HEIGHT ZERO CONJECTURE WITH GALOIS AUTOMORPHISMS

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Abstract. We prove a strengthening of Brauer's height zero conjecture for principal 2-blocks with Galois automorphisms. This requires a new extension of the Itô–Michler theorem for the prime 2, again with Galois automorphisms. We close, this time for odd primes p, with a new characterisation of p-closed groups via the decomposition numbers of certain characters.

1. Introduction

Is there a strengthening of Brauer's Height Zero Conjecture with Galois automorphisms? If G is a finite group, p is a prime and B a p-block of G with defect group D, Brauer's conjecture from 1955 asserts that D is abelian if and only if all complex irreducible characters in B have height zero. (Recall that $\chi \in \mathrm{Irr}(B)$ has height zero if $\chi(1)_p = |G|_p/|D|$, and that $\mathrm{Irr}_0(B)$ denotes the set of irreducible characters in B which have height zero.) The proof of this statement has recently been completed [22].

The fundamental local/counting conjectures in representation theory of finite groups were successfully strengthened in [27] using the *Frobenius elements* of $\mathrm{Gal}(\mathbb{Q}^{\mathrm{ab}}/\mathbb{Q})$: those which send roots of unity of order not divisible by p to some fixed p-power. For $H \leq \mathrm{Gal}(\mathbb{Q}^{\mathrm{ab}}/\mathbb{Q})$, let $\mathrm{Irr}_H(B)$ denote the set of H-fixed irreducible characters of B. Then, is there an $H \neq 1$ such that $\mathrm{Irr}_H(B) \subseteq \mathrm{Irr}_0(B)$ if and only if D is abelian? If we pause to think on this for a moment, in the trivial case where G is a p-group, and therefore $\mathrm{Irr}(B)$ consists of all the irreducible characters of G, it seems natural to assume that H fixes the p-power roots of unity.

Let $\sigma \in \mathrm{Gal}(\mathbb{Q}^{\mathrm{ab}}/\mathbb{Q})$ be the automorphism that fixes 2-power roots of unity and complex-conjugates odd-order roots of unity. If B is a 2-block, let us denote by $\mathrm{Irr}_\sigma(B)$ the irreducible characters of B that are fixed by σ. The following strengthening of Brauer's height zero conjecture is the main result of this paper.

Theorem A. *Let B be the principal 2-block of a finite group G, and let $P \in \mathrm{Syl}_2(G)$. Then all characters in $\mathrm{Irr}_\sigma(B)$ have odd degree if and only if P is abelian.*

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Theorem A does not extend to arbitrary 2-blocks, even if these possess σ-invariant height zero characters, as shown by the non-principal 2-block of `SmallGroup`$(96, 13)$ in [34]. Also, we cannot replace σ by complex conjugation (as shown by many 2-groups, for instance). It might have some interest to remark that our σ needs not be a *Frobenius* automorphism because it is not necessarily true that if $m = |G|_{2'}$, then there exists an integer n such that $2^n \equiv -1 \pmod{m}$. (For instance, if $m = 7$ or 15.)

In order to obtain Theorem A, we shall need to prove the following extension of the Itô–Michler theorem for $p = 2$. Here, for G a finite group, we let $\mathrm{Irr}_\sigma(G)$ denote the set of σ-invariant irreducible characters of G.

Theorem B. *Let G be a finite group, and let $P \in \mathrm{Syl}_2(G)$. Then all characters in $\mathrm{Irr}_\sigma(G)$ have odd degree if and only if P is normal in G and is abelian.*

Theorem B resembles the main result of [6], where it is proved that if all the real-valued irreducible characters of G, $\mathrm{Irr}_{\mathbb{R}}(G)$, have odd degree, then G has a normal Sylow 2-subgroup. Let us point out now that the sets $\mathrm{Irr}_{\mathbb{R}}(G)$ and $\mathrm{Irr}_\sigma(G)$ are in general different: $\mathrm{Irr}_\sigma(G)$ can be properly contained in $\mathrm{Irr}_{\mathbb{R}}(G)$ (for instance, if $G = D_{24}$), or the other way around (if $G = \mathrm{GL}_2(3)$).

Unfortunately, we have not been able to find versions of Theorem A or B for odd primes and general finite groups, if they exist. (For solvable groups and only for the normality of the Sylow subgroups, see [13].)

In the final part of the paper, now for odd primes p, we derive a new characterisation for a finite group to be p-closed. In [20], we isolated the normality condition in the Itô–Michler theorem by proving that a Sylow p-subgroup P of G is normal if and only if the irreducible constituents of the permutation character $(1_P)^G$ have degree not divisible by p. Now we use the p-rational characters that lift modular characters of G to prove the following result. Recall that an irreducible complex character χ of G is *p-rational* if it has values in a cyclotomic field $\mathbb{Q}_m$, where m is not divisible by p.

Theorem C. *Let G be a finite group, and let p be an odd prime number. Let $P \in \mathrm{Syl}_p(G)$. Then $P \trianglelefteq G$ if and only if every p-rational $\chi \in \mathrm{Irr}(G)$ with $\chi^0 \in \mathrm{IBr}(G)$ has degree not divisible by p.*

Theorem C generalises the main result of [29] (for odd primes). It does not extend to $p = 2$, though: for instance, the Mathieu groups M_{22} and M_{24} do not have non-trivial characters lifting irreducible 2-modular characters. Due to the result in [20] already mentioned, it is tempting to explore possible relationships between the irreducible constituents of $(1_P)^G$ and the p-rational characters that lift irreducible Brauer characters. As shown by $\mathrm{SL}_2(3)$ for $p = 3$, these sets do not seem well related.

The p-rational characters that lift irreducible Brauer characters constitute an interesting object of study. I. M. Isaacs proved, for p-solvable groups and p odd, that every irreducible p-Brauer character has a unique p-rational lift ([14]). (So in p-solvable groups, these are abundant characters.) Moreover, continuing in this case, the subnormal irreducible constituents of these lifts are again p-rational lifts. Of course, Brauer characters do not admit lifts in general, although it is not uncommon to find p-rational characters that lift irreducible Brauer characters. In Example 5.6 we exhibit an infinite family of cases showing that p-rational lifts of modular characters are not necessarily unique. On the other hand, for $p > 3$, we have not been able to find a p-rational character χ such that $\chi^0 \in \mathrm{IBr}(G)$ with a subnormal constituent $\theta \in \mathrm{Irr}(N)$ such that θ^0 is not irreducible.

We have said above that Theorem A does not extend to arbitrary blocks. It is not unreasonable, though, to think that in any 2-block with a σ-invariant canonical character (these are the defect zero character of any root of the block, see [26] for a definition), perhaps one can strengthen Brauer's Height Zero Conjecture only using σ-invariant characters. We have not attempted this here.

We prove Theorem B in Section 3, then Theorem A in Section 4 using Theorem B, and finally Theorem C in Section 5. In Section 2 we derive some facts about characters of almost simple groups needed along the way.

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2. Characters of almost simple groups

In this section we derive some results on characters of finite almost simple groups with respect to the Galois automorphism $\sigma \in \mathrm{Gal}(\mathbb{Q}^{\mathrm{ab}}/\mathbb{Q})$ that fixes 2-power roots of unity and complex-conjugates odd-order roots of unity. These will be needed in the proofs of our first two main theorems. Throughout, $B_0(G)$ denotes the principal 2-block of a finite group G.

2.1. Existence of σ-invariant characters.

We first discuss simple groups with abelian Sylow 2-subgroups; for this we recall the following well-known facts:

Lemma 2.1. *Let S be non-abelian simple with abelian Sylow 2-subgroups.*

(a) *If G is quasi-simple with $G/\mathbf{Z}(G) = S$ and $|\mathbf{Z}(G)| = 2$, then G has non-abelian Sylow 2-subgroups.*
(b) *If $S < G \le \mathrm{Aut}(S)$ with G/S a 2-group, then G has non-abelian Sylow 2-subgroups.*

Proof. By the classification theorem of Walter [35], S is one of $\mathrm{L}_2(q)$ with $q \equiv \pm 3 \pmod 8$, $\mathrm{L}_2(2^f)$ $(f \ge 3)$, ${}^2G_2(3^{2f+1})$ $(f \ge 1)$ or J_1. Now the latter three do not have proper Schur covers, so in (a) there is nothing to prove for these. For $S = \mathrm{L}_2(q)$ with $q \equiv \pm 3 \pmod 8$, the only relevant central extension is $G = \mathrm{SL}_2(q)$, and this has quaternion Sylow 2-subgroups of order 8. If S has an even order outer automorphism, then either $S = \mathrm{L}_2(q)$ with $q \equiv \pm 3 \pmod 8$ and $G = \mathrm{PGL}_2(q)$ with dihedral Sylow 2-subgroups of order 8, or $S = \mathrm{L}_2(2^f)$ with f even, and here the field automorphisms act as Galois automorphisms on a Sylow 2-subgroup $P \cong \mathbb{F}_{2^f}^+$ of S. $\square$

Proposition 2.2. *Let S be non-abelian simple with abelian Sylow 2-subgroups.*

(a) *There exist $\chi_1 \in \mathrm{Irr}_\sigma(S)$ of even degree and $1_S \neq \chi_2 \in \mathrm{Irr}_\sigma(B_0(S))$.*
(b) *If G is quasi-simple with $G/\mathbf{Z}(G) = S$ and $|\mathbf{Z}(G)| = 2$, then there is a faithful $\chi \in \mathrm{Irr}_\sigma(B_0(G))$ (hence of even degree).*
(c) *If $S < G \leq \mathrm{Aut}(S)$ with G/S a 2-group, then there exists $\chi \in \mathrm{Irr}_\sigma(B_0(G))$ of even degree not lying over 1_S.*

Proof. In (a), for $S = \mathrm{L}_2(q)$ with $3 < q \equiv \pm 3 \pmod 8$ we take for χ_1 a (real) character of degree $q-1$ labelled by an element of odd order in a torus of order $q+1$ in the dual group $\mathrm{PGL}_2(q)$, for χ_2 the Steinberg character; for ${}^2G_2(q^2)$ with $q^2 \geq 27$ for χ_1 a (real) character of degree $(q^4-1)(q^2-\sqrt{3}q+1)$ (see [9]) and for χ_2 the Steinberg character; for J_1 for χ_1 one of the two (real) characters of degree 56, and for χ_2 the rational character of degree 209; and for $\mathrm{L}_2(q)$, $q = 2^f \geq 8$, for χ_1 the Steinberg character and for χ_2 a real character of degree $q-1$ labelled by an element of order $q+1$ in $\mathrm{PGL}_2(q)$. Then in all cases, χ_1 is σ-invariant of even degree, and $\chi_2 \in \mathrm{Irr}_\sigma(B_0(S))$.

For (b), the only case is $G = \mathrm{SL}_2(q)$ with $q \equiv \pm 3 \pmod 8$. Here the claim follows from the known character table of G (see [10, Tab. 2.6]). Alternatively, the dual group $\mathrm{PGL}_2(q)$ contains an element s of order 4 not lying in $\mathrm{L}_2(q)$. Since s is regular semisimple, by Lusztig's Jordan decomposition [10, 2.6.22] it labels a faithful irreducible Deligne–Lusztig character χ of $\mathrm{SL}_2(q)$. Since s is a rational element, χ is rational (by [10, Cor. 3.3.14]) and thus σ-invariant. Also, as s is a 2-element, $\chi \in \mathrm{Irr}(B_0(G))$ by [3, Thm. 21.14].

In (c), for $S = \mathrm{L}_2(2^f)$ all outer automorphisms are field automorphisms. Here, the irreducible induction of a character $\chi \in \mathrm{Irr}(S)$ corresponding to an element of order $q+1$ in $\mathrm{PGL}_2(q)$ is as desired. For $\mathrm{L}_2(q)$, $3 < q \equiv \pm 3 \pmod 8$, we have $G = \mathrm{PGL}_2(q)$. Here we take for χ an irreducible character of degree $q-1$ labelled by an element of order 4 in the dual group $\mathrm{SL}_2(q)$. Again, χ is rational by [10, Cor. 3.3.14] and lies in $B_0(G)$ by [3, Thm. 21.14]. □

The main result of this section is the following; its proof will occupy the remainder of this section:

Theorem 2.3. *Let S be simple with non-abelian Sylow 2-subgroups and $S \leq G \leq \mathrm{Aut}(S)$ with G/S a 2-group. Then there exists $\chi \in \mathrm{Irr}_\sigma(B_0(G))$ of even degree not lying over 1_S.*

In this paper, if N is a normal subgroup of G and $\theta \in \mathrm{Irr}(N)$, we will denote by G_θ or $I_G(\theta)$ the stabilizer of θ in G. We will make use of the following criterion:

Lemma 2.4. *In the situation of Theorem 2.3, let $1_S \neq \theta \in \mathrm{Irr}(S)$ of odd degree be such that $\theta^\sigma = \theta^g$ for some $g \in G$. If θ is not G-invariant and lies in the principal 2-block of S, then θ^G has a constituent χ satisfying the conclusion of Theorem 2.3.*

Proof. Since S is perfect, the determinantal order $o(\theta)$ is 1, so $|G:S|$ is prime to $o(\theta)\theta(1)$. Let G_θ be the stabiliser of θ in G. By [15, Cor. 6.28] there exists a unique $\eta \in \mathrm{Irr}(G_\theta)$ extending θ with $o(\eta) = 1$. Since $\theta^\sigma = \theta^g$ for some $g \in G$ we have $\theta^{\sigma g^{-1}} = \theta$, so $\eta^{\sigma g^{-1}}$ is an extension of θ with determinantal order 1. Thus $\eta^{\sigma g^{-1}} = \eta$, and therefore $\eta^\sigma = \eta^g$. Now

$\chi := \eta^G \in \mathrm{Irr}(G)$ has even degree because $G_\theta < G$, and $\chi^\sigma = (\eta^\sigma)^G = (\eta^g)^G = \chi^g = \chi$. Finally χ lies in $B_0(G)$ since this is the only 2-block covering $B_0(S)$ by [26, Cor. 9.6]. $\square$

2.2. **Alternating, sporadic, Ree and Suzuki groups.** Let's first get some easy cases out of the way.

Proposition 2.5. *Theorem* 2.3 *holds for S either an alternating group $\mathfrak{A}_n$ with $n \geq 5$ or a sporadic simple group.*

Proof. For the sporadic groups and their automorphism groups, the known character tables [34] allow one to check the claim directly. Let now $G = \mathfrak{S}_n$ with $n \geq 5$. Let λ be the partition $(n-2, 2)$ when $n \equiv 0, 3 \pmod 4$ and $(n-2, 1^2)$ when $n \equiv 1, 2 \pmod 4$. Then by the hook formula, the irreducible character $\chi \in \mathrm{Irr}(G)$ labelled by λ has even degree. Since λ has 2-core of size 1 or 0, χ lies in the principal 2-block of G. Since all characters of $\mathfrak{S}_n$ are rational valued, this proves our claim in this case.

The group $\mathfrak{A}_5$ has abelian Sylow 2-subgroups. For $n \geq 6$, λ is not self-conjugate and so χ restricts irreducibly to $S = \mathfrak{A}_n$ and hence also proves our claim there. The only remaining case is $S = \mathfrak{A}_6$ and G one of $\mathrm{PGL}_2(9)$, M_{10} or $\mathrm{Aut}(\mathfrak{S}_6)$. For these groups, the extensions of χ_S to G have values in $\mathbb{Q}(\zeta_8)$ and thus are σ-invariant. $\square$

Proposition 2.6. *Theorem* 2.3 *holds for S a Suzuki or Ree group.*

Proof. The groups ${}^2G_2(3^{2f+1})$ have abelian Sylow 2-subgroups and thus need not be considered. For $S = {}^2F_4(2)'$ the assertion is easily checked. Otherwise, $\mathrm{Out}(S)$ has odd order whence $G = S$. Here, all characters apart from the Steinberg character lie in the principal 2-block by [3, Thm. 6.18]. For ${}^2B_2(2^{2f+1})$ the two cuspidal unipotent characters have values in $\mathbb{Q}(\sqrt{-1})$ and thus work; for ${}^2F_4(2^{2f+1})$, we may take any of the three unipotent characters of even degree in the principal series. $\square$

2.3. **Groups of Lie type in characteristic 2.** In this section we work in the following setting: $\mathbf{G}$ is a simple algebraic group of adjoint type over an algebraically closed field of characteristic 2 with a Frobenius endomorphism F such that $S = [\mathbf{G}^F, \mathbf{G}^F]$ is non-abelian simple. We denote by $\mathbf{G}^*$ a group in duality with $\mathbf{G}$, with corresponding Frobenius map also denoted F. Thus, $\mathbf{G}^*$ is of simply connected type. The characters whose existence is stipulated in Theorem 2.3 will mainly be constructed as follows:

Lemma 2.7. *Let $\mathbf{G}, \mathbf{G}^*$ and S be as above. Let $1 \neq s \in \mathbf{G}^{*F}$ be a semisimple element with connected centraliser in $\mathbf{G}^*/\mathbf{Z}(\mathbf{G}^*)$ and let $\chi \in \mathrm{Irr}(\mathbf{G}^F)$ be the associated regular character. Then*

(a) *χ restricts irreducibly to S;*
(b) *$\chi_S \in \mathrm{Irr}(B_0(S))$;*
(c) *if s is real then χ is σ-stable;*
(d) *if s is not regular then $\chi(1)$ is even; and*
(e) *for any automorphism γ of $\mathbf{G}^F$ induced by a Frobenius map $F_0 : \mathbf{G} \to \mathbf{G}$ commuting with F, χ has a σ-stable extension to $\mathbf{G}^F\langle\gamma\rangle_\chi$.*

Proof. As $\mathbf{G}$ has trivial centre, there is a unique (irreducible) regular character χ of $\mathbf{G}^F$ attached to s, see [5, 12.4.10]. Let $\mathbf{H} \twoheadrightarrow \mathbf{G}$ be a simply connected covering of $\mathbf{G}$, and $\mathbf{H} \hookrightarrow \tilde{\mathbf{H}}$ a regular embedding and choose corresponding Frobenius maps on $\mathbf{H}$ and $\tilde{\mathbf{H}}$, also denoted F. The natural maps $\mathbf{H} \hookrightarrow \tilde{\mathbf{H}} \twoheadrightarrow \tilde{\mathbf{H}}/\mathbf{Z}(\tilde{\mathbf{H}}) \cong \mathbf{H}/\mathbf{Z}(\mathbf{H}) \cong \mathbf{G}$ induce F-equivariant maps $\mathbf{G}^* \hookrightarrow \tilde{\mathbf{H}}^* \twoheadrightarrow \mathbf{H}^*$ of dual groups, where $\mathbf{H}^* \cong \mathbf{G}^*/\mathbf{Z}(\mathbf{G}^*)$. By assumption, the image $\bar{s} \in \mathbf{H}^*$ of s has connected centraliser. Let $\tilde{\chi}$ denote the inflation of χ to $\tilde{\mathbf{H}}^F$. By [10, Prop. 2.6.17 and Lemma 2.6.20] the number of constituents of $\tilde{\chi}|_{\mathbf{H}^F}$ equals the number of connected components of $C_{\mathbf{H}^*}(\bar{s})$, that is, $\tilde{\chi}$ restricts irreducibly to $\mathbf{H}^F$. Thus χ restricts irreducibly to $S = [\mathbf{G}^F, \mathbf{G}^F] \cong \mathbf{H}^F/\mathbf{Z}(\mathbf{H})^F$ whence we obtain (a).

Now, in defining characteristic all irreducible characters of $\mathbf{G}^F$ apart from those above the Steinberg character of S lie in the principal 2-block [3, Thm. 6.18]. Since the Steinberg character is unipotent while $\chi \in \mathcal{E}(G, s)$ with $s \neq 1$, we have $\chi \in \mathrm{Irr}(B_0(\mathbf{G}^F))$ and hence also $\chi_S \in \mathrm{Irr}(B_0(S))$. For (c) observe that χ is uniform by its very definition as a linear combination of Deligne–Lusztig characters [10, Thm. 3.4.16]. Then by the character formula for Deligne–Lusztig characters [10, Thm. 2.2.16] the values of χ lie in the field generated by values of linear characters of order dividing $o(s)$ of various tori of $\mathbf{G}^F$. Thus, since s is semisimple and hence of odd order, the character values of χ lie in an extension generated by odd-order roots of unity. Moreover, if s is real then χ is real valued using [10, Prop. 3.3.15], whence it is σ-stable. By the degree formula [10, Thm. 3.4.16(c) and Cor. 2.6.6] the degree $\chi(1)$ is divisible by $|C_{\mathbf{G}^*}(s)^F|_2$, hence even if s is not regular.

For (e) note that we are in the setting of [31, §6.5]: Since χ is regular, its Alvis–Curtis dual χ^0 is semisimple and hence lies in the image of the map f from [31, Thm 5.7]. Now a maximal unipotent subgroup of $\mathbf{G}^F$ is a (Sylow) 2-group, hence all of its characters are σ-invariant. This shows that our Galois automorphism σ satisfies the Assumption 6.6 of [31] with $\tilde{t} = 1$. In particular μ_σ in [31, Rem. 6.7] is then the trivial character. Our assertion (e) is now [31, Prop. 6.8]. For this note that its proof applies to σ since it only uses [31, Prop. 6.2] which is formulated for arbitrary Galois automorphisms. Moreover, the proof actually first shows that the Alvis–Curtis dual χ of χ_0 has the desired property. □

Proposition 2.8. *Theorem* 2.3 *holds for S a simple group of Lie type in characteristic* 2.

Proof. Let S be as in the statement. By Proposition 2.6 we may assume S is not a Suzuki or Ree group. Thus there is a simple algebraic group $\mathbf{G}$ of adjoint type in characteristic 2 with a Frobenius endomorphism $F : \mathbf{G} \to \mathbf{G}$ such that $S = [\mathbf{G}^F, \mathbf{G}^F]$. We have $\mathbf{G}$ is not of type A_1 since Sylow 2-subgroups of S are non-abelian.

For $S = \mathrm{L}_3(2)$ or $\mathrm{L}_4(2)$ the claim is easily verified. For all other cases we claim $\mathbf{G}^{*F}$ contains a non-trivial real non-regular semisimple element with connected centraliser in $\mathbf{G}^*/\mathbf{Z}(\mathbf{G}^*)$. Indeed, in groups of type A we choose an element with non-trivial eigenvalues the two primitive third roots of unity, and all other eigenvalues 1, in groups of type E_6 we take an element of order five in a subsystem subgroup of type A_3, so centralised at least by an A_1-subgroup, and in all other types we take any element of order 3 in an SL_2-subgroup (this has connected centraliser by application of [24, Prop. 14.20]). Now a Sylow 2-subgroup of the outer automorphism group of S is generated by graph automorphisms together with Frobenius endomorphisms of $\mathbf{G}$ commuting with F (see [11, Thm. 2.5.1]). Thus, by Lemma 2.7 we are done unless G induces non-trivial graph automorphisms, that is, S is of type A_{n-1} ($n \geq 3$), D_n ($n \geq 4$) or E_6. So now assume we are in the latter cases.

For $S = \mathrm{L}_n(q)$ let $t \in \mathrm{PGL}_n(q)$ be the image of an element $\tilde{t} \in \mathrm{GL}_n(q)$ of order $q^n - 1$ (if n is odd), respectively $q^{n-1} - 1$ (if n is even), and let $s := t^d \in [\mathrm{PGL}_n(q), \mathrm{PGL}_n(q)]$ with $d = \gcd(n, q-1)$. Then s has connected centraliser in $\mathrm{PGL}_n \cong \mathbf{G}^*/\mathbf{Z}(\mathbf{G}^*)$. It parametrises a semisimple character χ of S (see [10, 2.6.10]) whose values lie in a cyclotomic field generated by odd-order roots of unity. The eigenvalues of $\tilde{t}$ form an orbit under $\mathrm{Gal}(\mathbb{F}_{q^n}/\mathbb{F}_q)$ of a generator of $\mathbb{F}_{q^n}^\times$ if n is odd, respectively an orbit under $\mathrm{Gal}(\mathbb{F}_{q^{n-1}}/\mathbb{F}_q)$ of a generator of $\mathbb{F}_{q^{n-1}}^\times$ together with 1, if n is even. From this it follows that the class of s is not invariant under non-trivial 2-power order field automorphisms, nor under their product with the transpose-inverse automorphism, sending s to its inverse (up to conjugation). Thus, using [33, Prop. 7.2], χ is not invariant under the dual automorphisms of S, so $\chi^G \in \mathrm{Irr}(B_0(G))$ is of even degree and real, whence σ-stable.

Let S be of type $D_n(q)$. Since we are in characteristic 2 we may identify $\mathbf{G}^{*F}$ with $\mathrm{SO}_{2n}^+(q)$. Let s be an element of order $q^n - 1$ in the stabiliser $\mathrm{GL}_n(q)$ of a maximal totally isotropic subspace of $\mathbf{G}^{*F}$. Being semisimple, s has connected centraliser in $\mathbf{G}^*/\mathbf{Z}(\mathbf{G}^*)$. The eigenvalues of s in this representation are then a Galois orbit of a generator of $\mathbb{F}_{q^n}^\times$ together with their inverses. It follows that the set of eigenvalues is not invariant under any non-trivial automorphism of $\mathbb{F}_q$, hence the class of s is not invariant under any non-trivial field automorphism of $\mathbf{G}^{*F}$. Also, if n is even, then s is real but the graph automorphism interchanges the two conjugacy classes of stabilisers of totally singular subspaces, hence does not fix the class of s. If n is odd, s is non-real but the graph automorphism of S induces the transpose-inverse automorphism of $\mathrm{GL}_n(q)$, so conjugates s to its inverse. So in either case the semisimple character χ of S labelled by s has $\chi^G \in \mathrm{Irr}_\sigma(B_0(G))$ of even degree.

Finally, for $S = E_6(q)$ let s be the third power of a generator of a maximal torus T of $\mathbf{G}^{*F}$ of order $q^6 + q^3 + 1$, contained in a subsystem subgroup $H = A_2(q^3)$. Since the order of s is prime to 3, its image has connected centraliser in $\mathbf{G}^*/\mathbf{Z}(\mathbf{G}^*)$ by [24, Prop. 14.20]. By definition the order of s is divisible by a Zsigmondy prime for $q^9 - 1$, so s lies in a unique maximal torus of $\mathbf{G}^{*F}$ and so is regular semisimple. Thus $N_S(\langle s \rangle) = N_S(T)$. Since $|N_S(T)/T| = 9$, s is non-real. A computation in the root system of $\mathbf{G}^*$ using Chevie [9] shows that the graph automorphism of order 2 of $\mathbf{G}^*$ induces on $H = A_2(q^3)$ the transpose-inverse automorphism and thus inverts s. Computation in A_2 shows that the class of s is also not invariant under non-trivial field automorphisms of 2-power order or products of these with the graph automorphism, so we conclude as before. □

2.4. **Groups of Lie type in odd characteristic.** We now prove Theorem 2.3 for simple groups of Lie type in odd characteristic. Let $\mathbf{G}$ be a simple algebraic group of adjoint type over a field of odd characteristic, with a Frobenius endomorphism F with respect to an $\mathbb{F}_q$-structure, and let $(\mathbf{G}^*, F)$ be dual to $(\mathbf{G}, F)$. We set $S := [\mathbf{G}^F, \mathbf{G}^F]$. Similar to the approach in the even characteristic case, the following will be our main source of suitable characters:

Lemma 2.9. *Let $\mathbf{G}, \mathbf{G}^*, S$ be as above. Let $1 \neq s \in \mathbf{G}^{*F}$ be a 2-element with connected centraliser in $\mathbf{G}^*/\mathbf{Z}(\mathbf{G}^*)$. Then the associated semisimple character $\chi \in \mathrm{Irr}(\mathbf{G}^F)$ satisfies:*

(a) *χ restricts irreducibly to S;*
(b) *$\chi_S \in \mathrm{Irr}(B_0(S))$;*
(c) *χ is σ-stable;*
(d) *$\chi(1)$ is odd if and only if s lies in the centre of a Sylow 2-subgroup of $\mathbf{G}^{*F}$; and*
(e) *for any automorphism γ of $\mathbf{G}^F$ induced by a Frobenius map $F_0 : \mathbf{G} \to \mathbf{G}$ commuting with F, χ has a σ-stable extension to $\mathbf{G}^F\langle\gamma\rangle_\chi$.*

Proof. Since $\mathbf{G}$ is of adjoint type we have $\mathbf{Z}(\mathbf{G}) = 1$ and thus there is a unique semisimple character $\chi \in \mathrm{Irr}(\mathbf{G}^F)$ in the Lusztig series $\mathcal{E}(\mathbf{G}^F, s)$ (see [10, 2.6.10]). Since the centraliser of s in $\mathbf{G}^*/\mathbf{Z}(\mathbf{G}^*)$ is connected, the argument in the proof of the corresponding statement in Lemma 2.7 gives (a). For s a 2-element, χ lies in the principal 2-block by [7, Thm. B]. Since $\mathbf{G}^*$ is of simply connected type, $C_{\mathbf{G}^*}(s)$ is connected, so by the definition of χ as a linear combination of Deligne–Lusztig characters [5, Def. 12.4.2], the values of χ only involve 2-power roots of unity, and thus χ is σ-stable, whence (c). Part (d) is a direct consequence of the degree formula [10, Cor. 2.6.6] for Lusztig's Jordan decomposition.

For (e) we again use [31]. Observe that $\mathbf{G}$ is of adjoint type. In the setting of [31, 3.1] choose $\phi_0 \in \mathrm{Irr}((\mathbb{F}_{q^N}, +))$ by decomposing $(\mathbb{F}_{q^N}, +) = (\mathbb{F}_p, +) \oplus C$ such that $-C = C$ and defining

$$\phi_0(a) := \begin{cases} \exp(2\pi\mathbf{i}\, a/p) & \text{if } a \in \mathbb{F}_p, \\ 1 & \text{if } a \in C. \end{cases}$$

As σ complex conjugates odd order roots of unity, ${}^\sigma\phi_0(a) = \overline{\phi_0(a)} = \phi_0(-a)$ for all $a \in \mathbb{F}_{q^N}$ and so in the notation of [31, Lemma 3.2],

$${}^\sigma\phi_i(x_i(a)) = {}^\sigma\phi_0(c_i a) = \phi_0(-c_i a) = \phi_i(x_i(-a)) \qquad \text{for all } a \in \mathbb{F}_{q^N}.$$

Let $F_0 : \mathbf{G} \to \mathbf{G}$ denote a Frobenius map with respect to an $\mathbb{F}_p$-structure and such that $F = F_0^f\rho$ for some $f \geq 1$ and some graph automorphism ρ of $\mathbf{G}$. Let $\mathbf{T} \leq \mathbf{G}$ be a maximally F_0-split torus. Since $\mathbf{G}$ is of adjoint type there is $\tilde{t} \in \mathbf{T}^{F_0}$ such that $\alpha_i(t) = -1$ for all simple roots α_i and thus $x_i(a)^{\tilde{t}} = x_i(-a)$ for all $a \in \mathbb{F}_{q^N}$. This means that ${}^\sigma\phi_i(x_i(a)) = \phi_i(x_i(a)^{\tilde{t}})$ for all i and so ${}^\sigma\phi = \phi^{\tilde{t}}$ for all $\phi \in \mathrm{Irr}(\mathbf{U}/[\mathbf{U},\mathbf{U}]^F)$.

Now note that $\tilde{t}$ is invariant under all graph automorphisms of $\mathbf{G}$ and so $d(\tilde{t})\tilde{t}^{-1} = 1$ for all graph and field automorphisms d of $\mathbf{G}^F$. Thus Assumption 6.6 in [31] is satisfied. Then [31, Prop. 6.8] shows that the semisimple character χ has a σ-stable extension as claimed. $\square$

Proposition 2.10. *Theorem 2.3 holds for $S = \mathrm{L}_n(q)$ and $\mathrm{U}_n(q)$ with $n \geq 2$ and q odd.*

Proof. When $n = 2$, since Sylow 2-subgroups of S are assumed non-abelian we have $q \equiv \pm 1 \pmod 8$. Here from the character table in [10, Tab. 2.6] one sees that any element $s \in \mathbf{G}^{*F} \cong \mathrm{SL}_2(q)$ of order 8 will satisfy the assumptions of Lemma 2.9.

So now assume $n \geq 3$. Let $\zeta \in \mathbb{F}_{q^2}^{\times}$ be a generator of the Sylow 2-subgroup. Let $s \in \mathrm{SL}_n(q)$ be an element with eigenvalues $\zeta, \zeta^q, \zeta^{-q-1}$ and all other eigenvalues equal to 1. Then s is a 2-element that is conjugate to its qth power, but not to any other of its primitive powers modulo scalars, unless $n = 4$ and $q \equiv 3 \pmod 4$. Let us for the moment exclude that latter case. Then the image of the conjugacy class of s in $\mathrm{L}_n(q)$ is not invariant under non-trivial field automorphisms of $\mathrm{L}_n(q)$. The transpose-inverse graph automorphism acts by inverting the eigenvalues of semisimple elements, and thus it ensues that the class of s is neither invariant under products of field automorphisms with this graph automorphism. Furthermore, the image of s has connected centraliser in $\mathbf{G}^*/\mathbf{Z}(\mathbf{G}^*) \cong \mathrm{PGL}_n$. Let $\chi \in \mathrm{Irr}(\mathrm{PGL}_n(q))$ be the semisimple character associated to s. Then $\chi \in \mathrm{Irr}_\sigma(B_0(\mathrm{PGL}_n(q)))$ by Lemma 2.9 and χ restricts irreducibly to $S = \mathrm{L}_n(q)$. By what we said above about s, $\psi := \chi_S \in \mathrm{Irr}_\sigma(B_0(S))$ is not invariant under 2-power order field or graph automorphisms of S. Thus, the full stabiliser of ψ in $\mathrm{Out}(S)$ is the group of diagonal automorphisms. So for any G as in Theorem 2.3, ψ extends to $\tilde{\chi} := \chi_{G \cap \mathrm{PGL}_n(q)} \in \mathrm{Irr}(G \cap \mathrm{PGL}_n(q))$ and from there induces irreducibly to G, whence $\tilde{\chi}^G$ is as desired.

Now assume that $n = 4$ and $q \equiv 3 \pmod 4$. Since q is not a square, $S = \mathrm{L}_4(q)$ does not have even order field automorphisms. Let χ be the unipotent character of $\mathrm{PGL}_4(q)$ labelled by the partition (2^2). Then χ has even degree, lies in the principal 2-block and restricts irreducibly to S. Furthermore, χ extends to a rational character $\tilde{\chi}$ of the extension of $\mathrm{PGL}_4(q)$ by the graph automorphism, by [4, Thm. II.3.3] combined with [32, Thm 3.8]. Then $\tilde{\chi}_G$ is as desired.

Similarly, we let $s \in \mathrm{SU}_n(q)$ be an element with non-trivial eigenvalues $\zeta, \zeta^{-q}, \zeta^{q-1}$; then precisely the same argument as before applies to show our claim for $S = \mathrm{U}_n(q)$. $\square$

Proposition 2.11. *Theorem 2.3 holds for S of exceptional Lie type in odd characteristic, not of type ${}^{(2)}E_6$ or E_7.*

Proof. Let S be simple of exceptional type. By Proposition 2.6 we may assume that $S = [\mathbf{G}^F, \mathbf{G}^F]$ for $\mathbf{G}, F$ as above. Let $s \in \mathbf{G}^{*F}$ be a 2-element that is not 2-central. Since $\mathbf{Z}(\mathbf{G}^*) = 1$ the centraliser of s in the simply connected group $\mathbf{G}^*/\mathbf{Z}(\mathbf{G}^*) = \mathbf{G}^*$ is connected. If moreover G only contains automorphisms induced by Frobenius maps, then by Lemma 2.9, the semisimple character $\chi \in \mathcal{E}(\mathbf{G}^F, s)$ does the job. So we are left with the groups of type G_2 in characteristic 3.

For $S = G_2(3^n)$ we need to discuss extensions G involving the exceptional graph automorphism σ of order 2, or its product with a field automorphism. Let $H \leq \mathbf{G}^{*F} \cong \mathbf{G}^F = S$ be a subsystem subgroup of type A_1^2. Let $s \in H$ be an element of maximal 2-power order in the short root A_1-factor. Then s centralises a long root element, thus the image of s under σ centralises a short root element, whence neither σ nor its product with a field automorphism can fix the class of s. Moreover, since s is of maximal 2-power order in

$A_1(q)$, its class is not fixed by any non-trivial 2-power order field automorphism of S. Thus, the semisimple character of S labelled by s induces irreducibly to G, lies in the principal block, and is real as any semisimple element of an A_1-type group is conjugate to its inverse. □

Proposition 2.12. *Theorem* 2.3 *holds for S of Lie type in odd characteristic if q is not a square.*

Proof. If q is not a square then S does not possess even order field automorphisms. So in this case only products of diagonal and graph automorphisms can be present in G. Then we let χ be a rational unipotent character of $\mathbf{G}^F$ of even degree in the principal 2-block of $\mathbf{G}^F$. More specifically, for $\mathbf{G}$ of type E_6 we take χ to be $\phi_{20,2}$ for the untwisted and $\phi_{4,1}$ for the twisted version, and for $\mathbf{G}$ of type E_7 we take $\phi_{210,6}$. In view of Proposition 2.11 these are the only groups of exceptional type we need to discuss. For $\mathbf{G}$ of classical type we take χ labelled by the symbol $\binom{0,1,n}{-}$ for types B_n or C_n ($n \geq 2$), by $\binom{0,1,2,n-1}{-}$ for type D_n ($n \geq 4$), by $\binom{1,n-1}{-}$ for type ${}^2D_n$ with $n \geq 4$ even, and by $\binom{0,1,n}{1}$ when $n \geq 5$ is odd. These are of even degree by [10, Prop. 4.4.15].

All of the above lie in the principal block by [7]. The restriction of the unipotent character χ to $S = [\mathbf{G}^F, \mathbf{G}^F]$ is still irreducible, hence of even degree, and lies in the principal block of S. Finally, by [4, Cor. II.3.4] combined with [32, Thm 3.8], χ extends to a rational character of the extension of $\mathbf{G}^F$ with a graph automorphism of order 2, if such exists. Thus $\chi|_S$ is as required. □

Proposition 2.13. *Theorem* 2.3 *holds for S of Lie type in odd characteristic if q is a square.*

Proof. Here we have $q \equiv 1 \pmod 8$. For S of type E_7, B_n ($n \geq 2$) or C_n ($n \geq 3$), let $\mathbf{L}$ be a split Levi subgroup of $\mathbf{G}^*$ of type E_6, C_{n-1}, B_{n-1} respectively. Let $s \in \mathbf{Z}(\mathbf{L})^F$ of order 8. As $|\mathbf{Z}(\mathbf{G}^*)| = 2$ the image of s in $\mathbf{G}^*/\mathbf{Z}(\mathbf{G}^*)$ has order at least 4 and hence $C_{\mathbf{G}^*}(s)$ is connected (otherwise, since $\mathbf{L}$ is a maximal Levi subgroup, s would have to be quasi-isolated, but by [2, Tab. 2] there do not exist quasi-isolated elements of order at least 4 in $\mathbf{G}^*$. Inspection of the order formulae shows s is not 2-central, so we may conclude with Lemma 2.9.

Assume $\mathbf{G}$ is of type E_6. If G does not involve a graph automorphism (of order 2) then we may argue exactly as in the proof of Proposition 2.11. If G does induce a graph automorphism, and hence $\mathbf{G}^F = E_6(q)$, then let $s \in \mathbf{G}^{*F}$ be an element of order 8 in the centre of a split Levi subgroup $\mathbf{L}$ of $\mathbf{G}^*$ of type D_5. Since $\mathbf{L}^F$ contains a Sylow 2-subgroup of $\mathbf{G}^{*F}$ (by inspection of the order formulae), the corresponding semisimple character χ of S lies in $\mathrm{Irr}_\sigma(B_0(S))$ and has odd degree by Lemma 2.9. Also, since $C_{\mathbf{G}^*}(s)^F = L = N_{\mathbf{G}^{*F}}(L)$, and s is central in L, s is not conjugate to its inverse in $\mathbf{G}^*$. But a calculation in the root system shows that the graph automorphism of $\mathbf{G}^*$ acts by inversion on $\mathbf{Z}(\mathbf{L})$ and hence inverts s. Thus Lemma 2.4 applies.

For $\mathbf{G}$ of type D_n, $n \geq 4$, let $\mathbf{L}$ be a split Levi subgroup of $\mathbf{G}^*$ of type D_{n-1} and again $s \in \mathbf{Z}(\mathbf{L})^F$ of order 8. If either n is even, or $\mathbf{G}^F$ is of twisted type, then $\mathbf{Z}(\mathbf{G}^*)^F$ is elementary abelian and so the image $\bar{s}$ of s in $\mathbf{G}^*/\mathbf{Z}(\mathbf{G}^*)$ has order at least 4. According to [2, Tab. 2] there are no quasi-isolated elements of that order with centraliser containing $\mathbf{L}$, so $\bar{s}$ has connected centraliser. When n is odd and $\mathbf{G}^F$ is untwisted, let $\mathbf{L}$ be a split Levi subgroup of $\mathbf{G}^*$ of type A_{n-1} and again $s \in \mathbf{Z}(\mathbf{L})^F$ of order 8. In either case we can then conclude as before unless G also induces a graph automorphism of order 2.

In the latter case, $\mathbf{G}^F$ is untwisted. Let $\zeta \in \mathbb{F}_{q^2}^\times$ be a generator of the Sylow 2-subgroup (of order at least 16). Let $t \in H := \mathrm{SO}_{2n}^+(q)$ be an element with eigenvalues ζ, ζ^q and $n-2$ times ζ^2, and their inverses, in the natural matrix representation. Then $s := t^2$ is not invariant under non-trivial 2-power order field automorphisms. Furthermore s has centraliser $\mathrm{GL}_1(q)^2\,\mathrm{GL}_{n-2}(q)$ in H as well as in $\mathrm{GO}_{2n}^+(q)$, since s has no eigenvalues ± 1, which means that its class is not invariant under the graph automorphism of order 2 induced by $\mathrm{GO}_{2n}^+(q)$. Since the graph automorphism preserves the eigenvalues of semisimple elements, this then shows that the class of s is also not invariant under products of field automorphisms with the graph automorphism. Let $\chi \in \mathrm{Irr}_\sigma(B_0(\mathrm{SO}_{2n}^+(q)))$ be the corresponding semisimple character (noting that $H = \mathrm{SO}_{2n}^+(q)$ is self-dual), see Lemma 2.9. Since s has connected centraliser, χ restricts irreducibly to the derived subgroup of $\mathrm{SO}_{2n}^+(q)$. Moreover, as s is a square in H, it lies in $[H,H]$, so χ has $\mathbf{Z}(\mathrm{SO}_{2n}^+(q))$ in its kernel and can be considered as a character of $S = \mathrm{O}_{2n}^+(q)$. Again since s has connected centraliser, χ also extends to a σ-stable character of the adjoint type group $\mathbf{G}^F$. By our earlier remarks on χ, all of these characters have trivial stabiliser in the group generated by graph and field automorphisms. So their induction to G provides a character as claimed. □

By the classification of finite simple groups, the proof of Theorem 2.3 is now complete.

3. 2-closed groups

In this section, we improve on the Itô–Michler Theorem for $p = 2$ which asserts that a finite group G has a normal and abelian Sylow 2-subgroup if and only if all the irreducible complex characters of G have odd degree. As we have mentioned, our main result in this section and its proof resemble Theorem A of [6], where it is shown that if all real-valued irreducible characters of G have odd degree then G is 2-closed. Of course our Galois automorphism σ behaves like complex conjugation but only on odd order roots of unity. This difference allows us to fully generalise Itô–Michler in both directions (which cannot be done by using complex conjugation) but poses additional complications. One of them arises from the fact that, unlike complex conjugation, σ does not act on p-Brauer characters for p odd, as shown by the group $2.\mathfrak{A}_6.2_2$ in characteristic 3, for example.

Our notation for characters follows [15] and [28], and our notation for blocks and Brauer characters follows [26].

We shall use below that if $\gamma \in \mathrm{Gal}(\mathbb{Q}^{\mathrm{ab}}/\mathbb{Q})$ complex-conjugates odd-order roots of unity, $n \geq 1$ is any integer, and $\mathbb{Q}_n$ is the n-th cyclotomic field, then the restriction $\tau = \gamma|_{\mathbb{Q}_n}$ has order a power of 2. This follows from the fact that τ^2 fixes $2'$-roots of unity and the Galois group $\mathrm{Gal}(\mathbb{Q}_n/\mathbb{Q}_{n_{2'}})$ is a 2-group. (In this paper, n_2 is the largest power of 2 dividing n, and $n_{2'} = n/n_2$.)

Recall that $\mathrm{Gal}(\mathbb{Q}^{\mathrm{ab}}/\mathbb{Q})$ permutes the p-blocks of any finite group G (but as we said, not the p-Brauer characters).

Lemma 3.1. *Suppose that $\sigma \in \mathrm{Gal}(\mathbb{Q}^{\mathrm{ab}}/\mathbb{Q})$ complex-conjugates odd-order roots of unity. Let G be a finite group. Then the principal 2-block of G is the only σ-invariant 2-block of G with maximal defect.*

Proof. Suppose that B is a σ-invariant 2-block of G with defect group $P \in \mathrm{Syl}_2(G)$. Let b be its Brauer first main correspondent. It easily follows that $(b^\sigma)^G = B^\sigma$, since Galois action commutes with Brauer induction. By the uniqueness in Brauer's first main theorem, and the third main theorem, we may assume that $P \trianglelefteq G$. Now, let $\theta \in \mathrm{Irr}(\mathbf{O}_{2'}(G))$ such that the 2-block $\{\theta\}$ is covered by B. Then $\{\theta^\sigma\}$ is covered by $B^\sigma = B$, and it follows that there exists $g \in G$ such that $\theta^\sigma = \bar{\theta} = \theta^g$. Now, using that the restriction of σ to $\mathbb{Q}_{|G|}$ has 2-power order, we may assume that g has 2-power order. Then $g \in P$ and $[g, \mathbf{O}_{2'}(G)] = 1$. Therefore, θ is the trivial character, by Burnside's theorem (see Problem 3.16 of [15]). Then B is the principal block of G, for instance by Fong's theorem [26, Thm. 10.20]. $\square$

Lemma 3.2. *Let $N \trianglelefteq G$ be finite groups. Suppose that G/N has order not divisible by some prime p. Let $\sigma \in \mathrm{Gal}(\mathbb{Q}_{|G|}/\mathbb{Q})$ have order a power of p.*

(a) *If $\chi \in \mathrm{Irr}_\sigma(G)$, then all the irreducible constituents of χ_N are σ-invariant.*

(b) *Suppose that $p = 2$, and σ complex-conjugates odd-order roots of unity. Suppose that $\theta \in \mathrm{Irr}_\sigma(N)$. Then there exists a unique $\chi \in \mathrm{Irr}_\sigma(G)$ over θ. Furthermore, if θ is G-invariant, then $\chi_N = \theta$. Also, χ belongs to the principal 2-block of G if and only if θ belongs to the principal 2-block of N.*

Proof. For (a), let $\theta \in \mathrm{Irr}(N)$ be an irreducible constituent of χ_N, and let Ω be the set of all the distinct G-conjugates of θ. Notice that $|\Omega|$ divides $|G : N|$, and therefore has size not divisible by p. Since χ is $\langle\sigma\rangle$-invariant, we have that $\langle\sigma\rangle$ acts on Ω. Therefore, there is $\eta \in \Omega$ which is σ-invariant. Now, if $g \in G$, then $(\eta^g)^\sigma = (\eta^\sigma)^g = \eta^g$, and so all elements of Ω are σ-invariant.

Next, we prove (b) by induction on $|G : N|$. To prove the first two assertions, by the Clifford correspondence, we may assume that θ is G-invariant. If $N = G$, there is nothing to prove. Else, since G/N has odd order, there is $N \le M \trianglelefteq G$ of prime index in G. By induction, there exists a unique σ-invariant $\psi \in \mathrm{Irr}(M)$ over θ. Furthermore, $\psi_N = \theta$. By uniqueness, we have that ψ is G-invariant, and therefore ψ (and θ) extends to G, because G/M is cyclic. Let $\tau \in \mathrm{Irr}(G)$ be any extension of θ. Then $\tau^\sigma = \lambda\tau$, for a unique linear $\lambda \in \mathrm{Irr}(G/N)$ by the Gallagher correspondence [15, Cor. 6.17]. Since $|G/N|$ is odd, there exists a linear $\nu \in \mathrm{Irr}(G/N)$ such that $\nu^2 = \lambda$. Now, for $\chi = \nu\tau \in \mathrm{Irr}(G)$ we have

$$\chi^\sigma = (\nu\tau)^\sigma = \nu^{-1}\tau^\sigma = \nu^{-1}\lambda\tau = \chi\,.$$

Also, if $\rho \in \mathrm{Irr}(G/N)$ and $\rho\chi$ is σ-invariant, then ρ is σ-invariant, by the uniqueness in the Gallagher correspondence. However, $\rho^\sigma = \bar{\rho}$, and since G/N has odd order, we obtain that $\rho = 1$, by Burnside's theorem (see Problem 3.16 of [15]).

If χ belongs to the principal block of G, then θ belongs to the principal block of N by [26, Thm. 9.2]. Conversely, assume that θ belongs to the principal block of N. Since G/N has odd order, it follows that χ lies in a block of maximal defect of G (using, for instance, [26, Thm. 9.26]). Now, the result follows from Lemma 3.1. □

From now on, $\sigma \in \mathrm{Gal}(\mathbb{Q}^{\mathrm{ab}}/\mathbb{Q})$ is the Galois automorphism that fixes 2-power roots of unity and complex conjugates odd-order roots of unity. Of course, $\sigma^2 = 1$.

Assuming Proposition 2.2(a) and (c), and Theorem 2.3 (on almost-simple groups), the proof of Theorem B is standard and follows the proof of [6, Thm. 4.2].

Theorem 3.3. *Let G be a finite group, and let $P \in \mathrm{Syl}_2(G)$. Then all characters in $\mathrm{Irr}_\sigma(G)$ have odd degree if and only if P is normal in G and abelian.*

Proof. Suppose that $P \trianglelefteq G$ is abelian. Then all elements of $\mathrm{Irr}(G)$ have odd degree by Itô's theorem [15, Thm. 6.15].

Now suppose that all characters in $\mathrm{Irr}_\sigma(G)$ have odd degree. We claim that if P is normal in G, then P is abelian. To see this, let $\theta \in \mathrm{Irr}(P)$. By Lemma 3.2(b), let $\chi \in \mathrm{Irr}_\sigma(G)$ be over θ. By hypothesis, χ has odd degree, and by Clifford's theorem we conclude that θ is linear.

We prove by induction on $|G|$ that P is normal in G. We may assume that $P > 1$. Arguing by induction, we may assume that G has a unique minimal normal subgroup N and that $PN \trianglelefteq G$. In particular, we may assume that N is not a 2-group. Using Lemma 3.2(b), we may assume that $\mathbf{O}^{2'}(G) = G$. Therefore, we have $PN = G$ and $\mathbf{C}_P(N) = 1$.

Assume that N has odd order. Let $x \in P$ be an involution, and notice that there exists $\lambda \in \mathrm{Irr}(N)$ such that $\nu := \lambda^{-1}\lambda^x \neq 1$. Then $\nu^x = \bar{\nu} = \nu^\sigma$, and if $\mu \in \mathrm{Irr}(G_\nu)$, is the canonical extension of ν to G_ν [15, Cor. 6.28], then $\mu^\sigma = \mu^x$, by uniqueness. It follows that $\mu^G \in \mathrm{Irr}(G)$ is σ-invariant. By hypothesis, μ^G has odd degree, and therefore $G_\nu = G$, $\bar{\nu} = \nu$, and this is not possible because $|N|$ is odd.

Hence, we may assume that $N = S^{x_1} \times \cdots \times S^{x_t}$, where $S \trianglelefteq N$ is non-abelian simple, and $\{x_1, \ldots, x_t\}$ is a complete set of representatives of the right cosets of $H = \mathbf{N}_G(S)$ in G, with $x_1 = 1$. Write $C = \mathbf{C}_G(S)$. We know that H/C is almost simple with socle SC/C, and with H/SC a 2-group. We claim that there exists $\tau \in \mathrm{Irr}_\sigma(H/C)$ of even degree such that S is not in the kernel of τ. Suppose first that S has non-abelian Sylow 2-subgroups. Then the claim follows from Theorem 2.3. Suppose now that S has abelian Sylow 2-subgroups. If $SC < H$, then the claim follows from Proposition 2.2(c). We are left with the case where S has abelian Sylow 2-subgroups and $H = SC$. In this case, we take $\tau = \chi_1$ from Proposition 2.2(a). Now, if $1 \neq \theta \in \mathrm{Irr}(S)$ is an irreducible constituent of τ_S, we have that $\eta = \theta \times 1_{S^{x_2}} \times \cdots \times 1_{S^{x_t}}$, lies under τ and that $G_\eta \subseteq H$. By the

Clifford correspondence, we have that $\tau^G \in \mathrm{Irr}(G)$ is σ-invariant of even degree and this is a contradiction. $\square$

If G is solvable, we can obtain the normality of P by using the main result of [13], where it is proved that if G is solvable, p is a prime, σ is a Galois automorphism of $\mathrm{Gal}(\mathbb{Q}_{|G|}/\mathbb{Q})$ of order p, and all $\mathrm{Irr}_\sigma(G)$ have p'-degree, then G has a normal Sylow p-subgroup.

4. Proof of Theorem A

In this section we prove Theorem A, using Proposition 2.2 and Theorem 2.3. Of course, Theorem A improves on Brauer's Height Zero conjecture for 2-principal block (shown in [30], and more recently in [21] for every prime). We adapt several arguments from [30] and [21] to our present case.

In the first two results, p is an arbitrary prime. For B a p-block of G, $Z \trianglelefteq G$ and $\lambda \in \mathrm{Irr}(Z)$, we set $\mathrm{Irr}(B|\lambda) := \mathrm{Irr}(G|\lambda) \cap \mathrm{Irr}(B)$.

Lemma 4.1. *Suppose that G is the central product of subgroups U_i, $1 \le i \le t$. Let $Z = \bigcap U_i$, $\lambda \in \mathrm{Irr}(Z)$, and assume that Z is a p-group. Suppose that B_i is the principal p-block of U_i, and that B is the principal p-block of G. Then there is a natural bijection*

$$\mathrm{Irr}(B_1|\lambda) \times \cdots \times \mathrm{Irr}(B_t|\lambda) \to \mathrm{Irr}(B|\lambda)\,.$$

Proof. Given $\psi_i \in \mathrm{Irr}(U_i|\lambda)$, we know that there is a unique $\chi \in \mathrm{Irr}(G|\lambda)$ such that χ lies over every ψ_i, and that $(\psi_1, \ldots, \psi_t) \mapsto \chi$ is a bijection. Furthermore, $\chi(u_1 \cdots u_k) = \psi_1(u_1) \cdots \psi_k(u_k)$ for $u_i \in U_i$. (See [28, Thm. 10.7].) If χ lies in B, then each ψ_i lies in B_i by [26, Thm. 9.2], so we only need to prove the converse. We prove it for $t = 2$, which is clearly sufficient. Since $[U_1, U_2] = 1$, $G = U_1U_2$ and $Z \subseteq \mathbf{Z}(G)$ is a p-group, it easily follows that the map $U_1^0 \times U_2^0 \to G^0$ given by $(x, y) \mapsto xy$ is a bijection, where G^0 is the set of p-regular elements of G. Now

$$\sum_{(x,y)\in U_1^0\times U_2^0} \chi(xy) = \left(\sum_{x\in U_1^0} \psi_1(x)\right)\left(\sum_{y\in U_2^0} \psi_2(y)\right) \neq 0,$$

using [26, Thm. 3.19 and Cor. 3.25]. Therefore χ is in the principal block, again by [26, Thm. 3.19 and Cor. 3.25]. $\square$

The character χ under the bijection in Lemma 4.1 is usually denoted by

$$\chi = \psi_1 \cdots \psi_t\,.$$

We shall use the following surely well-known result (which we used in the proof of [23, Thm. 5.4]), and whose argument we write down here for the reader's convenience. Recall that the layer $\mathbf{E}(G)$ of a finite group G is the product of the components of G, and that $\mathbf{F}^*(G) = \mathbf{E}(G)\mathbf{F}(G)$ is the generalised Fitting subgroup of G.

Lemma 4.2. *Let G be a finite group, let $E = \mathbf{E}(G)$. Assume that $E \neq 1$ and that $\mathbf{F}(G)$ is central in G.*

(a) *We have that*

$$\bigcap_U U\mathbf{C}_G(U) = \mathbf{F}^*(G),$$

where U runs over the components of G.

(b) *Suppose that p is a prime dividing $|E|$ but not $|\mathbf{Z}(E)|$ and $\mathbf{O}_{p'}(G) \subseteq \mathbf{Z}(G)$. If $Q \in \mathrm{Syl}_p(E)$, then $\mathbf{F}^*(G)\mathbf{C}_G(Q)/\mathbf{F}^*(G)$ is solvable.*

Proof. Suppose that $\Omega = \{U_1, \ldots, U_n\}$ is the set of the different components of G, so $E = U_1 \cdots U_n$. Write $C_i = \mathbf{C}_G(U_i)$ and $K = \mathbf{F}^*(G)$. Let

$$L = \bigcap_{i=1}^n U_i C_i\,.$$

It is well-known that $[U_i, U_j] = 1$ ([16, Thm. 9.4]) and $[U_i, \mathbf{F}(G)] = 1$ ([16, Thm. 9.7.c]). In particular, $K \subseteq L$. Suppose that $g \in L$. Write $g = c_i u_i$, where $c_i \in C_i$ and $u_i \in U_i$. Let $x = u_1 \cdots u_n \in E$. We claim that $e^g = e^x$ for all $e \in E$. Indeed, if $e = v_1 \cdots v_n$, then

$$e^g = v_1^g \cdots v_n^g = v_1^{c_1 u_1} \cdots v_n^{c_n u_n} = v_1^{u_1} \cdots v_n^{u_n} = (v_1 \cdots v_n)^{u_1 \cdots u_n} = e^x\,.$$

Then $gx^{-1} \in \mathbf{C}_G(E)$. Since $\mathbf{F}(G) \subseteq \mathbf{Z}(G)$, we have $gx^{-1}\mathbf{C}_G(E\mathbf{F}(G)) = \mathbf{C}_G(K) \subseteq K$ (by using [16, Thm. 9.8]). Therefore $g \in K$, showing (a).

Let $Z = \mathbf{Z}(E)$. By hypothesis, p does not divide $|Z|$. For (b), by the proof of Theorem 9.7 and Lemma 9.6 of [16], we know that

$$E/Z = \prod_{i=1}^n (U_i Z)/Z$$

is a direct product (of non-abelian simple groups). Let $Q_i = Q \cap U_i Z = Q \cap U_i \in \mathrm{Syl}_p(U_i)$. Since $\mathbf{O}_{p'}(G) \subseteq \mathbf{Z}(G)$, we have $Q_i \neq 1$. Write $C = \mathbf{C}_G(Q)$. Let $c \in C$. We claim that c normalises U_i for all i. We have that c permutes the set Ω. Let $1 \neq y_i \in Q_i$. Since $C \subseteq \mathbf{C}_G(Q_i)$, we have $(y_i Z)^c = y_i Z$. Using that the product is direct, we have that necessarily $c \in \mathbf{N}_G(U_i Z)$. Since $(U_i Z)' = U_i$, we conclude that $c \in \mathbf{N}_G(U_i)$. By the Schreier conjecture, $\mathbf{N}_G(U_i)/U_i C_i$ is solvable. (Recall that $\mathrm{Out}(U_i)$ is a subgroup of $\mathrm{Out}(U_i/Z_i)$, where $Z_i = Z \cap U_i = \mathbf{Z}(U_i)$.) Therefore $C/(C \cap U_i C_i)$ is solvable. Hence,

$$C/\Big(C \cap \bigcap_{i=1}^n (U_i C_i)\Big)$$

is solvable. By part (a), we conclude that $C/(C \cap \mathbf{F}^*(G))$ is solvable, as required. ☐

We are now ready to prove Theorem A, which we restate:

Theorem 4.3. *Let B be the principal 2-block of a finite group G, and let $P \in \mathrm{Syl}_2(G)$. Then all characters in $\mathrm{Irr}_\sigma(B)$ have odd degree if and only if P is abelian.*

Proof. Write $B_0(G)$ for the principal 2-block of G. If $P \in \mathrm{Syl}_2(G)$ is abelian, then all irreducible characters in $B_0(G)$ have odd degree by [30].

For the converse, we argue by induction on $|G|$. We may assume that G is not a 2-group. Assume that $N \trianglelefteq G$ has odd index. We claim that all characters in $\mathrm{Irr}_\sigma(B_0(G))$ have odd degree if and only if those in $\mathrm{Irr}_\sigma(B_0(N))$ have odd degree. But this easily follows from Lemma 3.2, and the fact that if $\chi \in \mathrm{Irr}(G)$ and $\theta \in \mathrm{Irr}(N)$ lies under χ, then $\chi(1)/\theta(1)$ divides $|G : N|$ ([15, Cor. 11.29]). Hence, we may assume that $\mathbf{O}^{2'}(G) = G$. Also, since $\mathbf{O}_{2'}(G)$ is in the kernel of the characters of $B_0(G)$ ([26, Thm. 6.10]), we may also assume that $\mathbf{O}_{2'}(G) = 1$.

Since $B_0(G/N) \subseteq B_0(G)$ if N is normal in G, by induction, we deduce that G has a unique minimal normal subgroup N, and that G/N has an abelian Sylow 2-subgroup. Let $K/N = \mathbf{O}_{2'}(G/N)$.

Now, let $Q = P \cap N \in \mathrm{Syl}_2(N)$ and $M = N\mathbf{C}_G(Q) \trianglelefteq G$. We know that all irreducible characters of G/M are in $B_0(G)$, by [30, Lemma 3.1]. By Theorem B, we have that G/M has a normal Sylow 2-subgroup. Since $\mathbf{O}^{2'}(G) = G$, we conclude that G/M is a 2-group. Hence, $K \subseteq M$.

Suppose that N is a 2-group. Then N is elementary abelian (because N is a minimal normal subgroup) and $M = \mathbf{C}_G(N)$. Then $K = N \times \mathbf{O}_{2'}(K)$, by using the Schur–Zassenhaus theorem. Since $\mathbf{O}_{2'}(G) = 1$, we have that $K = N$. Then G/N is a direct product of non-abelian simple groups and a 2-group (by [35, Thm. 1]). Write $G/N = L/N \times E/N$, where E/N is a 2-group, and L/N is the direct product of non-abelian simple groups X_i/N. Since $L'N = L$ and G has a unique minimal normal subgroup N, we conclude that $L' = L$ is perfect. Since E is a 2-group, then $N \cap \mathbf{Z}(E) > 1$, and therefore $E \leq M$. Since G/M is a 2-group, necessarily $M = G$, and N is central. Thus $|N| = 2$. Now, $[L, E, L] = 1 = [E, L, L] = 1$ and by the three subgroups lemma, we have that $[E, L] = 1$. Now, X_i is a normal subgroup of G, and since G has a unique minimal normal subgroup N, and X_i' is normal in G, we conclude that X_i is perfect for all i. Hence X_i is quasi-simple. By the same argument as before, $[X_i, X_j] = 1$, if $i \neq j$. By Lemma 2.1(a), every X_i has non-abelian Sylow 2-subgroups. Let $1 \neq \lambda \in \mathrm{Irr}(N)$. By Proposition 2.2(b), let $\chi_i \in \mathrm{Irr}_\sigma(B_0(X_i)|\lambda)$ with even degree. Also, let $\mu \in \mathrm{Irr}(E|\lambda)$. Then, using Lemma 4.1, we have that $\chi = \chi_1 \cdots \chi_t \cdot \mu \in \mathrm{Irr}(B_0(G)|\lambda)$ has even degree and is σ-invariant. This is a contradiction, hence N cannot be a 2-group.

We thus may assume that $N = \mathbf{E}(G) = \mathbf{F}^*(G)$ is a minimal normal subgroup of G.

Now, M/N is solvable by Lemma 4.2(b). Thus G/N is solvable and since it has abelian Sylow 2-subgroups, it follows that G/N has a normal 2-complement K/N (by using the so called Hall–Higman's Lemma 1.2.3 and the fact that $\mathbf{O}^{2'}(G/N) = G/N$). Recall that $K \subseteq M$, so $K = N\mathbf{C}_K(Q)$.

We claim that we may assume that $G = NP$. Let $\tau \in \mathrm{Irr}_\sigma(B_0(NP))$, and let $\theta \in \mathrm{Irr}(N)$ be under τ. Then $\theta^\sigma = \theta^g$ for some $g \in P$, by Clifford's theorem, and θ lies in the principal 2-block of N. Let $\eta \in \mathrm{Irr}(B_0(K))$ be the unique extension of θ in the principal block of K (using Alperin isomorphic blocks, [1]). By uniqueness, $\eta^g = \eta^\sigma$. (To prove this, use that $(\eta^\sigma)^{g^{-1}}$ and η are extensions in the principal block of K of θ, so they coincide.) Let I be the stabiliser of η in G. Notice that $J = I \cap PN$ is the stabiliser of θ in PN (again using uniqueness in Alperin's theorem). Let $\mu \in \mathrm{Irr}(J)$ be the Clifford correspondent of τ over θ. Again by uniqueness, $\mu^\sigma = \mu^g$. By the Isaacs restriction correspondence [28, Lemma 6.8(d)], let $\rho \in \mathrm{Irr}(I|\eta)$ be such that $\rho_J = \mu$. By uniqueness of this restriction map, we again have $\rho^\sigma = \rho^g$. By the Clifford correspondence, $\chi = \rho^G = (\rho^\sigma)^G \in \mathrm{Irr}(G)$ is σ-invariant, and lies in the principal 2-block of G (because η does and G/K is a 2-group). By hypothesis, we have that χ has odd degree. Thus $I = G$, $\rho = \chi$, and $\chi_{PN} = \tau$ has odd degree. By induction, we may assume that $G = PN$, as claimed.

Write $N = S^{x_1} \times \cdots \times S^{x_t}$ where the x_i are representatives of the right cosets of $H = \mathbf{N}_G(S)$ in G, and S is non-abelian simple. Write $C = \mathbf{C}_G(S)$. Assume that $t > 1$ (so that $H < G$). We claim that there exists $\gamma \in \mathrm{Irr}_\sigma(H/C)$ in the principal block of H/C (and therefore of H), such that γ_S does not contain the trivial character. If the Sylow 2-subgroups of S are not abelian, this follows from Theorem 2.3. Assume that S has abelian Sylow 2-subgroups. If $SC < H$, then this follows from Proposition 2.2(c), while if $SC = H$, it follows from the second part of Proposition 2.2(a). Now, if $1 \neq \xi \in \mathrm{Irr}(S)$ is under γ, then the stabiliser $G_\tau \subseteq H$, where $\tau = \xi \times 1 \times \cdots \times 1 \in \mathrm{Irr}(N)$. By the Clifford correspondence, we obtain $\gamma^G = \chi \in \mathrm{Irr}_\sigma(B_0(G))$ (by [26, Cor. 6.2] and Brauer's third main theorem [26, Thm. 6.7]), with even degree (using that $H < G$ and that $|G : H|$ is a 2-power). This is not possible by hypothesis. We conclude that $t = 1$. In this case, the assertion follows by Theorem 2.3. $\square$

5. p-RATIONAL CHARACTERS

In this final section, we prove Theorem C. We start with the following. Recall that $\chi \in \mathrm{Irr}(G)$ is *p-rational* if the field of values $\mathbb{Q}(\chi) = \mathbb{Q}(\chi(g) \mid g \in G)$ of χ is contained in a cyclotomic field $\mathbb{Q}_m$, with m not divisible by p.

Theorem 5.1. *Suppose that $\chi \in \mathrm{Irr}(G)$ is p-rational, with $\chi^0 = \varphi \in \mathrm{IBr}(G)$. Let $N \trianglelefteq G$ and let $\theta \in \mathrm{Irr}(N)$ be under χ. Then θ is p-rational.*

Proof. We argue by induction on $|G|$. Let T^* be the set of all $g \in G$ such that $\theta^g = \theta^\sigma$ for some $\sigma \in \mathrm{Gal}(\mathbb{Q}(\theta)/\mathbb{Q})$. Let $T \subseteq T^*$ be the stabiliser of θ in G, and let $\psi \in \mathrm{Irr}(T)$ be the Clifford correspondent of χ over θ. Let $\eta = \psi^{T^*}$. It is well-known that $\mathbb{Q}(\eta) = \mathbb{Q}(\chi)$ (see for instance, Problem 3.9 of [28]). Since $\chi^0 = (\eta^G)^0 = (\eta^0)^G = \varphi \in \mathrm{IBr}(G)$, we deduce that $\eta^0 \in \mathrm{IBr}(T^*)$. If $T^* < G$, then we are done by induction. Hence, we may assume that $T^* = G$. Notice that in this case $T \trianglelefteq G$, since the stabilisers $G_\theta = G_{\theta^\sigma}$ coincide.

Assume that ψ is not p-rational. Then there exists $\sigma \in \mathrm{Gal}(\mathbb{Q}_{|G|}/\mathbb{Q}_{|G|_{p'}})$ such that $\psi^\sigma \neq \psi$. Now, $\chi^\sigma = \chi$, and thus $\theta^\sigma = \theta^g$ for some $g \in G$, by Clifford's theorem. Hence $\psi^\sigma = \psi^g$ by uniqueness in the Clifford correspondence. Now $(\psi^0)^\sigma = \psi^0$ because σ fixes p'-roots of unity, and thus $\psi^0 = (\psi^\sigma)^0 = (\psi^g)^0 = (\psi^0)^g$. However, $(\psi^0)^G = (\psi^G)^0 = \chi^0 \in \mathrm{IBr}(G)$, and therefore we deduce that $g \in T$ (by Problem 8.3 in [26].) Then $\psi^\sigma = \psi$, contrary to assumption. Therefore ψ is p-rational. Now, $\psi_N = e\theta$ for some $e \geq 1$, and we deduce that θ is p-rational. $\square$

Corollary 5.2. *Suppose that $\chi \in \mathrm{Irr}(G)$ is p-rational for some odd prime p, with $\chi^0 = \varphi \in \mathrm{IBr}(G)$. Then $\ker\varphi \le \ker\chi$.*

Proof. We argue by induction on $|G|$. By induction, we may assume that $\ker\varphi\cap\ker\chi = 1$. Let $N = \ker\varphi$. Let $\theta \in \mathrm{Irr}(N)$ be under χ. Also, $\theta^0 = e1_N$. Then all p-regular elements of N are in $\ker\theta\cap\ker\varphi$, and therefore N is a p-group. By Theorem 5.1 we have that θ is p-rational. Since N is a p-group, we have that θ is rational valued. Since p is odd, $\theta = 1_N$, as wanted. □

We shall prove Theorem C, based on the following consequence of the classification of finite simple groups. Here we use $I_G(\theta) = G_\theta$ to denote the stabiliser in G of the character θ of a normal subgroup N of G, as mentioned before Lemma 2.4.

Theorem 5.3. *Suppose that p is an odd prime number. If S is a finite non-abelian simple group of order divisible by p, then there exists a p-rational $\eta \in \mathrm{Irr}(S)$ such that p divides $\eta(1)$, $\eta^0 \in \mathrm{IBr}(S)$, and $I_{\mathrm{Aut}(S)}(\eta) = I_{\mathrm{Aut}(S)}(\eta^0)$.*

Proof. Assume that S has a character $\eta \in \mathrm{Irr}(S)$ of p-defect zero. Then $\eta^0 \in \mathrm{IBr}(S)$, p divides $\eta(1)$, and clearly the stabilisers of η and η^0 in $\mathrm{Aut}(S)$ agree. Now it is known that all non-abelian simple groups have blocks of p-defect zero for all $p \ge 5$, and for $p = 3$ the only exceptions to this statement are certain alternating groups and the two sporadic groups Suz and Co_3 (see [12, Cor. 2]). For Suz and Co_3 our claim is checked from the known decomposition matrices in [34]: Suz has a unique irreducible character η with $\eta(1) = 15795$, and that remains irreducible modulo 3, and Co_3 has a unique irreducible character η with $\eta(1) = 91125$ in a block of defect 1 that remains irreducible modulo 3.

The alternating group $\mathfrak{A}_5$ has two characters of 3-defect zero. For $\mathfrak{A}_n$ with $n \ge 6$ let χ be the irreducible character of $\mathfrak{S}_n$ labelled by the partition $\lambda = (n-1,1)$ if $n \equiv 1 \pmod 3$, $\lambda = (n-2,1^2)$ if $n \equiv 2 \pmod 3$ and $\lambda = (n-2,2)$ if $n \equiv 0 \pmod 3$. Then by the hook formula, $\chi(1)$ is $n-1$, $(n-1)(n-2)/2$, $n(n-3)/2$ respectively, so divisible by 3. Furthermore, since λ is not self-conjugate, χ restricts to an irreducible character η of $\mathfrak{A}_n$ that is rational. Finally, λ is a 3-JM partition in each case, so $\eta^0 \in \mathrm{IBr}(\mathfrak{A}_n)$ by [8, Thm. 3.1]. This completes the proof. □

We also need this lemma.

Lemma 5.4. *Suppose that p is an odd prime. Let G be a finite group, $N \trianglelefteq G$, and let $\theta \in \mathrm{Irr}(N)$ be p-rational such that $\theta^0 \in \mathrm{IBr}(N)$ and $I_G(\theta) = I_G(\theta^0)$. Assume that G/N is p-solvable. Then there exists a p-rational $\chi \in \mathrm{Irr}(G|\theta)$ such that $\chi^0 \in \mathrm{IBr}(G)$.*

Proof. Let $T = I_G(\theta) = I_G(\theta^0)$, and assume that $T < G$. Arguing by induction on $|G : N|$, there exists a p-rational $\psi \in \mathrm{Irr}(T|\theta)$ such that $\psi^0 \in \mathrm{IBr}(T)$. Now, $\psi^0 \in \mathrm{IBr}(T|\theta^0)$. By the Clifford correspondence for Brauer characters, we have $(\psi^0)^G \in \mathrm{IBr}(G)$. Now, $(\psi^0)^G = (\psi^G)^0$, and therefore $(\psi^G)^0$ is irreducible (and $\psi^G \in \mathrm{Irr}(G)$ is p-rational). So we may assume that $T = G$. By [23, Thm. 4.5], we may then assume that N is a central p'-subgroup. In particular, G is p-solvable. Now, let $\varphi \in \mathrm{IBr}(G)$ lying over θ, and let

$\chi \in \mathrm{Irr}(G)$ be the p-rational lift of χ (by the main result of [14]). Then χ is over θ and we are done. $\square$

This is Theorem C:

Theorem 5.5. *Let G be a finite group, and let p be an odd prime number. Let $P \in \mathrm{Syl}_p(G)$. Then $P \trianglelefteq G$ if and only if every p-rational $\chi \in \mathrm{Irr}(G)$ with $\chi^0 \in \mathrm{IBr}(G)$ has degree not divisible by p.*

Proof. Assume that $P \trianglelefteq G$, and let $\chi \in \mathrm{Irr}(G)$ be p-rational with $\chi^0 \in \mathrm{IBr}(G)$. We know that $P \subseteq \ker \chi^0$ (by [26, Lemma 2.32]). Let $\theta \in \mathrm{Irr}(P)$ be an irreducible constituent of χ_P. By Corollary 5.2, we have $\theta = 1_P$. Therefore, $\chi \in \mathrm{Irr}(G/P)$, and p does not divide $\chi(1)$.

We prove the converse by induction on $|G|$. Arguing by induction, we have that G has a unique minimal normal subgroup N, and that G/N has a normal Sylow p-subgroup PN/N. If N is a p-group, then $P \trianglelefteq G$, and we are done. If N is a p'-group, then G is p-solvable and the theorem follows. (For instance, using that every $\varphi \in \mathrm{IBr}(G)$ has a p-rational lift, by [14], and [25, Thm. 13.1(c)].)

So we assume that $N = S^{x_1} \times \cdots \times S^{x_t}$, where S is a non-abelian simple group of order divisible by p, and $\{x_1, \ldots, x_t\}$ is a complete set of representatives of the right cosets of $H = \mathbf{N}_G(S)$ in G. By Theorem 5.3, there exists a p-rational $\eta \in \mathrm{Irr}(S)$ such that $\eta^0 \in \mathrm{IBr}(S)$, p divides $\eta(1)$, and $I_{\mathrm{Aut}(S)}(\eta) = I_{\mathrm{Aut}(S)}(\eta^0)$. Let

$$\theta = \eta^{x_1} \times \cdots \times \eta^{x_t} \in \mathrm{Irr}(N)\,.$$

Then $\theta \in \mathrm{Irr}(N)$ and

$$\theta^0 = (\eta^0)^{x_1} \times \cdots \times (\eta^0)^{x_t} \in \mathrm{IBr}(N)\,.$$

We claim that $I_G(\theta) = I_G(\theta^0)$. Suppose $x \in G$ fixes θ^0. We have

$$S^{x_i x^{-1}} = S^{x_{\sigma(i)}}$$

for a permutation $\sigma \in \mathfrak{S}_t$. Thus $x_{\sigma(i)}x = h_i x_i$ for some $h_i \in \mathbf{N}_G(S)$. Then it is easy to check that

$$\big((\eta^0)^{x_1} \times \cdots \times (\eta^0)^{x_t}\big)^x = (\eta^0)^{x_{\sigma(1)}x} \times \cdots \times (\eta^0)^{x_{\sigma(t)}x} = (\eta^0)^{h_1 x_1} \times \cdots \times (\eta^0)^{h_t x_t}\,.$$

We conclude that

$$(\eta^0)^{h_i} = \eta^0$$

for all i. Now, $\mathbf{N}_G(S)/S\mathbf{C}_G(S) \leq \mathrm{Aut}(S)$, and we conclude that $\eta^{h_i} = \eta$. Then

$$\theta^x = (\eta^{x_1} \times \cdots \times \eta^{x_t})^x = \eta^{x_{\sigma(1)}x} \times \cdots \times \eta^{x_{\sigma(t)}x} = \eta^{h_1 x_1} \times \cdots \times \eta^{h_t x_t} = \eta^{x_1} \times \cdots \times \eta^{x_t} = \theta\,.$$

By Lemma 5.4, there exists a p-rational $\chi \in \mathrm{Irr}(G|\theta)$ such that $\chi^0 \in \mathrm{IBr}(G)$. Since p divides $\theta(1)$ and $\chi(1)$ is not divisible by p, by hypothesis, we get a contradiction. $\square$

The following gives an example where p-rational lifts of an irreducible Brauer character are not necessarily unique. Furthermore, it is an example of an irreducible p-rational character lifting an irreducible Brauer character χ which is not *automorphic* (see Definition 5.3 of [14]). These are characters $\chi \in \mathrm{Irr}(G)$ such that $\chi^0 \in \mathrm{IBr}(G)$ and $\chi^a = \chi$ whenever $a \in \mathrm{Aut}(G)$ fixes χ^0.

Example 5.6. Let $p \geq 3$ be a prime and $G = \mathfrak{A}_n$ the alternating group of degree $n := p^2$. Let χ be the irreducible character of $\mathfrak{S}_n$ labelled by the hook $\lambda = ((n+1)/2, 1^{(n-1)/2})$. Since λ is self-conjugate, χ splits into two distinct characters η_1, η_2 upon restriction to $\mathfrak{A}_n$. By [17, Thm. 2.5.13] the η_i only differ in their value on elements whose cycle type is made up of the hook lengths on the main diagonal of λ, hence on n-cycles, and if the values there are irrational, then they involve $\sqrt{(-1)^{(n-1)/2}n}$, which in our situation is an integer. Thus, both η_i are rational.

Since η_1 and η_2 only differ on n-cycles, that is, elements of order p^2, $\eta_1^0 = \eta_2^0$. Now the hook length of λ in the $(1,1)$-node is $n = p^2$, hence has positive p-valuation. This means that λ is an R-partition of type I in the sense of [8, §2.6]. Then $\eta_1^0 = \eta_2^0$ is an irreducible Brauer character by [8, Thm. 3.1].

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