be the adjoint quotient and $\pi^* : \mathbf{G}^*_{\mathrm{sc}} \to \mathbf{G}^*$ its dual. Let $F : \mathbf{G} \to \mathbf{G}$ be a Frobenius map with dual lifted to $\mathbf{G}_{\mathrm{sc}}$ also denoted F. Let $s \in \mathbf{G}^{*F}$ be semisimple. Then the character formula [10, Prop. 12.2] shows that the restriction of any Deligne-Lusztig character $R_{\mathbf{T}^*}(s)$ lies above a unique character $\hat{s}$ of $Z(\mathbf{G}^F)$, whence all characters in $\mathcal{E}(\mathbf{G}^F, s)$ lie above $\hat{s}$.

Lemma 2.2. *Let $s, s' \in \mathbf{G}^{*F}$ semisimple. Then $\hat{s} = \hat{s'}$ if and only if $s^{-1}s' \in \pi^*(\mathbf{G}^{*F}_{\mathrm{sc}})$.*

Proof. Let $z \in Z(\mathbf{G}^F)$. Then z lies in any F-stable maximal torus $\mathbf{T}$ of $\mathbf{G}$, hence it induces a character $\hat{z} \in \mathrm{Irr}(\mathbf{T}^{*F})$. As z is central, any root of $\mathbf{G}$ relative to $\mathbf{T}$ vanishes on z, so $\hat{z}$ is trivial on the subgroup generated by the F-fixed points of the images of the coroots in $\mathbf{T}^*$. But these are just the images $\pi^*(\mathbf{T}^{*F}_{\mathrm{sc}})$ in $\mathbf{G}^{*F}$, so, as $\mathbf{G}^{*F}_{\mathrm{sc}}$ is generated by its maximal tori, $\hat{z}$ is trivial on $\pi^*(\mathbf{G}^{*F}_{\mathrm{sc}}) \cap \mathbf{T}^{*F}$.

Now by the character formula the central character associated to $R_{\mathbf{T}^*}(s)$ is $z \mapsto \hat{s}(z) = \hat{z}(s)$, hence it is constant on cosets of $\pi^*(\mathbf{G}^{*F}_{\mathrm{sc}})$. As clearly above any central character there is some irreducible character χ of $\mathbf{G}^F$, this map from cosets to central characters is surjective. It is injective as $Z(\mathbf{G}^F) = \ker(\pi)$ and $\mathbf{G}^{*F}/\pi^*(\mathbf{G}^{*F}_{\mathrm{sc}}) = \mathrm{coker}(\pi^*)$ have the same order by duality. □

Proposition 2.3. *Let $\mathbf{G}$ be simple, simply connected, not of type A_1, and $Z \leq \mathbf{G}^F$ be a central subgroup such that $S = \mathbf{G}^F/Z$ is quasi-simple of Lie type in characteristic p. Then any p-block of S of positive defect contains characters of positive height.*

Proof. By assumption, $S/Z(S) \not\cong \mathrm{L}_2(q)$. By the result of Humphreys [14, Concluding Remarks], the p-blocks of $\mathbf{G}^F$ of positive defect are in bijection with $\mathrm{Irr}(Z(\mathbf{G}^F))$ and are of full defect. The principal block of $\mathbf{G}^F$ contains all the unipotent characters of $\mathbf{G}^F$, hence a character of positive height e.g. by [20, Thm. 6.8] (except when $S = \mathrm{S}_4(2) = \mathfrak{S}_6$ where the statement can be checked directly).

Now assume that $Z(\mathbf{G}^F) \neq 1$, and B is the p-block of $\mathbf{G}^F$ lying over a non-trivial character $\lambda \in \mathrm{Irr}(Z(\mathbf{G}^F))$. As $\mathbf{G}^{*F}/\pi^*(\mathbf{G}_{\mathrm{sc}}^{*F})$ is abelian, according to Lemma 2.2 there is a coset of $\pi^*(\mathbf{G}_{\mathrm{sc}}^{*F})$ in $\mathbf{G}^{*F}$ such that for any s in that coset all characters of $\mathcal{E}(\mathbf{G}^F, s)$ lie over λ, hence in B. Now by Lemma 2.1 this coset contains a semisimple element s_λ centralising a root subgroup of $\mathbf{G}^{*F}$. Then $C_{\mathbf{G}^{*F}}(s_\lambda)$ contains a root subgroup, hence has semisimple rank at least 1. By Lusztig's Jordan decomposition of characters, the regular character in $\mathcal{E}(\mathbf{G}^F, s_\lambda)$ corresponds to the Steinberg character of $C_{\mathbf{G}^{*F}}(s_\lambda)$, so has positive p-height, and it lies in B. □

2.2. **Unipotent pairs and e-cuspidality.** We now turn to the investigation of ℓ-blocks for primes $\ell \neq p$, which is considerably more involved. For the rest of this section we assume that $F : \mathbf{G} \to \mathbf{G}$ is a Frobenius morphism with respect to some $\mathbb{F}_q$-structure on $\mathbf{G}$. Let ℓ be a prime not dividing q and let $e = e_\ell(q)$, where $e_\ell(q)$ is the order of q modulo ℓ if ℓ is odd and is the order of q modulo 4 if $\ell = 2$.

By a unipotent pair for $\mathbf{G}^F$ we mean a pair $(\mathbf{L}, \lambda)$, where $\mathbf{L}$ is an F-stable Levi subgroup of $\mathbf{G}$ and $\lambda \in \mathcal{E}(\mathbf{L}^F, 1)$. If $\mathbf{L}$ is d-split in $\mathbf{G}$, then $(\mathbf{L}, \lambda)$ is said to be a unipotent d-pair and if in addition λ is a unipotent d-cuspidal character of $\mathbf{L}^F$, then $(\mathbf{L}, \lambda)$ is said to be a unipotent d-cuspidal pair.

Recall that if $\mathbf{L}$ is an F-stable Levi subgroup of $\mathbf{G}$, then $\bar{\mathbf{L}} := \mathbf{L}/Z(\mathbf{G})$ is an F-stable Levi subgroup of $\mathbf{G}/Z(\mathbf{G})$ and $\mathbf{L}_0 := \mathbf{L} \cap [\mathbf{G}, \mathbf{G}]$ is an F-stable Levi subgroup of $[\mathbf{G}, \mathbf{G}]$; the maps $\mathbf{L} \mapsto \bar{\mathbf{L}}$ and $\mathbf{L} \mapsto \mathbf{L}_0$ give bijections between the sets of F-stable Levi subgroups of $\mathbf{G}$ and of $\mathbf{G}/Z(\mathbf{G})$ and between the sets of F-stable Levi subgroups of $\mathbf{G}$ and of $[\mathbf{G}, \mathbf{G}]$. Also recall that the natural maps $\mathbf{L} \to \mathbf{L}/Z(\mathbf{G})$ and $\mathbf{L} \cap [\mathbf{G}, \mathbf{G}] \to \mathbf{L}$ induce degree preserving bijections between $\mathcal{E}(\mathbf{L}^F, 1)$, $\mathcal{E}(\bar{\mathbf{L}}^F, 1)$ and $\mathcal{E}(\mathbf{L}_0^F, 1)$. Hence there are natural bijections between the sets of unipotent pairs of $\mathbf{G}^F$, $(\mathbf{G}/Z(\mathbf{G}))^F$ and of $[\mathbf{G}, \mathbf{G}]^F$ and these preserve the properties of being d-split and of being d-cuspidal (see [6, Sec. 3]).

Lemma 2.4. *Let $(\mathbf{L}, \lambda)$, $(\mathbf{L}_0, \lambda_0)$ and $(\bar{\mathbf{L}}, \bar{\lambda})$ be corresponding unipotent pairs for $\mathbf{G}^F$, $[\mathbf{G}, \mathbf{G}]^F$ and $(\mathbf{G}/Z(\mathbf{G}))^F$. Then,*

$$W_{[\mathbf{G},\mathbf{G}]^F}(\mathbf{L}_0, \lambda_0) \cong W_{\mathbf{G}^F}(\mathbf{L}, \lambda) \cong W_{(\mathbf{G}/Z(\mathbf{G}))^F}(\bar{\mathbf{L}}, \bar{\lambda}).$$

Proof. Let $\bar{\mathbf{G}} = \mathbf{G}/Z(\mathbf{G})$. The canonical map $\mathbf{G} \to \bar{\mathbf{G}}$ induces an injective map from $W_{\mathbf{G}^F}(\mathbf{L}, \lambda)$ into $W_{\bar{\mathbf{G}}^F}(\bar{\mathbf{L}}, \bar{\lambda})$. Conversely, let $x \in \mathbf{G}$ be such that its image $\bar{x} \in \bar{\mathbf{G}}$ is in $N_{\bar{\mathbf{G}}^F}(\bar{\mathbf{L}}, \bar{\lambda})$. Then x normalises $\mathbf{L}$ as well as $\mathbf{L}^F$ and stablises λ. Further, by the Lang–Steinberg theorem, $xt \in \mathbf{G}^F$ for some t lying in an F-stable maximal torus $\mathbf{T}$ of $\mathbf{L}$. Since $N_{\mathbf{T}}(\mathbf{L}^F)$ stabilises λ, we have that $xt \in N_{\mathbf{G}^F}(\mathbf{L}, \lambda)$. Further, since $\bar{x} \in \bar{\mathbf{G}}^F$, $\bar{t} \in \bar{\mathbf{L}}^F$, and hence $xt\mathbf{L}^F \mapsto \bar{x}\bar{\mathbf{L}}^F$ under the inclusion of $W_{\mathbf{G}^F}(\mathbf{L}, \lambda)$ in $W_{\bar{\mathbf{G}}^F}(\bar{\mathbf{L}}, \bar{\lambda})$. The proof for the isomorphism

$$W_{[\mathbf{G},\mathbf{G}]^F}(\mathbf{L}_0, \lambda_0) \cong W_{\mathbf{G}^F}(\mathbf{L}, \lambda)$$

is similar. □

Next, we note the following consequence of [6, Prop. 1.3].

Lemma 2.5. *Suppose that $\mathbf{G} = [\mathbf{G}, \mathbf{G}]$ is simply connected. Let $\mathbf{G}_1, \ldots, \mathbf{G}_r$ be a set of representatives for the F-orbits on the set of simple components of $\mathbf{G}$ and for each i let d_i denote the length of the F-orbit of $\mathbf{G}_i$. For a Levi subgroup $\mathbf{L}$ of $\mathbf{G}$, let $\mathbf{L}_i = \mathbf{L} \cap \mathbf{G}_i$.*

Then $\mathbf{L}$ *is* F*-stable if and only if* $\mathbf{L}_i$ *is* F^{d_i}*-stable for all* i *and in this case*

$$\mathbf{L} = (\mathbf{L}_1 F(\mathbf{L}_1)\cdots F^{d_1-1}(\mathbf{L}_1))\cdots(\mathbf{L}_r F(\mathbf{L}_r)\cdots F^{d_r-1}(\mathbf{L}_r)).$$

Further, projecting onto the $\mathbf{G}_i$ *component in each* F*-orbit induces an isomorphism*

$$\mathbf{L}^F \cong \mathbf{L}_1^{F^{d_1}} \times \cdots \times \mathbf{L}_r^{F^{d_r}}.$$

If, under the above isomorphism, $\lambda \in \mathcal{E}(\mathbf{L}^F, 1)$ *corresponds to* $\lambda_1 \times \cdots \times \lambda_r$*, with* $\lambda_i \in \mathcal{E}(\mathbf{L}^{F^{d_i}}, 1)$*, then* $(\mathbf{L}, \lambda)$ *is an* e*-cuspidal pair for* $\mathbf{G}^F$ *if and only if* $(\mathbf{L}_i^{F^{d_i}}, \lambda_i)$ *is an* $e_\ell(q^{d_i})$*-cuspidal pair for* $\mathbf{G}_i^{F^{d_i}}$ *for each* i*.*

Lemma 2.6. *Suppose that either* ℓ *is odd or that* $\mathbf{G}$ *has no components of classical type* $A, B, C,$ *or* D*. Let* $(\mathbf{L}, \lambda)$ *be a unipotent* e*-cuspidal pair of* $\mathbf{G}^F$*. Then,* $\mathbf{L} = C^\circ_{\mathbf{G}}(Z(\mathbf{L})^F_\ell)$*.*

Proof. We claim that it suffices to prove the result in the case that $\mathbf{G}$ is semisimple. Indeed, let $\mathbf{G}_0 = [\mathbf{G}, \mathbf{G}]$, $\mathbf{L}_0 = \mathbf{L} \cap \mathbf{G}_0$ and λ_0 be the restriction of λ to $\mathbf{L}_0^F$. Then, $(\mathbf{L}_0, \lambda_0)$ is a unipotent e-cuspidal pair of $\mathbf{G}_0^F$. Suppose that $\mathbf{L}_0 = C^\circ_{\mathbf{G}_0}(Z(\mathbf{L}_0)^F_\ell)$. Since $\mathbf{G} = Z^\circ(\mathbf{G})\mathbf{G}_0$, we have that

$$C_{\mathbf{G}}(Z(\mathbf{L}_0)^F_\ell) = Z^\circ(\mathbf{G}) C_{\mathbf{G}_0}(Z(\mathbf{L}_0)^F_\ell),$$

hence

$$C^\circ_{\mathbf{G}}(Z(\mathbf{L}_0)^F_\ell) = Z^\circ(\mathbf{G}) C^\circ_{\mathbf{G}_0}(Z(\mathbf{L}_0)^F_\ell) = Z^\circ(\mathbf{G})\mathbf{L}_0 = \mathbf{L}.$$

Here the first equality holds since $C_{\mathbf{G}}(Z(\mathbf{L}_0)^F_\ell)/Z^\circ(\mathbf{G})C^\circ_{\mathbf{G}_0}(Z(\mathbf{L}_0)^F_\ell)$ is a surjective image of $C_{\mathbf{G}_0}(Z(\mathbf{L}_0)^F_\ell)/C^\circ_{\mathbf{G}_0}(Z(\mathbf{L}_0)^F_\ell)$ and hence is finite. On the other hand, we have that $Z(\mathbf{L}_0)^F_\ell \le Z(\mathbf{L})^F_\ell$ whence $C_{\mathbf{G}}(Z(\mathbf{L})^F_\ell) \le C_{\mathbf{G}}(Z(\mathbf{L}_0)^F_\ell)$ and the claim follows.

We assume from now on that $\mathbf{G} = [\mathbf{G}, \mathbf{G}]$. We claim that it suffices to prove the result in the case that $\mathbf{G}$ is simply connected. Indeed, let $\hat{\mathbf{G}} \to \mathbf{G}$ be an F-compatible simply connected covering of $\mathbf{G}$, with finite central kernel, say Z. Let $\hat{\mathbf{L}}$ be the inverse image of $\mathbf{L}$ in $\hat{\mathbf{G}}$ and let $\hat{\lambda}_0 \in \mathrm{Irr}(\hat{\mathbf{L}}^F)$ be the (unipotent) inflation of λ. By Lemma 2.4 $(\hat{\mathbf{L}}, \hat{\lambda})$ is an e-cuspidal unipotent pair of $\hat{\mathbf{L}}^F$. Let $\hat{A} = Z(\hat{\mathbf{L}})^F_\ell$ and suppose that $C^\circ_{\hat{\mathbf{G}}}(\hat{A}) = \hat{\mathbf{L}}$. Let $A = \hat{A}Z/Z$ and let $\mathbf{C}$ be the inverse image in $\hat{\mathbf{G}}$ of $C_{\mathbf{G}}(A)$. Then $C_{\hat{\mathbf{G}}}(\hat{A}) = C_{\hat{\mathbf{G}}}(\hat{A}Z)$ is a normal subgroup of $\mathbf{C}$ and $\mathbf{C}/C_{\hat{\mathbf{G}}}(\hat{A})$ is isomorphic to a subgroup of the automorphism group of $\hat{A}Z$. Since $\hat{A}Z$ is finite, it follows that $\mathbf{C}/C_{\hat{\mathbf{G}}}(\hat{A})$ is finite and hence $C_{\hat{\mathbf{G}}}(\hat{A})/Z$ has finite index in $\mathbf{C}/Z = C_{\mathbf{G}}(\hat{A})$. On the other hand, $\hat{A} \le Z(\mathbf{L})^F_\ell$, hence $C_{\hat{\mathbf{G}}}(\hat{A})/Z$ has finite index in $C_{\mathbf{G}}(Z(\mathbf{L})^F_\ell)$. So,

$$C^\circ_{\mathbf{G}}(Z(\mathbf{L})^F_\ell) \le (C_{\hat{\mathbf{G}}}(\hat{A})/Z)^\circ = C^\circ_{\hat{\mathbf{G}}}(\hat{A})/Z = \hat{\mathbf{L}}/Z = \mathbf{L}$$

which proves the claim.

Thus, we may assume that $\mathbf{G} = [\mathbf{G}, \mathbf{G}]$ is simply connected. By [15, Lemma 7.1] and Lemma 2.5 we may assume that $\mathbf{G}$ is simple. If ℓ is good for $\mathbf{G}$ and odd, then the result is contained in [6, Prop. 3.3(ii)]. If $\mathbf{G}$ is of exceptional type and ℓ is bad for $\mathbf{G}$ then the result is proved case by case in [11]. □

2.3. **On heights of unipotent characters.** We now collect some results on heights of unipotent characters. We first need the following observation:

Lemma 2.7. *Let ℓ be a prime and $n \geq \ell$.*

(a) *The symmetric group $\mathfrak{S}_n$ has an irreducible character of degree divisible by ℓ unless $n = \ell \in \{2, 3\}$.*

(b) *The complex reflection group $G(2e, 1, n) \cong C_{2e} \wr \mathfrak{S}_n$ and its normal subgroup $G(2e, 2, n)$ of index 2 (with $e > 1$ if $n < 4$) have an irreducible character of degree divisible by ℓ.*

Proof. (a) By the hook formula for the character degrees of $\mathfrak{S}_n$ it suffices to produce a partition $\lambda \vdash n$ with no ℓ-hook, for $\ell \leq n \leq 2\ell - 1$. For $\ell \geq 5$ the partition $(\ell - 2, 2) \vdash \ell$ and suitable hook partitions for $\ell < n \leq 2\ell - 1$ are as claimed. For $\ell \leq 3$ the symmetric groups $\mathfrak{S}_m$, $\ell + 1 \leq m \leq 2\ell$, have suitable characters.

For (b) note that both $G(2e, 1, n)$ and $G(2e, 2, n)$ have $\mathfrak{S}_n$ as a factor group, so we are done by (a) unless $n = \ell \in \{2, 3\}$. In the latter two cases the claim is easily checked. □

Recall that an irreducible character χ of a finite group G is said to be of *central ℓ-defect* if the ℓ-block of G containing χ has a defect group contained in the centre of G; in this case we say that the pair (G, χ) is of central ℓ-defect.

Lemma 2.8. *Let $(\mathbf{L}, \lambda)$ be a unipotent e-cuspidal pair of $\mathbf{G}^F$ of central ℓ-defect, where $e = e_\ell(q)$. Suppose that $|W_{\mathbf{G}^F}(\mathbf{L}, \lambda)|_\ell \neq 1$ and all irreducible characters of $W_{\mathbf{G}^F}(\mathbf{L}, \lambda)$ are of degree prime to ℓ. Then, $\ell \leq 3$. Suppose in addition that $\mathbf{G}$ is simple and simply connected. Then $W_{\mathbf{G}^F}(\mathbf{L}, \lambda) \cong \mathfrak{S}_\ell$ and the following holds:*

(a) *If $\ell = 3$, then either $\mathbf{G}^F = \mathrm{SL}_3(q)$ with $3|(q-1)$ or $\mathrm{SU}_3(q)$ with $3|(q+1)$ or $\mathbf{G}$ is of type E_6 and $(\mathbf{L}, \lambda)$ corresponds to Line 8 of the E_6-tables of* [11, pp. 351, 354].

(b) *If $\ell = 2$, then either $\mathbf{G}$ is of classical type, or $\mathbf{G}$ is of type E_7 and $(\mathbf{L}, \lambda)$ corresponds to one of Lines 3 or 7 of the E_7-table of* [11, p. 354].

Proof. The first statement easily reduces to the case that $\mathbf{G}$ is simple, which we will assume from now on. We go through the various cases. First assume that $\mathbf{G}$ is of exceptional type, or that $\mathbf{G}^F = {}^3D_4(q)$. The relative Weyl groups $W_{\mathbf{G}^F}(\mathbf{L}, \lambda)$ of unipotent e-cuspidal pairs are listed in [3, Table 1], and an easy check shows that they possess characters of degree divisible by ℓ whenever ℓ divides $|W_{\mathbf{G}^F}(\mathbf{L}, \lambda)|$, unless either $\ell = 3$, $\mathbf{G}$ is of type E_6 and we are in case (a), or $\ell = 2$ and $W_{\mathbf{G}^F}(\mathbf{L}, \lambda) \cong C_2$ in $\mathbf{G}$ of type E_6, E_7 or E_8. According to the tables in [11, pp. 351, 354, 358], the only case with λ of central ℓ-defect is in E_7 with $\mathbf{L}$ of type E_6 and λ one of the two cuspidal characters as in (b).

Next assume that $\mathbf{G}^F$ is of type A. The relative Weyl groups have the form $C_e \wr \mathfrak{S}_a$ for some $a \geq 1$. By definition, $e < \ell$, so if ℓ divides $|W_{\mathbf{G}^F}(\mathbf{L}, \lambda)|$ then $\ell \leq a$. Then by Lemma 2.7 we arrive at either (a) or (b) of the conclusion. If $\mathbf{G}^F$ is a unitary group, the same argument applies, except that here the relative Weyl groups have the form $C_d \wr \mathfrak{S}_a$ with $d = e_\ell(-q)$. For $\mathbf{G}$ of type B or C, the relative Weyl groups have the form $C_d \wr \mathfrak{S}_a$, with $d \in \{e, 2e\}$ even, and again by Lemma 2.7 no exceptions arise. The relative Weyl groups have the same structure for $\mathbf{G}$ of type D, unless $\mathbf{G}^F$ is untwisted and λ is parametrised by a degenerate symbol, and either $e \in \{1, 2\}$, $\lambda = 1$, $W_{\mathbf{G}^F}(\mathbf{L}, \lambda) = W$ and so is of type D_n with $n \geq 4$, or $W_{\mathbf{G}^F}(\mathbf{L}, \lambda) \cong G(2d, 2, n)$ with $d \geq 2$, so again we are done by Lemma 2.7. □

Recall that by [11, Thm. A] if $(\mathbf{L},\lambda)$ is a unipotent e-cuspidal pair of $\mathbf{G}$, then all irreducible constituents of $R_{\mathbf{L}}^{\mathbf{G}}(\lambda)$ lie in the same ℓ-block, say $b_{\mathbf{G}^F}(\mathbf{L},\lambda)$ of $\mathbf{G}^F$.

Lemma 2.9. *Let $(\mathbf{L},\lambda)$ be a unipotent e-cuspidal pair of $\mathbf{G}^F$ and let $B = b_{\mathbf{G}^F}(\mathbf{L},\lambda)$. Suppose that λ is of central ℓ-defect and that $\mathbf{L} = C^\circ_{\mathbf{G}}(Z(\mathbf{L})^F_\ell)$. If B has non-abelian defect groups, then $|W_{\mathbf{G}^F}(\mathbf{L},\lambda)|$ is divisible by ℓ.*

Proof. Let $Z = Z(\mathbf{L})^F_\ell$ and let b be the block of $\mathbf{L}^F$ containing λ. Since $\mathbf{L} = C^\circ_{\mathbf{G}}(Z)$, and Z is an ℓ-subgroup of $\mathbf{L}$ contained in a maximal torus of $\mathbf{G}$, $C_{\mathbf{G}}(Z)/\mathbf{L}$ is an ℓ-group. Hence, $\mathbf{L}^F$ is a normal subgroup of $C_{\mathbf{G}^F}(Z)$ of ℓ-power index and consequently, there is a unique block, say $\tilde b$ of $C_{\mathbf{G}^F}(Z)$ covering b. Further, by [15, Props. 2.12, 2.13(1), 2.15] and [3, Thm. 3.2], $(Z,\tilde b)$ is a B-Brauer pair.

Since $I_{C_{\mathbf{G}^F}(Z)}(\lambda)/\mathbf{L}^F \le W_{\mathbf{G}^F}(\mathbf{L},\lambda)$ and since $C_{\mathbf{G}^F}(Z)/\mathbf{L}^F$ is an ℓ-group, we may assume by way of contradiction that $I_{C_{\mathbf{G}^F}(Z)}(\lambda) \le \mathbf{L}^F$. Further, since λ is of central ℓ-defect in $\mathbf{L}^F$, λ is the unique character of b with Z in its kernel. Thus, $I_{C_{\mathbf{G}^F}(Z)}(b) = I_{C_{\mathbf{G}^F}(Z)}(\lambda) \le \mathbf{L}^F$. Consequently, Z is a defect group of $\tilde b$. Now the defect groups of B are non-abelian, whereas Z is abelian. Hence $N_{\mathbf{G}^F}(Z,\tilde b)/C_{\mathbf{G}^F}(Z)$ is not an ℓ'-group. On the other hand, $N_{\mathbf{G}^F}(Z,\tilde b)$ normalises $\mathbf{L}^F$ and therefore acts by conjugation on the set of ℓ-blocks of $\mathbf{L}^F$ covered by $\tilde b$. Since $C_{\mathbf{G}^F}(Z)$ acts transitively on the set of the ℓ-blocks of $\mathbf{L}^F$ covered by $\tilde b$, by the Frattini argument, $N_{\mathbf{G}^F}(Z,\tilde b) = C_{\mathbf{G}^F}(Z)N_{\mathbf{G}^F}(Z,b)$. Hence,

$$N_{\mathbf{G}^F}(Z,b)/\mathbf{L}^F = N_{\mathbf{G}^F}(Z,b)/(N_{\mathbf{G}^F}(Z,b)\cap C_{\mathbf{G}^F}(Z)) \cong N_{\mathbf{G}^F}(Z,\tilde b)/C_{\mathbf{G}^F}(Z)$$

is not an ℓ' group. But again since λ is of central ℓ defect, $N_{\mathbf{G}^F}(Z,b) \le N_{\mathbf{G}^F}(\mathbf{L},\lambda)$. Hence $N_{\mathbf{G}^F}(\mathbf{L},\lambda)/\mathbf{L}^F$ is not an ℓ' group, contradicting our assumption. □

Recall that by the fundamental result of e-Harish-Chandra theory [3, Thm. 3.2], for any unipotent e-cuspidal pair $(\mathbf{L},\lambda)$ of $\mathbf{G}$ there is a bijection

$$\rho_{\mathbf{L},\lambda} : \mathcal{E}(\mathbf{G}^F,(\mathbf{L},\lambda)) \xrightarrow{1-1} \mathrm{Irr}(W_{\mathbf{G}^F}(\mathbf{L},\lambda))$$

between the set $\mathcal{E}(\mathbf{G}^F,(\mathbf{L},\lambda))$ of irreducible constituents of $R_{\mathbf{L}}^{\mathbf{G}}(\lambda)$ and $\mathrm{Irr}(W_{\mathbf{G}^F}(\mathbf{L},\lambda))$. Moreover we have the following relationship between the degrees of corresponding characters.

Lemma 2.10. *Let $(\mathbf{L},\lambda)$ be a unipotent e-cuspidal pair of $\mathbf{G}^F$ and let $\chi \in \mathcal{E}(\mathbf{G}^F,(\mathbf{L},\lambda))$. Then*

$$\chi(1)_\ell = \frac{|\mathbf{G}^F|_\ell\,\lambda(1)_\ell}{|\mathbf{L}^F|_\ell\cdot|W_{\mathbf{G}^F}(\mathbf{L},\lambda)|_\ell}\,(\rho_{\mathbf{L},\lambda}(\chi))(1)_\ell.$$

In particular, there exist $\chi_1,\chi_2 \in \mathcal{E}(\mathbf{G}^F,(\mathbf{L},\lambda))$ with $\chi_1(1)_\ell \ne \chi_2(1)_\ell$ if and only if there exists an irreducible character of $W_{\mathbf{G}^F}(\mathbf{L},\lambda)$ with degree divisible by ℓ.

Proof. This follows from [20, Thm. 4.2 and Cor. 6.3]. □

Lemma 2.11. *Let $\mathbf{G}$ be connected reductive and let B be a unipotent ℓ-block of $\mathbf{G}^F$. Then B has an irreducible unipotent character of height zero.*

Proof. We may assume that $\mathbf{G} = [\mathbf{G},\mathbf{G}]$. Indeed, set $\mathbf{G}_0 = [\mathbf{G},\mathbf{G}]$ and let B_0 be the unipotent block of $\mathbf{G}_0^F$ covered by B. Then the degrees in $\mathrm{Irr}(B)\cap\mathcal{E}(\mathbf{G}^F,1)$ are the same as the degrees in $\mathrm{Irr}(B_0)\cap\mathcal{E}(\mathbf{G}_0^F,1)$. On the other hand, if $\chi \in \mathrm{Irr}(B_0)$ and $\chi' \in \mathrm{Irr}(B)$

covers χ, then $\chi'(1)$ is divisible by $\chi(1)$. Since every $\chi' \in \mathrm{Irr}(B)$ covers some $\chi \in \mathrm{Irr}(B_0)$ and vice versa (see for example [25, Ch. 5, Lemmas 5.7, 5.8]), we may assume that $\mathbf{G} = \mathbf{G}_0$.

We next claim that we may assume that $\mathbf{G}$ is simple. Indeed, let $\bar{\mathbf{G}} = \mathbf{G}/Z(\mathbf{G})$ and $\bar{B}$ the block of $\bar{\mathbf{G}}^F$ dominated by B. Let $H \cong \mathbf{G}^F/Z(\mathbf{G}^F)$ be the image of $\mathbf{G}^F$ in $\bar{\mathbf{G}}^F$ under the canonical map from $\mathbf{G}$ to $\bar{\mathbf{G}}$ and let C be the block of H dominated by B. Then H is normal in $\bar{\mathbf{G}}^F$ and C is covered by $\bar{B}$. The degrees in $\mathrm{Irr}(\bar{B}) \cap \mathcal{E}(\bar{\mathbf{G}}^F, 1)$ are the same as the degrees in $\mathrm{Irr}(B) \cap \mathcal{E}(\mathbf{G}^F, 1)$ and by the same arguments as above every irreducible character degree of $\bar{B}$ is divisible by an irreducible character degree of C and the set of irreducible character degrees of C is contained in the set of irreducible character degrees of B. Thus, if the result is true for B, it holds for $\bar{B}$. So, we may assume that $\mathbf{G} = [\mathbf{G}, \mathbf{G}]$ is simply connected, and hence also that $\mathbf{G}$ is simple.

If $\mathbf{G}$ is of type A and ℓ is odd and divides the order of $Z(\mathbf{G}^F)$, then by [6, Theorem, Prop. 3.3] B is the principal block and the result holds. If $\ell = 2$ and $\mathbf{G}$ is of classical type, then by [4, Thm. 13] again B is the principal block. In the remaining cases by the results of [6] and [11] there exists an e-cuspidal pair $(\mathbf{L}, \lambda)$ for B such that λ is of central ℓ-defect and a defect group of B is an extension of $Z(\mathbf{L}^F)_\ell$ by a Sylow ℓ-subgroup of $W_{\mathbf{G}^F}(\mathbf{L}, \lambda)$ (see [15, Thm. 7.12(a) and (d)]). Now the result follows from Lemma 2.10 by considering the character in $\mathcal{E}(\mathbf{G}^F, (\mathbf{L}, \lambda))$ corresponding to the trivial character of $W_{\mathbf{G}^F}(\mathbf{L}, \lambda)$. □

Lemma 2.12. *Suppose that* $\mathbf{G}$ *is simple and let* λ *be a unipotent* e*-cuspidal character of* $\mathbf{G}^F$ *of central* ℓ*-defect. Then* λ *is of* ℓ*-defect zero. Moreover, any diagonal automorphism of* $\mathbf{G}^F$ *of* ℓ*-power order is an inner automorphism of* $\mathbf{G}^F$.

Proof. Let $\mathbf{G} \hookrightarrow \tilde{\mathbf{G}}$ be a regular embedding and set $\bar{\mathbf{G}} := \mathbf{G}/Z(\mathbf{G})$. If ℓ is odd, good for $\mathbf{G}$ and $\ell \neq 3$ if $\mathbf{G}^F = {}^3D_4(q)$, then by [6, Prop. 4.3], every unipotent e-cuspidal character of $\bar{\mathbf{G}}^F$ and of $\tilde{\mathbf{G}}^F$ is of central ℓ-defect. The first assertion follows since $\bar{\mathbf{G}}^F$ has trivial center and since $\bar{\mathbf{G}}^F$ and $\mathbf{G}^F$ have the same order. For the second assertion, note the central ℓ-defect property of λ as a character of $\mathbf{G}^F$ and $\tilde{\mathbf{G}}^F$ implies that $|\tilde{\mathbf{G}}^F : Z(\tilde{\mathbf{G}}^F)|_\ell = |\mathbf{G}^F : Z(\mathbf{G}^F)|_\ell$, hence $Z(\tilde{\mathbf{G}}^F)\mathbf{G}^F$ is of ℓ'-index in $\tilde{\mathbf{G}}^F$, thus proving the result.

If $\ell = 2$ and $\mathbf{G}$ is of classical type A, B, C or D then by [4, Thm. 13] the principal block of $\mathbf{G}^F$ is the only unipotent block of $\mathbf{G}^F$, and the Sylow 2-subgroups of $\mathbf{G}^F$ are non-abelian, hence $\mathbf{G}^F$ has no unipotent character of central 2-defect. If ℓ is bad for $\mathbf{G}$ and $\mathbf{G}$ is of exceptional type, or if $\ell = 3$ and $\mathbf{G}^F = {}^3D_4(q)$, then the result follows by inspecting the tables in [11]. The last assertion follows as in type E_6 the outer diagonal automorphism is of order 3, but there are no unipotent e-cuspidals of central 3-defect, and similarly in type E_7, the outer diagonal automorphism has order 2, but there are no unipotent e-cuspidals of central 2-defect. □

2.4. **Some special blocks.** Here we investigate in some detail certain unipotent blocks for $\ell \leq 3$ related to the exceptions in Lemma 2.8.

Lemma 2.13. *Let* $\mathbf{G}^F = \mathrm{SL}_3(q)$, $3|(q-1)$, *and let* B *be the principal 3-block of* $\mathbf{G}^F$.

(a) *There exists an irreducible character of positive* 3*-height in* B. *This contains* $Z(\mathbf{G}^F)$ *in its kernel when* $q \equiv 1 \pmod 9$.

(b) *If* $q \not\equiv 1 \pmod 9$, *then there exists an irreducible character in* B *with* $Z(\mathbf{G}^F)$ *in its kernel and which is not stable under the outer diagonal automorphism of* $\mathbf{G}^F$.

The analogous result holds for $\mathbf{G}^F = \mathrm{SU}_3(q)$ *with* 3 *dividing* $q+1$.

Proof. Let $\mathbf{G}$ be simple, simply connected of type A_2 such that $\mathbf{G}^F = \mathrm{SL}_3(q)$ with $3|(q-1)$. Then the Sylow 3-subgroups of $\mathbf{G}^F$ are non-abelian and if $q \equiv 1 \pmod 9$, then the Sylow 3-subgroups of $\mathbf{G}^F/Z(\mathbf{G}^F)$ are non-abelian, hence (a) is a consequence of [1]. So we may assume that $q \not\equiv 1 \pmod 9$. Let η be a primitive third root of unity in $\mathbb{F}_q$ and let $t \in \mathbf{G}^{*F} = \mathrm{PGL}_3(q)$ be the image of $\mathrm{diag}(1, \eta, \eta^2)$ under the canonical surjection of $\mathrm{GL}_3(q)$ onto $\mathrm{PGL}_3(q)$. So, $C^\circ_{\mathbf{G}^*}(t)$ is a maximal torus of $\mathbf{G}^*$ and $|C_{\mathbf{G}^*}(t)/C^\circ_{\mathbf{G}^*}(t)| = 3$. Let $\mathbf{T}$ be an F-stable maximal torus of $\mathbf{G}$ in duality with $C^\circ_{\mathbf{G}^*}(t)$ and let $\hat{t}$ be the linear character of $\mathbf{T}^F$ in duality with t. Let ψ be an irreducible constituent of $R^{\mathbf{G}}_{\mathbf{T}}(\hat{t})$. Then, ψ is not stable under the outer diagonal automorphism of $\mathbf{G}^F$. Further, $\psi \in \mathrm{Irr}(B)$ as t is a 3-element and the principal block of $\mathbf{G}^F$ is the only unipotent block of $\mathbf{G}^F$. Finally, $Z(\mathbf{G}^F)$ is contained in the kernel of ψ as $t \in [\mathbf{G}^{*F}, \mathbf{G}^{*F}]$. The proof for the unitary case is entirely similar. □

Lemma 2.14. *Let* $\mathbf{G}$ *be simple, simply connected of type* E_6, $\mathbf{G}^F = E_6(q)$, $3|(q-1)$, *and let* $(\mathbf{L}, \lambda)$ *be a unipotent* 1*-cuspidal pair corresponding to Line 8 of the* E_6*-table in* [11].

(a) *There exists an irreducible character of positive* 3*-height in* $B = b_{\mathbf{G}^F}(\mathbf{L}, \lambda)$. *This contains* $Z(\mathbf{G}^F)$ *in its kernel when* $q \equiv 1 \pmod 9$.

(b) *If* $q \not\equiv 1 \pmod 9$, *then there exists an irreducible character in* B *with* $Z(\mathbf{G}^F)$ *in its kernel and which is not stable under the outer diagonal automorphism of* $\mathbf{G}^F$.

An analogous result holds for $\mathbf{G}^F = {}^2E_6(q)$ *with* 3 *dividing* $q+1$.

Proof. There exists $t \in \mathbf{G}^{*F}_3$ such that $\mathbf{M}^* := C_{\mathbf{G}^*}(t)$ is a 1-split Levi subgroup of $\mathbf{G}^*$ of type D_5 containing $\mathbf{L}^*$, which is contained in $[\mathbf{G}^{*F}, \mathbf{G}^{*F}]$ if and only if $q \equiv 1 \pmod 9$, see e.g. [17]. Denoting by $\mathbf{M} \geq \mathbf{L}$ an F-stable Levi subgroup of $\mathbf{G}$ in duality with $\mathbf{M}^*$ and by $\hat{t}$ the linear character of $\mathbf{M}^F$ corresponding to t we thus have that $Z(\mathbf{G}^F)$ is contained in the kernel of $\hat{t}$ if $q \equiv 1 \pmod 9$. Moreover there is an irreducible constituent η of $R^{\mathbf{M}}_{\mathbf{L}}(\lambda)$ such that $\psi := \epsilon_{\mathbf{M}}\epsilon_{\mathbf{G}} R^{\mathbf{G}}_{\mathbf{M}}(\hat{t}\eta)$ has $\psi(1)_3 > \chi(1)_3$ for any $\chi \in \mathcal{E}(\mathbf{G}^F, 1) \cap \mathrm{Irr}(B)$. Now

$$d^{1,\mathbf{G}^F}(\psi) = \pm d^{1,\mathbf{G}^F}(R^{\mathbf{G}}_{\mathbf{M}}(\hat{t}\eta)) = \pm R^{\mathbf{G}}_{\mathbf{M}}(d^{1,\mathbf{M}^F}(\hat{t}\eta)) = \pm R^{\mathbf{G}}_{\mathbf{M}}(d^{1,\mathbf{M}^F}(\eta)) = d^{1,\mathbf{G}^F}(R^{\mathbf{G}}_{\mathbf{M}}(\eta)).$$

Since η is a constituent of $R^{\mathbf{M}}_{\mathbf{L}}(\lambda)$ and $\mathbf{M}$ is 1-split in $\mathbf{G}$, the positivity of 1-Harish-Chandra theory yields that every constituent of $R^{\mathbf{G}}_{\mathbf{M}}(\eta)$ is a constituent of $R^{\mathbf{G}}_{\mathbf{L}}(\lambda)$ and hence in particular ψ is in $\mathrm{Irr}(B)$, proving (a).

Now assume that $q \not\equiv 1 \pmod 9$. Again by [17] there is $t' \in \mathbf{G}^{*F}_3$ such that $C^\circ_{\mathbf{G}^*}(t') = \mathbf{L}^*$, and $|C_{\mathbf{G}^*}(t')/C^\circ_{\mathbf{G}^*}(t')| = 3$. Let ψ' be an irreducible constituent of $R^{\mathbf{G}}_{\mathbf{L}}(\hat{t}'\lambda)$ for $\lambda \in \mathcal{E}(\mathbf{L}^F, 1)$ and $\hat{t}$ in duality with t. Then ψ' is not stable under the diagonal automorphism of $\mathbf{G}^F$, and it lies in B by the same argument as for ψ. The arguments for ${}^2E_6(q)$ are entirely similar. □

Lemma 2.15. *Let* $\mathbf{G}^F = \mathrm{SL}_2(q)$ *with* q *odd. The principal* 2*-block* B *of* $\mathbf{G}^F$ *contains an irreducible character of even degree. If* $q \equiv 1 \mod 4$, *then there exists an irreducible character of even degree in* B *which contains* $Z(\mathbf{G}^F)$ *in its kernel. If* $q \equiv 3 \mod 4$ *then there exists an irreducible character in* B *which contains* $Z(\mathbf{G}^F)$ *in its kernel and which is not stable under the outer diagonal automorphism of* $\mathbf{G}^F$.

Proof. This follows the lines of the proof of Lemma 2.13. □

Lemma 2.16. *Let* $\mathbf{G}$ *be simple, simply connected of type* E_7*,* $4|(q-1)$*, and let* $(\mathbf{L},\lambda)$ *be a unipotent* 1*-cuspidal pair corresponding to Line 3 of the* E_7*-table in* [11]*.*

(a) *There exists an irreducible character of positive* 2*-height in* $B = b_{\mathbf{G}^F}(\mathbf{L},\lambda)$*. This contains* $Z(\mathbf{G}^F)$ *in its kernel when* $q \equiv 1 \pmod 8$*.*

(b) *If* $q \not\equiv 1 \pmod 8$*, then there exists an irreducible character in* B *with* $Z(\mathbf{G}^F)$ *in its kernel and which is not stable under the outer diagonal automorphism of* $\mathbf{G}^F$*.*

An analogous result holds when $4|(q+1)$ *and* $(\mathbf{L},\lambda)$ *is a unipotent* 2*-cuspidal pair corresponding to Line 7 of the* E_7*-table in* [11]*.*

Proof. There exists $t \in \mathbf{G}_2^{*F}$ of order 4 such that $\mathbf{M}^* := C_{\mathbf{G}^*}(t)$ is a 1-split Levi subgroup of $\mathbf{G}^*$ of type E_6 containing $\mathbf{L}^*$, which is contained in $[\mathbf{G}^{*F},\mathbf{G}^{*F}]$ if and only if $q \equiv 1 \pmod 8$. As in the proof of Lemma 2.14, this gives rise to a character as in (a). For (b), consider the involution $t' \in \mathbf{L}^{*F}$ with $C^\circ_{\mathbf{G}^*}(t') = \mathbf{L}^*$ and $|C_{\mathbf{G}^*}(t')/C^\circ_{\mathbf{G}^*}(t')| = 2$. This lies in $[\mathbf{G}^{*F},\mathbf{G}^{*F}]$ (see [17]), and thus again arguing as before we find $\psi' \in \mathrm{Irr}(B)$ as in (b). The arguments for $4|(q+1)$ are entirely similar. □

2.5. **The height zero conjecture for unipotent blocks.** We need the following general observation on covering blocks.

Lemma 2.17. *Let* G *be a finite group,* b *an* ℓ*-block of* G*,* H *a normal subgroup of* G *and* c *a block of* H *covered by* b*.*

(a) *Suppose* H *has* ℓ'*-index in* G*. Then a defect group of* c *is a defect group of* b*. Further,* c *has irreducible character degrees with different* ℓ*-height if and only if* b *does.*

(b) *Suppose that* $H = XY$ *where* X *and* Y *are commuting normal subgroups such that* $X \cap Y$ *is a central* ℓ'*-subgroup of* H*. Let* c_X *be the block of* X *covered by* c *and let* c_Y *be the block of* Y *covered by* c*,* D_X *a defect group of* c_x *and* D_Y *a defect group of* c_Y*. Then* D_XD_Y *is a defect group of* c*. In particular,* D *is non-abelian if and only if at least one of* D_X *or* D_Y *is non-abelian. Further,* c *has irreducible character degrees with different* ℓ*-height if and only if one of* c_X *or* c_Y *does.*

(c) *Suppose* $G = HU$ *where* U *is a central* ℓ*-subgroup of* G*. Then* b *has abelian defect groups if and only if* c *has abelian defect groups and* b *has irreducible characters of different* ℓ*-height if and only if* c *does.*

Proof. Part (a) follows from the Clifford theory of characters and blocks (see for instance [25, Ch. 5, Thm. 5.10, Lem. 5.7 and 5.8]). Part (b) is immediate from the fact that $H = XY$ is a quotient of $X \times Y$ by a central ℓ'-subgroup. In (c), every irreducible character of H extends to a character of G, c is G-stable and b is the unique block of G covering c, and if D is a defect group of c, then DU is a defect group of b. □

Theorem 2.18. *Let* Z *be a central subgroup of* $\mathbf{G}^F$ *and let* $\bar{B}$ *be a block of* $\mathbf{G}^F/Z$ *dominated by a unipotent block* B *of* $\mathbf{G}^F$*. Suppose that* $\bar{B}$ *has non-abelian defect groups. Then* $\bar{B}$ *has irreducible characters of different height.*

Proof. By Lemma 2.11, B has a unipotent character of height zero. Since Z is contained in the kernel of every unipotent character of $\mathbf{G}^F$ it suffices to prove that there exists an irreducible character in $\mathrm{Irr}(B)$ of positive height and containing Z in its kernel.

By [11, Thm. A] there exists a unipotent e-cuspidal pair $(\mathbf{L},\lambda)$ of $\mathbf{G}^F$ such that $B = b_{\mathbf{G}^F}(\mathbf{L},\lambda)$ with λ of central ℓ-defect, unique up to $\mathbf{G}^F$-conjugacy. Here note that the existence of such a pair for bad primes is only proved for $\mathbf{G}$ simple and simply connected in [11], but by Lemma 2.12, the conclusion carries over to arbitrary $\mathbf{G}$. Suppose first that $\ell \geq 5$. By Lemmas 2.6 and 2.9, $W_{\mathbf{G}^F}(\mathbf{L},\lambda)$ is not an ℓ'-group. Thus, by Lemmas 2.10 and 2.8 there are irreducible unipotent characters of different height in $\mathcal{E}(\mathbf{G}^F,(\mathbf{L},\lambda))$. This proves the claim as Z is in the kernel of all unipotent characters.

We assume from now on that $\ell \leq 3$. Without loss of generality, we may assume that Z is an ℓ-group. We let $\mathbf{G}$ be a counter-example to the theorem of minimal semisimple rank. Let $\mathbf{X}$ be the product of an F-orbit of simple components of $[\mathbf{G},\mathbf{G}]$, and $\mathbf{Y}$ be the product of the remaining components of $[\mathbf{G},\mathbf{G}]$ (if any) with $Z^\circ(\mathbf{G})$. Then $\mathbf{G} = \mathbf{X}\mathbf{Y}$ and $\mathbf{X}^F\mathbf{Y}^F$ is a normal subgroup of $\mathbf{G}^F$ of index $|\mathbf{X}^F \cap \mathbf{Y}^F| = |Z(\mathbf{X}^F) \cap Z(\mathbf{Y}^F)|$. Denote by $B_\mathbf{X}$ the unique block (also unipotent) of $\mathbf{X}^F$ covered by B and let $B_\mathbf{Y}$ be defined similarly. Let $\bar{B}_\mathbf{X}$ be the block of $\mathbf{X}^F Z/Z \cong \mathbf{X}^F/(Z \cap \mathbf{X}^F)$ dominated by $B_\mathbf{X}$ and let $\bar{B}_\mathbf{Y}$ be defined similarly.

Let $\eta \in \mathrm{Irr}(B_\mathbf{X})$ with $Z \cap \mathbf{X}^F \leq \ker(\eta)$. We claim that η is $\mathbf{G}^F$-stable and is of height zero in $B_\mathbf{X}$. Indeed, let $\tau_\mathbf{X} \in \mathrm{Irr}(B_\mathbf{X}) \cap \mathcal{E}(\mathbf{X}^F,1)$ and $\tau_\mathbf{Y} \in \mathrm{Irr}(B_\mathbf{Y}) \cap \mathcal{E}(\mathbf{Y}^F,1)$ be of height zero (see Lemma 2.11) and let $\tau \in \mathrm{Irr}(B) \cap \mathcal{E}(\mathbf{G}^F,1)$ be the unique unipotent extension of $\tau_\mathbf{X}\tau_\mathbf{Y}$ to $\mathbf{G}^F$. Since Z is central, η extends to an irreducible character, say $\hat{\eta}$ of $\mathbf{X}^F Z$ with Z in its kernel. Since Z is an ℓ-group, there is a unique block of $\mathbf{X}^F Z$ covering $B_\mathbf{X}$, and this block is necessarily covered by B. Let ψ be an irreducible character of B lying above $\hat{\eta}$. Then $Z \leq \ker(\psi)$. Any irreducible constituent of the restriction of ψ to $\mathbf{X}^F\mathbf{Y}^F$ is of the form $\eta\eta'$, with $\eta' \in B_\mathbf{Y}$ and

$$\psi(1) = a|\mathbf{G}^F : I_{\mathbf{G}^F}(\eta\eta')|\eta(1)\eta'(1)$$

for some integer a (in fact $a = 1$ but we will not use this here). Since $\psi(1)_\ell = \tau(1)_\ell = \tau_\mathbf{X}(1)_\ell\tau_\mathbf{Y}(1)_\ell$ and since $\tau_\mathbf{X}$ and $\tau_\mathbf{Y}$ are of height zero, it follows from the above that η is of height zero and that $|\mathbf{G}^F : I_{\mathbf{G}^F}(\eta\eta')|$ is not divisible by ℓ. But $|\mathbf{G}^F : I_{\mathbf{G}^F}(\eta\eta')|$ is divisible by $|\mathbf{G}^F : I_{\mathbf{G}^F}(\eta)|$ and the latter index is a power of ℓ since $\eta \in \mathcal{E}_\ell(\mathbf{X}^F,1)$. Thus, η is $\mathbf{G}^F$-stable as claimed. Similarly, one sees that if $\zeta \in \mathrm{Irr}(B_\mathbf{Y})$ with $Z \cap \mathbf{Y}^F \leq \ker(\zeta)$, then ζ is $\mathbf{G}^F$-stable and is of height zero in $B_\mathbf{Y}$. In particular, all elements of $\mathrm{Irr}(\bar{B}_\mathbf{X})$ and of $\mathrm{Irr}(\bar{B}_\mathbf{Y})$ are of height zero.

Suppose that $\ell = 3$. By Lemma 2.6 and 2.9, $W_{\mathbf{G}^F}(\mathbf{L},\lambda)$ has order divisible by 3. Thus, by Lemma 2.4, there exists $\mathbf{X}$ such that $|W_{\mathbf{X}^F}(\mathbf{L}_\mathbf{X},\lambda_\mathbf{X})|$ is divisible by 3 where $(\mathbf{L}_\mathbf{X},\lambda_\mathbf{X})$ is the unipotent e-cuspidal pair of $\mathbf{X}^F$ corresponding to $(\mathbf{L},\lambda)$ by Lemmas 2.4 and 2.5, necessarily of central ℓ-defect. By Lemma 2.8, $W_{\mathbf{X}^F}(\mathbf{L}_\mathbf{X},\lambda_\mathbf{X}) \cong \mathfrak{S}_3$, $|Z(\mathbf{X}^F)|$ is divisible by 3 and either the components of $\mathbf{X}$ are of type A_2 or of type E_6. Without loss of generality, we may assume that $\mathbf{X}$ is simple. Suppose first that $\mathbf{X}$ is simple of type A_2. By Lemma 2.8, $\mathbf{X} = \mathbf{X_a}$ in the notation of [6]. Hence, by [4, Thm. 13], B is the principal block of $B_\mathbf{X}$. As has been shown above, every irreducible character of $\mathbf{X}^F$ which contains $\mathbf{X}^F \cap Z$ in its kernel has height zero and is stable under $\mathbf{G}^F$. By Lemma 2.13 it follows that $Z \cap \mathbf{X}^F \neq 1$, $3||(q-1)$ (respectively $3||(q+1)$) and that $\mathbf{G}^F$ induces inner automorphisms of $\mathbf{X}^F$, that is $\mathbf{G}^F = \mathbf{X}^F\mathbf{Y}^F U$ for some central subgroup U of $\mathbf{G}^F$. Since $Z \cap \mathbf{X}^F \neq 1$, $\mathbf{X}^F/(Z \cap \mathbf{X}^F) \cong \mathrm{L}_3(q)$ (respectively $\mathrm{U}_3(q)$) and $\mathbf{X}^F/(Z \cap \mathbf{X}^F)$ is a direct factor of $\mathbf{G}^F/Z$. Further, $\mathbf{X}^F/(Z \cap \mathbf{X}^F)$ has abelian Sylow 3-subgroups. Since U is central in $\mathbf{G}^F$, it follows by Lemma 2.17 that the block $\bar{B}_{\mathbf{Y}}$ of $\mathbf{Y}^F/(Z\cap\mathbf{Y}^F)$ has non-abelian defect groups. On the other hand, it has been shown above that all irreducible characters of $\bar{B}_{\mathbf{Y}}$ are of height zero. Hence, $\mathbf{Y}^F/(Z\cap\mathbf{Y}^F)$ is a counter-example to the theorem. But the semisimple rank of $\mathbf{Y}$ is strictly smaller than that of $\mathbf{G}$, a contradiction. Exactly the same argument works for the case that the components of $\mathbf{X}$ are of type E_6 by replacing Lemma 2.13 with Lemma 2.14.

Suppose now that $\ell=2$ and that the components of $\mathbf{X}$ are of classical type. Then $\mathbf{X}^F$ has a unique unipotent 2-block, namely the principal block and it follows by the above that all unipotent character degrees of $\mathbf{X}^F$ are odd. Thus, the components of $\mathbf{X}$ are of type A_1, so $\mathbf{X}^F$ is either $\mathrm{PGL}_2(q^d)$ or $\mathrm{SL}_2(q^d)$ for some d. Again we are done by the same arguments as above using Lemma 2.15. Thus, we may assume that all components of $\mathbf{G}$ are of exceptional type. By Lemmas 2.6 and 2.9, $W_{\mathbf{G}^F}(\mathbf{L},\lambda)$ has even order and by Lemma 2.4, there exists $\mathbf{X}$ such that $|W_{\mathbf{X}^F}(\mathbf{L_X},\lambda_{\mathbf{X}})|$ is divisible by 2 where $(\mathbf{L_X},\lambda_{\mathbf{X}})$ is the unipotent e-cuspidal pair of $\mathbf{X}^F$ corresponding to $(\mathbf{L},\lambda)$ necessarily of central ℓ-defect. Since $\mathbf{X}$ is of exceptional type, Lemma 2.8(b) gives that $\mathbf{L_X}$ is of type E_6 and $\lambda_{\mathbf{X}}$ corresponds to either line 3 or 7 of the E_7-table of [11, p. 354]. Then we are done by the same arguments as above using Lemma 2.16. □

2.6. **General blocks.** We also need to deal with the so-called quasi-isolated blocks of exceptional groups of Lie type.

Proposition 2.19. *Assume that $\mathbf{G}^F$ is of exceptional Lie type and ℓ is a bad prime different from the defining characteristic. Let Z be a central subgroup of $\mathbf{G}^F$ and let $\bar{B}$ be an ℓ-block of $\mathbf{G}^F/Z$ dominated by a quasi-isolated non-unipotent block B of $\mathbf{G}^F$. If $\bar{B}$ has non-abelian defect groups, then* $\mathrm{Irr}(\bar{B})$ *contains characters of positive height.*

Proof. We first deal with the case that $Z=1$, so $\bar{B}=B$. Here, the quasi-isolated blocks for bad primes were classified in [15, Thm. 1.2]. Any such block is of the form $B=b_{\mathbf{G}^F}(\mathbf{L},\lambda)$ for a suitable e-cuspidal pair $(\mathbf{L},\lambda)$ in $\mathbf{G}$, in such a way that all constituents of $R_{\mathbf{L}}^{\mathbf{G}}(\lambda)$ lie in $b_{\mathbf{G}^F}(\mathbf{L},\lambda)$, and the defect groups are abelian if and only if the relative Weyl group $W_{\mathbf{G}^F}(\mathbf{L},\lambda)$ has order prime to ℓ.

It is easily checked from the tables in [15] that all blocks B occurring in the situation of [15, Thm. 1.2] have the following property: either the characters in $B\cap\mathcal{E}(\mathbf{G}^F,\ell')$ lie in at least two different e-Harish-Chandra series, above e-cuspidal characters the ℓ-part of whose degrees is different, or the relative Weyl group has an irreducible character whose degree is divisible by ℓ. In the first case, the claim follows since then by inspection there are characters in $\mathrm{Irr}(B)\cap\mathcal{E}(\mathbf{G}^F,\ell')$ the ℓ-part of whose degrees is different. In the second case, let $s\in\mathbf{G}^{*F}$ be a semisimple (quasi-isolated) ℓ'-element such that $\mathrm{Irr}(B)\subseteq\mathcal{E}_\ell(\mathbf{G}^F,s)$. Lusztig's Jordan decomposition [18, Prop. 5.1] (see also [10, Rem. 13.24]) gives an ℓ-defect preserving bijection from $\mathcal{E}(\mathbf{G}^F,s)$ to the unipotent characters of the (possibly disconnected) centraliser $\mathbf{C}=C_{\mathbf{G}^*}(s)$ of s. By the results of [15, Thm. 1.2] this bijection sends $B\cap\mathcal{E}(\mathbf{G}^F,s)$ to a unipotent e-Harish-Chandra series in $\mathcal{E}(\mathbf{C}^F,1)$ (see [20, Thm 4.6] for e-Harish-Chandra series for disconnected groups), and the corresponding relative Weyl groups are isomorphic. Since the relative Weyl group has a character with degree divisible by ℓ, a straightforward generalisation of the arguments in [20, Cor. 6.6] shows that the relevant e-Harish-Chandra series in $\mathcal{E}(\mathbf{C}^F,1)$ contains characters the ℓ-part of whose degrees is different, and so there also exist characters in B of different height.

Now assume that $Z(\mathbf{G}^F) \neq 1$ and $Z = Z(\mathbf{G}^F)$, so that $\mathbf{G}$ is either of type E_6 and $\ell = 3$, or of type E_7 and $\ell = 2$. The only quasi-isolated block to consider for type E_6 is the one numbered 13 in [15, Tab. 3], respectively its Ennola dual in ${}^2E_6$. Since here the relative Weyl group has an irreducible character of degree divisible by 3, we get characters of different height in $\mathrm{Irr}(B) \cap \mathcal{E}(\mathbf{G}^F, \ell')$, which have the centre in their kernel. Similarly, the only cases in E_7 are the ones numbered 1 and 2 in [15, Tab. 4], for which the same argument applies. □

We can now show the Main Theorem for quasi-simple groups of Lie type. Let us write (BHZ2) for the assertion that blocks with all characters of height zero have abelian defect groups.

Theorem 2.20. *Suppose that* $\mathbf{G}$ *is simple and simply connected, not of type A, and* $\ell \neq p$. *Then (BHZ2) holds for* $\mathbf{G}^F/Z$ *for any central subgroup* Z *of* $\mathbf{G}^F$.

Proof. We may assume that Z is an ℓ-group. The Suzuki groups and the Ree groups ${}^2G_2(q^2)$ have no non-abelian Sylow subgroups for non-defining primes. The height zero conjecture for $G_2(q)$, Steinberg's triality groups ${}^3D_4(q)$ and the Ree groups ${}^2F_4(q^2)$ has been checked in [13, 9, 19]. Thus, we will assume that we are not in any of these cases.

Let B be an ℓ-block of $\mathbf{G}^F$ and $\bar{B}$ the ℓ-block of $\mathbf{G}^F/Z$ dominated by B. We assume that $\bar{B}$ has non-abelian defect groups. Let $s \in \mathbf{G}^{*F}$ be a semisimple ℓ'-element such that $\mathrm{Irr}(B) \subseteq \mathcal{E}_\ell(\mathbf{G}^F, s)$. Let $\mathbf{G}_1$ be a minimal F-stable Levi subgroup of $\mathbf{G}$ such that $C_{\mathbf{G}^*}(s) \leq \mathbf{G}_1^*$, thus s is quasi-isolated in $\mathbf{G}_1^*$. Let C be a Bonnafé–Rouquier correspondent of B in $\mathbf{G}_1^F$, and $\bar{C}$ the block of $\mathbf{G}_1^F/Z$ dominated by C. By [15, Thm. 1.3] Jordan decomposition induces a defect preserving bijection between $\mathrm{Irr}(\bar{B})$ and $\mathrm{Irr}(\bar{C})$ and $\bar{B}$ has abelian defect if and only if $\bar{C}$ does. Thus it suffices to prove the result for C. In particular, by Theorem 2.18, we may assume that s is not central in $\mathbf{G}_1$ and hence that $C_{\mathbf{G}_1^*}(s) = C_{\mathbf{G}^*}(s)$ is not a Levi subgroup of $\mathbf{G}_1^*$ (nor of $\mathbf{G}^*$).

We first consider the case that $Z(\mathbf{G})^F$ is an ℓ'-group. Let $\mathbf{G} \hookrightarrow \tilde{\mathbf{G}}$ be a regular embedding. If $\mathbf{G}$ has connected center we let $\mathbf{G} = \tilde{\mathbf{G}}$. Let $\tilde{B}$ be a block of $\tilde{\mathbf{G}}^F$ covering B and let $\tilde{s} \in \tilde{\mathbf{G}}^{*F}$ be a semisimple element such that $\mathrm{Irr}(\tilde{B}) \leq \mathcal{E}(\tilde{\mathbf{G}}^F, \tilde{s})$. Then by Lemma 2.17 it suffices to prove that $\tilde{B}$ has characters of different ℓ-heights (note that $Z = 1$ here). Further, let $\tilde{\mathbf{G}}_1$ be an F-stable Levi subgroup of $\tilde{\mathbf{G}}$ containing $C_{\tilde{\mathbf{G}}^*}(\tilde{s})$ such that $\tilde{s}$ is quasi-isolated in $\tilde{\mathbf{G}}_1$ and let $\tilde{C}$ be a Bonnafé–Rouquier correspondent of $\tilde{B}$ in $\tilde{\mathbf{G}}_1^F$. By [15, Thm. 7.12, Prop. 7.13(b)], $\tilde{C}$ has non-abelian defect groups. Hence it suffices to prove that $\tilde{C}$ has irreducible characters of different ℓ-heights. By the same reasoning as above, we may assume that s is not central in $\tilde{\mathbf{G}}_1$ and hence that $C_{\tilde{\mathbf{G}}_1^*}(s) = C_{\tilde{\mathbf{G}}^*}(s)$ is not a Levi subgroup of $\tilde{\mathbf{G}}_1^*$ (nor of $\tilde{\mathbf{G}}^*$).

If moreover ℓ is odd and good for $\tilde{\mathbf{G}}_1$, then by [12], there is a defect preserving bijection between $\mathrm{Irr}(\tilde{C})$ and $\mathrm{Irr}(C_0)$ for a unipotent block C_0 of $C_{\tilde{\mathbf{G}}_1^*}(\tilde{s})^F$ whose defect groups are isomorphic to those of $\tilde{C}$ and the result follows by Theorem 2.18. Enguehard has informed us that the prime 3 should have been excluded from the results of [12]. However, for classical groups with connected center Jordan decomposition commutes with Lusztig induction (see for instance appendix to latest version of [12]) and hence by [5, Thm. 2.5] and [7, 5.1, 5.2] the prime 3 may be included in the above.

Thus, we may assume that if ℓ is odd and $Z(\mathbf{G})$ is an ℓ'-group, then ℓ is bad for $\tilde{\mathbf{G}}_1$ and hence for $\tilde{\mathbf{G}}$ and $\mathbf{G}$. We now consider the various cases. Suppose that $\mathbf{G}$ is classical of type B, C, D. If $\ell = 2$, then s has odd order and $C_{\mathbf{G}^*}(s)$ is a Levi subgroup of $\mathbf{G}^*$, a contradiction. If ℓ is odd, then ℓ is good for $\mathbf{G}$. On the other hand, $Z(\mathbf{G})$ is a 2-group, a contradiction.

So, $\mathbf{G}$ is of exceptional type. If ℓ is good for $\mathbf{G}$, then $\ell \geq 5$, and in all cases $Z(\mathbf{G})$ is an ℓ'-group, a contradiction. Thus ℓ is bad for $\mathbf{G}$. Then by Proposition 2.19, $\mathbf{G}_1$ is proper in $\mathbf{G}$. Suppose that $\ell = 5$ and so $\mathbf{G}$ is of type E_8. Since $Z(\mathbf{G}) = 1$, 5 is bad for $\mathbf{G}_1$. Thus $\mathbf{G} = \mathbf{G}_1$, a contradiction.

Now assume that $\ell = 3$. Suppose that $\mathbf{G}$ is of type F_4. Then all components of $[\mathbf{G}_1, \mathbf{G}_1]$ are classical, hence 3 is good for $\mathbf{G}_1$ and $Z(\mathbf{G})$ is connected, a contradiction.

Suppose $\mathbf{G}$ is of type E_6. If all components of $\mathbf{G}_1$ are of type A, then $C^\circ_{\mathbf{G}_1^*}(s)$ is a Levi subgroup of $\mathbf{G}_1$. On the other hand, $Z(\mathbf{G}_1)/Z^\circ(\mathbf{G}_1) \leq Z(\mathbf{G})/Z^\circ(\mathbf{G})$ is a 3-group, and s is a $3'$-element, hence $C_{\mathbf{G}_1^*}(s)$ is connected. So, $C_{\mathbf{G}_1^*}(s)$ is a Levi subgroup of $\mathbf{G}_1^*$, a contradiction. Suppose $\mathbf{G}_1$ has a component, say $\mathbf{H}$ of type D_4 or D_5. So $\mathbf{G}_1 = \mathbf{H}Z^\circ(\mathbf{G}_1)$. Since the centre of $\mathbf{H}$ is a 2-group, by Lemma 2.17 we may replace $\mathbf{G}_1^F/Z$ with the direct product of $\mathbf{H}^F$ and $Z^\circ(\mathbf{G}_1)/Z$. Since (BHZ2) has been shown to be true for $\mathbf{H}^F$ above (here note that $\mathbf{H}$ is simply-connected), $\mathbf{H}^F$ has abelian Sylow 3-subgroups and we are done.

Suppose $\mathbf{G}$ is of type E_7. Then $|Z(\mathbf{G})| = 2$, hence 3 is bad for $\tilde{\mathbf{G}}_1$ and it follows that $[\mathbf{G}_1, \mathbf{G}_1]$ is of type E_6 (note that if $\mathbf{G}_1$ is proper in $\mathbf{G}$ then $\tilde{\mathbf{G}}_1$ is proper in $\tilde{\mathbf{G}}$). Denoting by $\bar{s}$ the image of s in $[\mathbf{G}_1, \mathbf{G}_1]^*$ and by D a block of $[\mathbf{G}_1, \mathbf{G}_1]^F$ covered by C, one sees that D corresponds to one of the lines 13, 14, 15 of Table 3 of [15]. If D corresponds to one of the lines 13 or 14, there are irreducible characters of different 3-heights in $\mathcal{E}([\mathbf{G}_1, \mathbf{G}_1]^F, \bar{s}) \cap \mathrm{Irr}(D)$. But since $\mathbf{G}_1$ has connected centre, and since $Z([\mathbf{G}_1, \mathbf{G}_1])/Z^\circ([\mathbf{G}_1, \mathbf{G}_1])$ is a 3-group and s has order prime to 3, all characters in $\mathcal{E}([\mathbf{G}_1, \mathbf{G}_1]^F, \bar{s})$ are $\mathbf{G}_1^F$-stable and extend to irreducible characters of $\mathbf{G}_1^F$ (see [2, Cor. 11.13]). All irreducible characters of $\mathbf{G}_1^F$ covering the same irreducible character of $[\mathbf{G}_1, \mathbf{G}_1]^F$ have the same degree and every element of $\mathrm{Irr}(D)$ is covered by an element of $\mathcal{E}(\mathbf{G}_1^F, s) \cap \mathrm{Irr}(C)$. Thus there exist elements in $\mathrm{Irr}(C) \cap \mathcal{E}(\mathbf{G}_1^F, s)$ of different 3-heights. If D corresponds to line 15, then 3 does not divide the order of $Z(\mathbf{G}_1^F)$. Hence, $\mathbf{G}_1^F = Z^\circ(\mathbf{G}_1^F) \times [\mathbf{G}_1, \mathbf{G}_1]^F$. By [15, Prop. 4.3], D has abelian defect groups hence so does C and there is nothing to prove.

If $\mathbf{G}$ is of type E_8, then exactly the same arguments as in the E_7 case apply hence we are left with one of the following cases: $[\mathbf{G}_1, \mathbf{G}_1]$ is of type $E_6 + A_1$ or of type E_7. In the former case, by Lemma 2.17 we may assume that the fixed point subgroup of the component of type A_1 is a direct factor of $\mathbf{G}_1^F$ and so has abelian Sylow 3-subgroups. Therefore, we may assume that $[\mathbf{G}_1, \mathbf{G}_1]$ is of type E_6 and we are done by the same argument as in the case that $\mathbf{G}$ is of type E_7. If $[\mathbf{G}_1, \mathbf{G}_1]$ has type E_7, then

$$|\mathbf{G}_1^F : [\mathbf{G}_1, \mathbf{G}_1]^F Z^\circ(\mathbf{G}_1)^F| = |[\mathbf{G}_1, \mathbf{G}_1]^F \cap Z^\circ(\mathbf{G}_1)^F| = 2,$$

hence by Lemma 2.17 we may assume that $\mathbf{G}_1$ is simple of type E_7, and we are done by Proposition 2.19.

Finally suppose that $\ell = 2$. In case $\mathbf{G}$ is of type E_6, we may replace $\mathbf{G}$ by $\tilde{\mathbf{G}}$ by Lemma 2.17 and still keep the assumption that $\tilde{\mathbf{G}}_1$ is proper in $\tilde{\mathbf{G}}$. Thus, either $Z(\mathbf{G})$ is connected or $Z(\mathbf{G})/Z^\circ(\mathbf{G})$ has order 2 (in case $\mathbf{G}$ is of type E_7). Consequently, since s

has odd order, $C_{\mathbf{G}_1^*}(s) = C_{\mathbf{G}^*}(s)$ is connected. Thus, if all components of $[\mathbf{G}_1, \mathbf{G}_1]$ are of classical type, then $C_{\mathbf{G}_1^*}(s)$ is a Levi subgroup of $\mathbf{G}_1^*$, a contradiction. We are left with the following cases: $\mathbf{G}$ is of type E_7 and $[\mathbf{G}_1, \mathbf{G}_1]$ is of type E_6, or $\mathbf{G}$ is of type E_8 and $[\mathbf{G}_1, \mathbf{G}_1]$ is of type E_6, $E_6 + A_1$ or E_7.

Suppose that $[\mathbf{G}_1, \mathbf{G}_1]$ is of type E_6. Since $C_{\mathbf{G}_1^*}(s)$ is connected and s is quasi-isolated in $\mathbf{G}_1^*$, $C^\circ_{\mathbf{G}_1^*}(s)$ has the same semisimple rank as $\mathbf{G}_1^*$. Thus, $\bar{s}$ and D correspond to one of the lines 1, 2, 6, 7, 8 or 12 of Table 3 of [15]. In all of these cases, there are characters in $\mathcal{E}([\mathbf{G}_1, \mathbf{G}_1]^F, \bar{s}) \cap \mathrm{Irr}(D)$ of different 2-heights. Since $Z(\mathbf{G})/Z^\circ(\mathbf{G})$ is a 2-group, every element of $\mathcal{E}([\mathbf{G}_1, \mathbf{G}_1]^F, \bar{s}) \cap \mathrm{Irr}(D)$ extends to an element of $\mathrm{Irr}(C) \cap \mathcal{E}(\mathbf{G}_1^F, s)$. Since Z is in the kernel of all characters in $\mathcal{E}(\mathbf{G}_1^F, s)$, $\bar{B}$ has characters of different 2-heights and we are done.

Suppose $\mathbf{G}$ is of type E_8 and $[\mathbf{G}_1, \mathbf{G}_1]$ is of type $E_6 + A_1$. Then by Lemma 2.17, we may assume that $\mathbf{G}_1^F = \mathbf{H}_1^F \times \mathbf{H}_2^F$, where $\mathbf{H}_1^F$ is isomorphic to $E_6(q)$ or ${}^2E_6(q)$, $\mathbf{H}_2$ has connected center and $[\mathbf{H}_2, \mathbf{H}_2]$ has a single component of type A_1. Since the block of $\mathbf{H}_2^F$ covered by C is quasi-isolated, we may assume that C covers a unipotent (in fact the principal) block of $\mathbf{H}_2^F$. If $\mathbf{H}_2^F/Z$ has non-abelian Sylow 2-subgroups, then we are done by Theorem 2.18. If the block of $\mathbf{H}_1^F$ covered by C has non-abelian defect groups, then we are done by Proposition 2.19.

Finally, assume that $\mathbf{G}$ is of type E_8 and $[\mathbf{G}_1, \mathbf{G}_1]$ is of type E_7. Since s is not central in $\mathbf{G}_1$, $1 \neq \bar{s}$ is a quasi-isolated element of $[\mathbf{G}_1, \mathbf{G}_1]^*$. By Table 5 of [15] the block D of $[\mathbf{G}_1, \mathbf{G}_1]^F$ has non-abelian defect groups. Now we are done by the same argument as given at the end of Proposition 2.19. □

3. Brauer's height zero conjecture for quasi-simple groups

Proof of the Main Theorem. We invoke the classification of finite simple groups. One direction of the assertion has been shown in [15, Thm. 1.1]. So we may now assume that all $\chi \in \mathrm{Irr}(B)$ have height zero. We need to show that B has abelian defect groups. If S is a covering group of a sporadic simple group or of ${}^2F_4(2)'$ it can be checked using the tables in [8] that the only ℓ-blocks with defect groups of order at least ℓ^3 and all characters in $\mathrm{Irr}(B)$ of height zero are the principal 2-block of J_1, the principal 3-block of $O'N$ and a 2-block of Co_3 with defect groups of order 2^7. For the first two groups, Sylow ℓ-subgroups are abelian, and the latter block has elementary abelian defect groups, see [16, §7].

Similarly, if S is an exceptional covering group of a finite simple group of Lie type, again by [8] there is no such block of positive defect at all.

The height zero conjecture for alternating groups $\mathfrak{A}_n$, $n \geq 7$, and their covering groups was verified in [24], for example, except for the 2-blocks of the double covering $2.\mathfrak{A}_n$. Since the height zero conjecture has been checked for the 2-blocks of $\mathfrak{A}_n$ we know that the only 2-blocks of $2.\mathfrak{A}_n$ which could possibly consist of characters of height zero are those whose defect groups in $\mathfrak{A}_n$ are abelian. But the latter have defect group of order at most 4, so the defect groups in $2.\mathfrak{A}_n$ have order at most 8, and for those the claim is again known by work of Olsson [23].

Now assume that S is of Lie type. If ℓ is the defining characteristic of S, then the result is contained in Proposition 2.3. We may hence suppose that ℓ is a non-defining prime. There, Brauer's height zero conjecture for groups of type A_n has been shown by Blau and Ellers [1]. For all the other types, the claim is shown in Theorem 2.20. □

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Department of Mathematical Sciences, City University London, Northampton Square, London EC1V 01B, U.K.
E-mail address: Radha.Kessar.1@city.ac.uk

FB Mathematik, TU Kaiserslautern, Postfach 3049, 67653 Kaiserslautern, Germany.
E-mail address: malle@mathematik.uni-kl.de