# ON THE NUMBER OF CHARACTERS IN BLOCKS OF QUASI-SIMPLE GROUPS

GUNTER MALLE

Abstract. We prove, for primes $p \geq 5$, two inequalities between the fundamental invariants of Brauer $p$-blocks of finite quasi-simple groups: the number of characters in the block, the number of modular characters, the number of height 0 characters, and the number of conjugacy classes of a defect group and of its derived subgroup. For this, we determine these invariants explicitly, or at least give bounds for them for several classes of classical groups.

## 1. Introduction

In this paper we study two inequalities for the number of simple modules in blocks of finite groups that had been proposed in joint work with G. Navarro [22].

Let $G$ be a finite group, $p$ be a prime, and $B$ a Brauer $p$-block of $G$ with defect group $D$. We write $D' = [D, D]$ for the derived subgroup of $D$. Let $k(B)$ denote the number of ordinary irreducible characters in $B$ and $k_0(B)$ the number of irreducible characters in $B$ of height 0 (see Section 2). We write $k(D)$ for the number of conjugacy classes of $D$, and $l(B)$ for the number of simple modules in characteristic $p$ of $B$. In Malle–Navarro [22] we proposed the following inequalities connecting these invariants:

**Conjecture.** *Let $B$ be a $p$-block of a finite group with defect group $D$. Then*

$$k(B)\,/\,k_0(B) \leq k(D') \quad \textit{and} \tag{C1}$$

$$k(B)\,/\,l(B) \leq k(D). \tag{C2}$$

It was shown in [22] that these statements are satisfied, for example, for blocks with a normal defect group, but also for blocks of symmetric groups. In the present paper we investigate a possible minimal counterexample to (C1) or (C2) for blocks of non-abelian simple groups and their covering groups. Our main result is:

**Theorem 1.** *Let $B$ be a $p$-block of a finite quasi-simple group $G$. Assume one of the following holds:*

(1) *$p \geq 5$, or*

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(2) *$G$ is a covering group of an alternating group, of a sporadic group or of a simple group of Lie type in characteristic $p$.*

*Then $B$ is a minimal counterexample to neither* (C1) *nor* (C2).

Here, we say that $(G, B)$ is a minimal counterexample to (C$i$) if (C$i$) holds for all $p$-blocks $B_1$ of all finite groups $G_1$ with $|G_1/Z(G_1)|$ strictly smaller than $|G/Z(G)|$ and with defect groups isomorphic to those of $B$. Note that we do not prove that all blocks of groups of Lie type in non-defining characteristic satisfy the conjecture.

While we obtain partial results for the primes $p = 2, 3$, the case of groups of Lie type in non-defining characteristic seems out of reach at the moment; the case $p = 3$ might be accessible to a similar but more tedious investigation, but the prime $p = 2$ will require a different approach.

Let us remark that no reduction of our conjecture to the case of (quasi-)simple groups has been proposed so far.

The conjectured inequalities are closely related to three long-standing conjectures in representation theory. Brauer's $k(B)$-conjecture claims that $k(B) \leq |D|$, while the Alperin–McKay conjecture relates the number $k_0(B)$ to the analogous number for the Brauer correspondent block of the normaliser $N_G(D)$ of a defect group $D$ of $B$. Finally, Brauer's height zero conjecture proposes that $k(B) = k_0(B)$ if and only if $D$ is abelian. In fact, we show in Theorem 2.1 that the statement of the known direction of this conjecture implies (C1) in the case of abelian defect groups. Let us point out one motivation for studying these: if the celebrated Alperin–McKay conjecture holds true, then $k_0(B) \leq |D/D'|$ by the proven $k(GV)$-conjecture. But then (C1) claims that $k(B) \leq |D|(k(D')/|D'|)$, which in general is much smaller than the bound $|D|$ stipulated by Brauer's $k(B)$-conjecture. Thus, if true, our conjecture would yield a better bound on $k(B)$ than Brauer's (yet unproven) one.

The paper is built up as follows: In Section 2 we present some first reductions. The covering groups of alternating groups are treated in Section 3. In Section 4 we verify the inequalities for blocks of quasi-simple groups of Lie type for the defining prime in Corollary 4.3. In Section 5 we first present some general results for groups of Lie type in cross characteristic and then show the conjecture for blocks of groups of classical type (Corollary 5.19). To this end we also derive explicit formulas for invariants of unipotent blocks which we believe to be of independent interest (see Theorems 5.12 and 5.17 and Proposition 5.13). The groups of exceptional type are then considered in Section 6, see Theorem 6.5.

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## 2. First reductions

Let $G$ be a finite group and $p$ a prime. Let $\mathbf{k}$ be an algebraically closed field of characteristic $p$. The decomposition of the group algebra $\mathbf{k}G$ into a sum of minimal 2-sided ideals (called Brauer $p$-blocks) induces a corresponding subdivision of the set of isomorphism classes of irreducible $\mathbf{k}G$-modules, as well as of the set $\mathrm{Irr}(G)$ of irreducible complex characters of $G$. If $B$ is such a $p$-block of $G$, then we write $l(B)$ for the number of isomorphism classes of irreducible $\mathbf{k}G$-modules belonging to $B$, and $k(B)$ for the number of irreducible complex characters in $B$. Associated to a block $B$ is a conjugacy class of $p$-subgroups of $G$, the so-called defect groups. If $D$ is a defect group of $B$, then $\mathbf{k}D$ is a single $p$-block, and so $k(\mathbf{k}D)$ coincides with the number of conjugacy classes of $D$, which we denote by $k(D)$. We write $k_0(B)$ for the number of characters in $B$ *of height 0*, that is, for the number of elements in

$$\{\chi \in \mathrm{Irr}(B) \mid \chi(1)|D|/|G|_p \not\equiv 0 \pmod p\}.$$

We denote by $H' = [H, H]$ the derived subgroup of a group $H$.

We start with the following consequence of the proven direction of Brauer's height zero conjecture:

**Theorem 2.1.** *Let $B$ be a $p$-block of a finite quasi-simple group with abelian defect groups. Then $B$ is not a counterexample to inequalities* (C1) *and* (C2).

*Proof.* Let $D$ denote a defect group of the block $B$. If $D$ is abelian, then $k(D') = 1$, and by the known direction of Brauer's height zero conjecture [18] we have $k_0(B) = k(B)$. So (C1) holds trivially (with equality). As pointed out by Sambale [31, p. 22], (C2) holds by a result of Feit. □

Sambale has proved the validity of our inequalities for several types of defect groups; we will need the following cases:

**Proposition 2.2.** *Let $B$ be a 2-block of a finite group whose defect group is either metacyclic or a central product of a metacyclic group with a cyclic group. Then* (C1) *and* (C2) *are satisfied for $B$.*

*Proof.* The claim for metacyclic groups is proved in [31, Cor. 8.2], the one for central products in [31, Thm. 9.1]. □

The following bound (see Pantea [30, Prop. 3.2]) on the number of conjugacy classes of $p$-groups will be useful in dealing with small cases:

**Proposition 2.3.** *Let $D$ be a $p$-group of order $|D| = p^n$. Then $k(D) \geq p^2 + (n-2)(p-1)$.*

**Proposition 2.4.** *Let $G$ be a covering group of a sporadic simple group or of ${}^2F_4(2)'$. Then inequalities* (C1) *and* (C2) *hold for all blocks of $G$.*

*Proof.* This is an elementary check based on the known character tables of the quasi-simple groups in question, using the bound from Proposition 2.3. Only the first inequality is not immediate in all cases, more precisely, not so for certain faithful 2-blocks of

$$4.M_{22},\ 2.HS,\ 2.Ru,\ 2.Suz,\ 6.Suz,\ 2.Fi_{22},\ 6.Fi_{22},\ Fi_{23} \text{ and } 2.Co_1.$$

In these cases it suffices to check that defect groups $D$ satisfy $k(D') \geq 8$. □

**Proposition 2.5.** *Let $G$ be an exceptional covering group of a simple group of Lie type or of the alternating group $\mathfrak{A}_7$. Then inequalities* (C1) *and* (C2) *hold for all blocks of $G$.*

*Proof.* Again, the ordinary character tables of all groups in question are known and the claim can be checked directly. The first inequality is not immediate for certain faithful 2-blocks of

$$12_1.\mathrm{U}_4(3),\ 12_2.\mathrm{U}_4(3),\ 2.\mathrm{U}_6(2),\ 6.\mathrm{U}_6(2),\ 2.\mathrm{O}_8^+(2),\ 2.F_4(2),\ 2.{}^2E_6(2) \text{ and } 6.{}^2E_6(2),$$

but again in these cases it suffices to verify that $k(D') \geq 8$. □

## 3. Alternating groups

In [22, Prop. 4.4 and 4.7] we proved that all blocks of symmetric groups $\mathfrak{S}_n$ satisfy our inequalities. The corresponding statement for blocks of the alternating groups $\mathfrak{A}_n$ and of the 2-fold covering groups $2.\mathfrak{A}_n$, which we will derive here, is not an immediate consequence of this latter result, though. Our proofs will crucially rely on various results of Olsson.

Recall that $p$-blocks of $\mathfrak{S}_n$ are parametrised by $p$-cores, and that their block theoretic invariants only depend on their weight. Let $\tilde{B} = \tilde{B}(w)$ be a $p$-block of $\mathfrak{S}_n$ of weight $w$. Then $k(p, w) := k(\tilde{B}(w))$ is the number of $p$-multipartitions of $w$. Let $P_i$ denote the $i$-fold wreath product of the cyclic group of order $p$, that is, $P_i = C_p \wr \cdots \wr C_p$ (with $i$ terms). Then a defect group of $\tilde{B}(w)$ is a direct product $\tilde{D} = \tilde{D}(w) = \prod_{i=0}^r P_{i+1}^{a_i}$, where $w = \sum_{i=0}^r a_i p^i$, with $0 \leq a_i < p$, is the $p$-adic decomposition of $w$ (see [29, Prop. 11.3]). In particular, defect groups are abelian if the weight $w$ is less than $p$.

**Proposition 3.1.** *Let $G = 2.\mathfrak{A}_n$ with $n \geq 5$, and $p$ be an odd prime. Then all $p$-blocks of $G$ satisfy inequalities* (C1) *and* (C2).

*Proof.* We start with some observations on $p$-blocks of $\mathfrak{S}_n$. Let $\tilde{B}$ be a $p$-block of $\mathfrak{S}_n$, of weight $w$, and let $D \leq \mathfrak{S}_n$ be a defect group of $\tilde{B}$. According to [22, Prop. 4.7] for $p \geq 5$ we have $k(\tilde{B}) \leq k(D')$, and in fact the given bound shows that also $k(\tilde{B}) \leq k(D)$. Now assume that $p = 3$. Here we still have $k(\tilde{B}) = k(p, w) \leq k(D')$ whenever $w \geq 18$ and $k(p, w) \leq k(D)$ when $w \geq 4$. The cases in which these inequalities are not satisfied are collected in Table 1. An entry "–" signifies that the quotient is at most 1 and hence our inequality holds.

Table 1. 3-blocks of small weight

| $w$ | 3 | 4 | 5 | 6 | 7 | 8 | 9 | 10 | 11 | 13 | 14 | 17 |
|---|---|---|---|---|---|---|---|---|---|---|---|---|
| $k(\tilde{B})/k(D') \leq$ | 3 | 6 | 12 | 3 | 6 | 10 | 2 | 3 | 5 | 2 | 3 | 2 |
| $k_0(\tilde{B})$ | 9 | 27 | 81 | 54 | 162 | 486 | 27 | 81 | 243 | 648 | 2187 | 13122 |
| $k(\hat{B})/k(D')$ | 2 | 3 | 4 | – | 2 | 2 | – | – | – | – | – | – |
| $k(\tilde{B})/k(D) \leq$ | 2 | – | – | – | – | – | – | – | – | – | – | – |

Now let $B$ be a $p$-block of $\mathfrak{A}_n$, and $\tilde{B}$ a $p$-block of $\mathfrak{S}_n$ covering $B$. As $p$ is odd, any defect group $D$ of $B$ is also a defect group of $\tilde{B}$. Now by [28, Prop. 4.10] we have $k(B) \leq k(\tilde{B})$, so by what we showed above

$$k(B)/k_0(B) \leq k(B) \leq k(\tilde{B}) \leq k(D') \quad \text{and} \quad k(B)/l(B) \leq k(B) \leq k(\tilde{B}) \leq k(D),$$

whence (C1) and (C2) hold for $B$, unless $p = 3$ and $w \leq 17$. Here, we certainly always have $k_0(B) \geq k_0(\tilde{B})/2$, and $k_0(\tilde{B}) = \prod_{i=0}^{r} k(3^{i+1}, a_i)$ by [29, Prop. 12.4]. The relevant values are given in Table 1 from which (C1) follows. Similarly we have $l(B) \geq l(\tilde{B})/2$, and $l(\tilde{B}) = k(2, w)$ by [29, Prop. 12.8] which is greater than 2 for $w = 3$, so (C2) is also satisfied.

Next, let $\hat{B}$ be a faithful $p$-block of $2.\mathfrak{A}_n$. As pointed out in [29, Rem. 13.18 and Prop. 3.19] for any spin block of $2.\mathfrak{A}_n$ there is some $m \geq 1$ and a spin block of $2.\mathfrak{S}_m$ having the same invariants, so we may assume that $\hat{B}$ is in fact a faithful block of $2.\mathfrak{S}_n$. Let $\tilde{B}$ denote a block of some symmetric group $\mathfrak{S}_m$ with the same weight as $\hat{B}$ and (hence) isomorphic defect groups. According to [29, Prop. 13.14] we again have $k(\hat{B}) \leq k(\tilde{B})$. We can thus argue as before unless $p = 3$ and $w \leq 17$. When $p = 3$ by [29, Cor. 13.6] we have that $k(\hat{B}) = \frac{3}{2}\sum_{i=0}^{w} q(i)p(w-i)$, where $q(i)$ is the number of strict partitions of $i$ (see [29, Prop. 9.6(i)]). Then $k(\hat{B})/k(D') \leq 1$ whenever $w \geq 9$, and the exceptions are again listed in Table 1. It is straightforward to check that the remaining five weights do not lead to a counterexample to (C1). By [29, Prop. 13.17], for all $w$ we have $l(\hat{B}) \geq k(1, w)$, the number of partitions of $w$. Visibly this is larger than $k(\tilde{B})/k(D)$ for $w \geq 3$ thus showing inequality (C2). □

*Remark* 3.2. The proof shows that $p$-blocks $B$ of covering groups of alternating groups for $p \geq 3$ always satisfy the strengthened form $k(B) \leq k(D)$ of Brauer's $k(B)$-conjecture, which trivially implies at least inequality (C2), unless $p = w = 3$.

**Proposition 3.3.** *All* 2*-blocks of* $\mathfrak{A}_n$ *with* $n \geq 5$ *satisfy inequalities* (C1) *and* (C2).

*Proof.* We first consider a block $\tilde{B}$ of $\mathfrak{S}_n$ of weight $w \geq 3$ and with defect group $\tilde{D}$. Then by [29, Prop. 11.4] we have $k(\tilde{B}) = k(2, w)$, while $k(\tilde{D}) = \prod_{i=0}^{r} k(\tilde{D}(2^i))^{a_i}$, where $w = \sum_i a_i 2^i$ is the 2-adic decomposition of $w$. Then using the estimates in [22, Lemma 4.3] when $w$ is large, and explicit values for small $w$, it is readily seen that

$$k(\tilde{B}) \leq k(\tilde{D}) \qquad \text{for all } w.$$

Furthermore, again with [22, Lemma 4.3] we get that

$$k(\tilde{B}) \leq 4\, k(\tilde{D}') \qquad \text{when } w \neq 3, 7.$$

In particular, (C2) is satisfied for all 2-blocks of $\mathfrak{S}_n$, and (C1) holds at least when $w \neq 3, 7$ as then $k_0(\tilde{B}) \geq 4$. For $w = 3, 7$, (C1) follows by using the explicit values of $k_0(\tilde{B})$ (see also [22, Prop. 4.4 and 4.7]).

Now let $B$ be a 2-block of $\mathfrak{A}_n$, and $\tilde{B}$ the 2-block of $\mathfrak{S}_n$ covering $B$. Let $D$ be a defect group of $B$ and $\tilde{D} \geq D$ a defect group of $\tilde{B}$. We may assume that $\tilde{B}$ has weight at least 3, as otherwise $|\tilde{D}| \leq 8$ and hence $D = \tilde{D} \cap \mathfrak{A}_n$ is abelian. According to [28, Prop. 4.13] we always have $k(B) \leq k(\tilde{B})$.

If $w$ is odd, then in fact $k(B) = k(\tilde{B})/2$ by [28, Prop. 4.5]. As $|\tilde{D} : D| = 2$ this implies inequality (C2), from the fact shown above that $k(\tilde{B}) \leq k(\tilde{D})$. For inequality (C1), by [28, Prop. 4.6] we have $k_0(B) = k_0(\tilde{B})/2$, while $D' = \tilde{D}'$ by [28, Prop. 4.15], so

$$k(B)/k_0(B) = k(\tilde{B})/k_0(\tilde{B}) \leq k(\tilde{D}') = k(D')$$

by the result for the block $\tilde{B}$ of $\mathfrak{S}_n$.

Now assume that $w$ is even. Then $l(B) \geq 2$ whenever $D$ is non-abelian, which gives

$$k(B)/l(B) \leq k(\tilde{B})/2 \leq k(\tilde{D})/2 \leq |\tilde{D} : D|k(D)/2 = k(D),$$

whence (C2). As for (C1), we still have

$$k(B) \leq k(\tilde{B}) \leq 4k(\tilde{D}') \leq 8k(D') \leq k_0(B)k(D')$$

for $w \neq 3, 7$, as $k_0(B) \geq 8$ for $w \geq 4$. It remains to check the two cases $w = 3, 7$, which is straightforward. □

**Proposition 3.4.** *All* 2*-blocks of* $2.\mathfrak{A}_n$ *with* $n \geq 5$ *satisfy inequalities* (C1) *and* (C2).

*Proof.* Let $\hat{B}$ be a 2-block of $G = 2.\mathfrak{A}_n$ and $B$ the block of $\mathfrak{A}_n$ contained in $\hat{B}$. If $\hat{D}$ is a defect group of $\hat{B}$, then $D = \hat{D}/Z$ is a defect group of $B$, where $Z = Z(G)$. In particular, $k(\hat{D}) \geq k(D)$ and $k(\hat{D}') \geq k(D')$. Let $\tilde{B}$ denote the block of $\mathfrak{S}_n$ covering $B$. By [25, Thm. C] we have $k(\hat{B}) \leq 2k(B)$. So using the results for blocks of $\mathfrak{S}_n$ shown in the proof of Proposition 3.3 we get

$$k(\hat{B}) \leq 2k(B) \leq 2k(\tilde{B}) \leq 2k(\tilde{D}) \leq 4k(D) \leq 4k(\hat{D}) \leq l(\hat{B})k(\hat{D})$$

as $l(\hat{B}) = l(B) \geq 4$ for $w \geq 4$. Thus we obtain (C2). For $w = 3$ we have $k(\hat{B}) = 9$, $l(\hat{B}) = 3$ and $k(\hat{D}) \geq 4$. Also, again using the result for $\mathfrak{S}_n$,

$$k(\hat{B}) \leq 2k(B) \leq 2k(\tilde{B}) \leq 8k(\tilde{D}') \leq 16k(D') \leq 16k(\hat{D}')$$

for $w \neq 3, 7$. As $k_0(\hat{B}) \geq 16$ for $w \geq 6$, this proves (C1) when $w \geq 8$. The cases of small $w$ can again be checked individually. □

Arguing along the same lines it is straightforward to show that the 2-blocks of $2.\mathfrak{S}_n$ also satisfy (C1) and (C2).

**Theorem 3.5.** *Let* $G$ *be a covering group of* $\mathfrak{A}_n$, $n \geq 5$*. Then all* $p$*-blocks of* $G$ *satisfy inequalities* (C1) *and* (C2) *for all primes* $p$.

*Proof.* The blocks of the exceptional covering groups of $\mathfrak{A}_6 \cong \mathrm{PSL}_2(9)$ and $\mathfrak{A}_7$ have been considered in Proposition 2.5 while the blocks of $\mathfrak{A}_n$ have been dealt with in Propositions 3.1 and 3.3. Finally, the faithful blocks of $2.\mathfrak{A}_n$ for odd primes were again handled in Proposition 3.1 and the 2-blocks in Proposition 3.4. This completes the proof. □

## 4. Groups of Lie type in their defining characteristic

In this section we verify the inequalities (C1) and (C2) for blocks of quasi-simple groups of Lie type in their defining characteristic. Partial results have already been obtained in [22, Prop. 3.2]. In particular, both inequalities were shown to hold for groups obtained from simple algebraic groups of adjoint type. Thus, neither the $p$-blocks of Suzuki and Ree groups nor those of groups of type $G_2$, ${}^3D_4$, $F_4$ or $E_8$ yield counterexamples. We can hence discard them from our discussion here.

Let $\mathbf{G}$ be a simple algebraic group of simply connected type over an algebraic closure of the finite field $\mathbb{F}_p$ and $F : \mathbf{G} \to \mathbf{G}$ a Frobenius endomorphism with respect to an $\mathbb{F}_q$-rational structure, where $q = p^f$. Let $G = \mathbf{G}^F$ be the finite group of fixed points. Let $\mathbf{T} \leq \mathbf{B} \leq \mathbf{G}$ be an $F$-stable maximal torus in an $F$-stable Borel subgroup of $\mathbf{G}$, and $\mathbf{U} = R_u(\mathbf{B})$ the unipotent radical of $\mathbf{B}$. Let $\Phi$ be the root system of $\mathbf{G}$ with respect to

$\mathbf{T}$ and $\Phi^+ \subset \Phi$ the positive system defined by $\mathbf{B}$, with base $\Delta \subset \Phi^+$. We write $r := |\Delta|$ for the rank of the algebraic group $\mathbf{G}$. For $\alpha \in \Phi^+$ let $\mathbf{U}_\alpha \leq \mathbf{U}$ denote the corresponding root subgroup. Set $U := \mathbf{U}^F$, a Sylow $p$-subgroup of $G$.

**Lemma 4.1.** *We have $k(U) \geq q^{|\Delta|}$ and $k(U') \geq q^{2|\Delta|-3}$.*

*Proof.* According to Chevalley's commutator formula (see e.g. [23, Thm. 11.8]), $\mathbf{U}'$ is contained in the subgroup $\mathbf{U}_1 := \langle \mathbf{U}_\alpha \mid \alpha \in \Phi^+ \setminus \Delta \rangle$, and $\mathbf{U}/\mathbf{U}_1 \cong \prod_{\alpha\in\Delta} \mathbf{U}_\alpha$. Clearly $\mathbf{U}_1$ is also $F$-stable, and we set $U_1 = \mathbf{U}_1^F$. But then

$$U/U_1 = \mathbf{U}^F/\mathbf{U}_1^F \cong (\mathbf{U}/\mathbf{U}_1)^F \cong (\prod_{\alpha\in\Delta} \mathbf{U}_\alpha)^F$$

has order $q^{|\Delta|}$, see [23, Cor. 23.9]. Now clearly $k(U) \geq k(U/U') \geq k(U/U_1)$, which proves the first claim.

For the second claim note that again by the commutator formula $U''$ is contained in the subgroup

$$\mathbf{U}_2 := \langle \mathbf{U}_{\alpha+\beta} \mid \alpha, \beta \in \Phi^+ \setminus \Delta \rangle$$

(where we let $\mathbf{U}_{\alpha+\beta} := 1$ if $\alpha + \beta \notin \Phi$). Now $\mathbf{U}_1/\mathbf{U}_2 \cong \prod_\gamma \mathbf{U}_\gamma$ where $\gamma$ runs over

$$\{\gamma \in \Phi^+ \setminus \Delta \mid \gamma \neq \alpha + \beta \text{ with } \alpha, \beta \in \Phi^+ \setminus \Delta\},$$

that is, $\mathbf{U}_1/\mathbf{U}_2$ contains a subgroup isomorphic to the product of the root subgroups for roots which are the sum of two or three simple roots. By assumption $\mathbf{G}$ is simple, so its root system $\Phi$ is indecomposable. It is easily seen that any indecomposable root system of rank $r$ has at least $2r - 3$ such roots. Now first assume that $U' = U_1$. Then the preceding argument shows that $|U'/U''| \geq |\mathbf{U}_1^F/\mathbf{U}_2^F| = |(\mathbf{U}_1/\mathbf{U}_2)^F| \geq q^{2|\Delta|-3}$.

Finally consider the case that $U' < U_1$. Then by [17, Lemma 7] we have $G$ is of type $B_n(2)$, $F_4(2)$, $G_2(2)$ or $G_2(3)$. In the first case, $U'$ has index 2 in $U_1$ by [4, Prop. 3], but in this case $|(\mathbf{U}_1/\mathbf{U}_2)^F| \geq q^{2|\Delta|-1}$, so we conclude as before. The other three cases can be checked by direct computation. □

An asymptotic version of the following result for principal blocks had already been obtained in [22, Thm. 3.1]:

**Theorem 4.2.** *Let $G = \mathbf{G}^F$ be quasi-simple of Lie type in characteristic $p$. Then the $p$-blocks of $G$ satisfy inequalities* (C1) *and* (C2).

*Proof.* Let $B$ be a $p$-block of $G$. Then by a result of Humphreys (see [6, Thm. 6.18]) either $B$ is of defect zero and $B$ contains just the Steinberg character of $G$ (whence our claim holds trivially), or it is of full defect. So in the latter case, the maximal unipotent subgroup $U$ is a defect group of $B$.

Let's first consider (C1). According to [11, Thm. 1.1] we have $k(B) \leq k(G) \leq 27.2q^r$, where $r = |\Delta|$ is the rank of $\mathbf{G}$. Moreover, $k(U') \geq q^{2r-3}$ by Lemma 4.1. The number of $p'$-characters of $G$ is bounded below by the number of semisimple characters (in the sense of Lusztig), hence of semisimple conjugacy classes of the dual group $G^*$. Observe that $\mathbf{G}^*$ is simple if $\mathbf{G}$ is. The number of semisimple conjugacy classes in the fixed points of simple algebraic groups under Frobenius maps was determined in [3, Thm. 4.1(b) and Table 2]. Let $d := |Z(G)|$. Then the description in [3, §4.2] shows that any of the sets

$\mathrm{Irr}(G|\theta)$, for $\theta \in \mathrm{Irr}(Z(G))$, contains at least $(q^r-1)/d$ irreducible characters of $p'$-degree. Observe that $d \le \min\{r+1, q+1\}$ in all cases. So we are done if we can show that

$$27.2\, d \le (q^r - 1)q^{r-3}.$$

This holds whenever $r \ge 4$. Moreover for $r = 3$ it holds whenever $q > 3$. For the finitely many groups of rank 3 with $q \in \{2, 3\}$ the assertion can be checked from their known character tables. On the other hand, for $r = 1$ we have $G = \mathrm{SL}_2(q)$ which has abelian Sylow $p$-subgroups, whence the claim holds by Theorem 2.1. Thus we are left with the case that $r = 2$. Then $\mathbf{G}$ is either of type $A_2$, $C_2$ or $G_2$. We have already dealt with the case $G_2$ in [22, Prop. 3.2]. For the other two types of groups, better bounds on $k(G)$ are available, namely $k(G) \le q^r + Aq^{r-1}$ for a certain constant $A$ which is at most 12 when $q \ge 5$ [11, Prop. 3.6, 3.10, Thm. 3.12, 3.13]. It is easily seen that this rules out the case $q \ge 5$. For the finitely many groups with $q \le 4$ the claim can be checked from their known character tables.

For inequality (C2) note that since $\mathbf{G}$ is of simply connected type we have that $G$ has precisely $l(G) = q^r$ semisimple, that is, $p'$-conjugacy classes (see e.g. [3, Thm. 4.1]). So there are $l(G) - 1 = q^r - 1$ simple $\mathbf{k}G$-modules in blocks of full defect. Let $d := |Z(G)|$. The description in [3, §4.2] shows that any of the $d$-blocks $B$ of full defect contains exactly $l(B) = (q^r - 1)/d$ modular irreducibles. Since $k(U) \ge q^r$ by Lemma 4.1, here it suffices to show that

$$27.2\, d \le q^r - 1.$$

As before, we have $d \le \min\{r+1, q+1\}$. Then the above inequality holds for all $r \ge 7$, and for $q \ge 8 - r$ for $r \ge 3$. The character tables of all remaining groups are available in GAP [32]. In rank 2, we can again replace the bound for $k(G)$ by the smaller values cited above to conclude unless $q \le 3$. These last few groups can again be checked individually. □

**Corollary 4.3.** *Let $H$ be a quasi-simple group of Lie type in characteristic $p$. Then all $p$-blocks of $H$ satisfy inequalities* (C1) *and* (C2).

*Proof.* By Proposition 2.5 we may assume that $H$ is not an exceptional covering group of the simple group $H/Z(H)$. But then $|Z(H)|$ is prime to $p$, so any $p$-block of $H$ is a $p$-block of a quasi-simple group $G = \mathbf{G}^F$ as in Theorem 4.2, and the claim follows. □

## 5. Groups of classical Lie type in non-defining characteristic

We now turn to $\ell$-blocks of groups of Lie type in characteristic $p$, where $\ell$ is different from $p$. We keep the algebraic group setup from Section 4 and let $G = \mathbf{G}^F$ for $\mathbf{G}$ simple of simply connected type defined over $\mathbb{F}_q$.

5.1. **On non-unipotent blocks.** Let $B$ be an $\ell$-block of $G$. Then by a fundamental result of Broué and Michel (see e.g. [6, Thm. 9.12(i)]) there is a semisimple $\ell'$-element $s \in G^* := \mathbf{G}^{*F}$, where $\mathbf{G}^*$ is dual to $\mathbf{G}$ with a Steinberg map also denoted $F$, such that

$$\mathrm{Irr}(B) \subseteq \mathcal{E}_\ell(G, s) := \coprod_t \mathcal{E}(G, st)$$

with the union running over $\ell$-elements $t \in C_{G^*}(s)$ up to conjugation.

**Lemma 5.1.** *Assume that* $\mathbf{G}$ *has cyclic centre. Let* $B$ *be an* $\ell$*-block of* $G$ *such that* $\mathrm{Irr}(B) \subseteq \mathcal{E}_\ell(G, s)$*. Assume that* $C^\circ_{\mathbf{G}^*}(s)$ *lies in a proper* $F$*-stable Levi subgroup of* $\mathbf{G}^*$*. Then* $B$ *is not a minimal counterexample to the inequalities* (C1) *and* (C2)*.*

*Proof.* Let $\mathbf{L}^* < \mathbf{G}^*$ be a proper $F$-stable Levi subgroup of $\mathbf{G}^*$ containing $C^\circ_{\mathbf{G}^*}(s)$, and let $\mathbf{L} \leq \mathbf{G}$ be dual to $\mathbf{L}^*$, with $F$-fixed points $L = \mathbf{L}^F$. By the main result of Bonnafé–Dat–Rouquier [1] any block $B$ in $\mathcal{E}_\ell(G, s)$ is Morita equivalent to a block $b$ of a subgroup of the normaliser of $L$ in $G$ above $\mathcal{E}_\ell(L, s)$ with isomorphic defect groups. Thus all block theoretic invariants occurring in our inequalities agree for $B$ and $b$, and as $L$ is proper in $G$ by assumption, $B$ cannot be a minimal counterexample. □

*Remark* 5.2. Recently, a gap has been pointed out in the proof of the main result of [1]. But this problem only concerns the case when $\mathbf{G}$ has non-cyclic centre, while the result in the cyclic centre case is undisputed.

Elements $s \in G^*$ for which $C_{\mathbf{G}^*}(s)$ does not lie in a proper $F$-stable Levi subgroup of $\mathbf{G}^*$ are called *quasi-isolated*, and *isolated* if that even holds for $C^\circ_{\mathbf{G}^*}(s)$. The most important example is when $s = 1$, that is, the unipotent blocks.

**Proposition 5.3.** *Let* $B$ *be a quasi-isolated, non-unipotent* $\ell$*-block of a quasi-simple group of Lie type* $H$ *for a prime* $\ell \geq 5$ *that is good for* $H$*. Then* $B$ *is not a minimal counterexample to either* (C1) *or* (C2)*.*

*Proof.* By Proposition 2.4 we may assume that $H$ is not an exceptional covering group. Thus, $H = G/Z$, where $G = \mathbf{G}^F$ is as above and $Z \leq Z(G)$. The only isolated element in type $A$ is the identity which corresponds to unipotent blocks, so since these groups have cyclic centre, we may assume by Lemma 5.1 that $\mathbf{G}$ is not of type $A$. But then $\ell$ good implies that $\ell$ does not divide $|Z(G)|$, so we may consider $B$ as an $\ell$-block of $G$. By the main result of Enguehard [10, Thm. 1.6] there is a group $G_1$, with $|G_1/Z(G_1)|$ strictly smaller than $|G/Z(G)|$ (since $B$ is not unipotent) with an $\ell$-block $B_1$ having the same invariants ($k(B)$, $l(B)$ and defect group) as $B$ and with a height preserving bijection $\mathrm{Irr}(B) \to \mathrm{Irr}(B_1)$. In particular, $B$ and any block of $H$ dominated by $B$ is not a minimal counterexample to (C1) or (C2). □

So as far as minimal counterexamples are concerned we only need to consider blocks in isolated series. We note one further reduction which will be used for isolated 5-blocks of $E_8(q)$:

**Lemma 5.4.** *Let* $s \in G^*$ *be a non-central semisimple* $\ell'$*-element with connected centraliser* $\mathbf{C}^* = C_{\mathbf{G}^*}(s)$ *and let* $\mathbf{C}$ *be dual to* $\mathbf{C}^*$*. Assume that both* $\mathcal{E}_\ell(G, s)$ *and* $\mathcal{E}_\ell(C, 1)$ *form single* $\ell$*-blocks* $B$*,* $b$ *respectively, with isomorphic defect groups. Then* $B$ *is not a minimal counterexample to* (C1)*. If in addition* $l(B) \geq l(b)$ *then* $B$ *is not a minimal counterexample to* (C2) *either.*

*Proof.* We have that $\mathrm{Irr}(B) = \coprod_t \mathcal{E}(G, st)$ and $\mathrm{Irr}(b) = \coprod_t \mathcal{E}(C, t)$ where both disjoint unions run over $C^* := C_{G^*}(s)$-conjugacy classes of $\ell$-elements $t$ in $C^*$. For any such $t$ the Jordan decompositions in $G$ as well as in $C$ establish bijections from both $\mathcal{E}(G, st)$ and $\mathcal{E}(C, t)$ to the same Lusztig series $\mathcal{E}(C_t, 1)$ of $C_t := C_{G^*}(st) = C_{C^*}(t)$. Thus, if $\chi \in \mathcal{E}(G, st)$ and $\chi' \in \mathcal{E}(C, t)$ correspond to the same character in $\mathcal{E}(C_t, 1)$ we have that

$\chi(1)/\chi'(1) = |G^* : C_{G^*}(s)|_{p'}$ is constant. So there is a height preserving bijection from $\mathrm{Irr}(B)$ to $\mathrm{Irr}(b)$, whence $k(B) = k(b)$ and $k_0(B) = k_0(b)$. As the defect groups of $B, b$ are isomorphic by assumption, this yields the first claim. If in addition $l(B) \geq l(b)$, then $k(B)/l(B) \leq k(b)/l(b)$, thus $B$ cannot be a minimal counterexample to (C2) either. □

5.2. **Estimates for numbers of multipartitions.** For integers $b, w \geq 0$, let $k(b, w)$ denote the number of $b$-multipartitions of $w$. These occur in the expression of block theoretic invariants of classical groups, and we will need upper and lower bounds for them. First, by [29, Lemma 11.11] and [22, Lemma 4.6] they satisfy:

**Lemma 5.5.** *For all $b \geq 3$ and $w, w_1, w_2 \geq 1$ we have:*

(a) *$k(b, w) \leq b^w$, and*

(b) *$k(b, w_1 + w_2) \leq k(b, w_1)k(b, w_2)$.*

We will need the following improvement of the first assertion:

**Lemma 5.6.** *Let $b \geq 4$ and $w \geq 5$. Then we have:*

(a) *$k(bx, w) \leq (bx)^w x^{-0.73\,w/\ln(x)}$ for all $x \geq 5$; and*

(b) *$k(b, w) \leq b^{w-0.47\,w/\ln(b)}$ unless $b = 4$, $w \leq 10$ or $b = 5$, $w \leq 7$.*

*Proof.* We first claim that

$$\binom{x+w-1}{w} \leq x^{w-0.73w/\ln(x)}$$

for all $w \geq 5$, $x \geq 5$, unless $w = 5$, $x \leq 7$, or $(w, x) = (6, 5)$. Indeed, this holds for $w = 5$, $x \geq 8$, by a direct check, as well as for the cases $w = 6$, $x = 6, 7$, and $(w, x) = (7, 5)$. Now

$$\binom{x+w}{w+1} = \frac{x+w}{w+1}\binom{x+w-1}{w}$$

and

$$\frac{x+w}{w+1} = 1 + \frac{x-1}{w+1} \leq 1 + \frac{x-1}{6} \leq \frac{x}{3} \leq x^{1-1/\ln(x)}$$

for $w, x \geq 5$, so we conclude by induction on $w$.

Now consider (a). Clearly the number $k(bx, w)$ of $bx$-tuples of partitions of $w$ can be split up as the sum of $k(b, i_1) \cdots k(b, i_x)$ over all compositions $(i_1, \ldots, i_x) \models w$ of $w$ of length $x$. Thus by Lemma 5.5(a) we have

$$k(bx, w) = \sum_{(i_1,\ldots,i_x)\models w} k(b, i_1) \cdots k(b, i_x) \leq \sum_{(i_1,\ldots,i_x)\models w} b^w = \binom{x+w-1}{w} b^w$$

and the assertion follows by our previous claim except in the excluded cases $w = 5$, $x \leq 7$, or $(w, x) = (6, 5)$. In those four cases the inequality can be verified directly.

For part (b), we use that $k(x, w) \leq \binom{x+w-1}{w}\alpha^w$ by [27, p. 43], where $\alpha = (1 + \sqrt{5})/2$. Now the first part of the proof can be recycled to show that even

$$\binom{b+w-1}{w} \leq b^{w-0.952\,w/\ln(b)}$$

unless $b = 4$, $w \leq 15$, or $b = 5$, $w \leq 10$, or $b = 6$, $w \leq 9$, or $b = 7$, $w \leq 8$. As $\ln(\alpha) < 0.482$ this proves the assertion under those conditions. The finitely many remaining cases can be checked directly. □

**Lemma 5.7.** *Let $\ell \geq 5$ be a prime, $a \geq 1$ and $d \leq \sqrt{\ell^a - 1}$ be a divisor of $\ell - 1$. Then for all multiples $w$ of $\ell$,*

$$k(d + (\ell^a - 1)/d, w) \leq \begin{cases} \ell^{aw - 0.83\, w \log_\ell d} & \text{when } d > 1, \\ \ell^{aw - 0.9\, w/\ln(\ell)} & \text{when } d = 1,\ a \geq 2, \\ \ell^{aw - 0.57 w/\ln(\ell)} & \text{when } d = a = 1, \end{cases}$$

*except when $d \leq 2$, $\ell^a = w = 5$, or $(\ell^a, d, w) = (25, 1, 5)$.*

*Proof.* Set $b := d + (\ell^a - 1)/d$. First assume that $d > 1$ and we are not in one of the excluded cases in Lemma 5.6(b). Then we have

$$k(b, w) \leq b^{w - 0.47w/\ln(b)} = b^w/\ell^{0.47w/\ln(\ell)}.$$

As $b = (d^2 + \ell^a - 1)/d \leq 2\ell^a/d$ this shows that

$$k(b, w) \leq (2\ell^a/d)^w\, \ell^{-0.47w/\ln(\ell)} = \ell^{w(a + \log_\ell 2 - \log_\ell d - 0.47/\ln(\ell))} \leq \ell^{aw - 0.83w \log_\ell d}$$

when $d \geq 4$. For $d = 2, 3$, actually $b \leq 1.62\, \ell^a/d$ and we can conclude as before.

In the excluded case $b = 4$, $6 \leq w \leq 10$, we necessarily have $\ell^a = 5$, $d = 2$, and the claim is checked directly. Similarly, when $b = 5$, $w \leq 7$ then $\ell^a = 7$, $d = 2$, and again the inequality holds.

For $d = 1$ with $a > 1$ we argue as in the proof of Lemma 5.6(a), avoiding the case that $(\ell^a, d, w) = (25, 1, 5)$, while for $a = 1$ we recycle the arguments in the proof of part (b) of that result, and first show that even

$$\binom{x + w - 1}{w} \leq x^{w - 1.052w/\ln(x)}$$

for all $w \geq x \geq 11$, as well as for $w \geq 10$ when $x = 7$ and for $w \geq 15$ when $x = 5$. The finitely remaining cases are again checked directly. □

For integers $\ell, a, w \geq 1$ and a divisor $d$ of $\ell - 1$ let us define

$$k(\ell, a, d, w) := \sum_{\mathbf{w}} k(d + (\ell^a - 1)/d, w_0) \prod_{i \geq 1} k((\ell^a - \ell^{a-1})/d, w_i)$$

where the sum runs over all $\ell$-*compositions of* $w$, that is, all tuples $\mathbf{w} = (w_0, w_1, \ldots)$ of non-negative integers satisfying $\sum w_i \ell^i = w$. We write $p_\ell(w)$ for the number of such.

**Lemma 5.8.** *Let $\ell \geq 5$, let $a, w \geq 1$ with $\ell | w$, let $d < \ell^a - 1$ be a divisor of $\ell - 1$. Then*

$$k(\ell, a, d, w) \leq \begin{cases} p_\ell(w)\, \ell^{aw - 0.83\, w \ln(d)/\ln(\ell)} & \text{if } d > 1, \\ p_\ell(w)\, \ell^{aw - 0.9\, w/\ln(\ell)} & \text{if } d = 1,\ a \geq 2, \\ p_\ell(w)\, \ell^{w - 0.57\, w/\ln(\ell)} & \text{if } d = a = 1. \end{cases}$$

*unless $(\ell^a, d, w) = (5, 1, 5)$.*

*Proof.* Observe that $\sum_{i \geq 1} w_i \leq (w - w_0)/\ell$, and $w_0$ is a multiple of $\ell$ (as $\ell$ divides $w$) for all $\ell$-compositions $(w_0, w_1, \ldots)$ of $w$.

We first treat the case that $d = 1$. Then by Lemmas 5.7 and 5.5(a) we get, with $c = 0.9$ when $a > 1$ and $c = 0.57$ when $a = 1$, that $k(\ell, a, 1, w)$ is bounded above by

$$\sum_{\mathbf{w}} k(\ell^a, w_0) \prod_{i \geq 1} k(\ell^a, w_i) \leq \sum \ell^{aw_0 - cw_0/\ln(\ell) + a(w-w_0)/\ell} + \sum k(\ell^a, w_0) \ell^{a(w-w_0)/\ell},$$

where the first sum ranges over $\ell$-compositions of $w$ with $w_0 \geq \ell$ (unless $\ell = 5$ and $a \leq 2$ in which case we take $w_0 \geq 10$), and the second one over those with $w_0 = 0$ (unless $\ell = 5$ and $a \leq 2$, in which case we take $w_0 \leq 5$). Now note that $a(w_0 + (w - w_0)/\ell) - cw_0/\ln(\ell) \leq aw - cw/\ln(\ell)$, and $aw/\ell \leq aw - cw/\ln(\ell)$ as well as

$$k(25, 5)\, 5^{(w-5)/5} \leq 5^{2w - 0.91w/\ln(5)} \quad \text{and}$$
$$k(5, 5)\, 5^{(w-5)/5} \leq 5^{w - 0.57w/\ln(5)}$$

for all $w \geq 10$. Thus we can bound all summands above by $\ell^{w - cw/\ln(\ell)}$ to find

$$k(\ell, a, 1, w) \leq p_\ell(w)\, \ell^{aw - cw/\ln(\ell)}$$

unless $w = \ell = 5$, $a \leq 2$. For $a = 2$ the inequality still holds, while it fails for $a = 1$.

Now consider the case when $d > 1$. If $a = 1$ then we may assume that $d \leq \sqrt{\ell - 1}$ as the value of $b := d + (\ell^a - 1)/d$ remains unchanged by replacing $d$ by $(\ell - 1)/d$. In particular, we thus can take $d \leq \sqrt{\ell^a - 1}$ in all cases. By Lemma 5.7 we obtain

$$\begin{aligned} k(\ell, a, d, w) &\leq \sum_{\mathbf{w}} k(d + (\ell^a - 1)/d, w_0) \prod_{i \geq 1} k((\ell^a - \ell^{a-1})/d, w_i) \\ &\leq \sum \ell^{aw_0 - 0.79 w_0 \log_\ell d} (\ell^a/d)^{(w-w_0)/\ell} + \sum k(b, w_0) (\ell^a/d)^{(w-w_0)/\ell}, \end{aligned}$$

where again the first sum ranges over the $\ell$-compositions with $w_0 \geq \ell$ (unless $\ell^a = 5$ in which case we take $w_0 \geq 10$), and the second one over those with $w_0 \leq 5$. Now note that

$$aw_0 - 0.83 w_0 \log_\ell d + (w - w_0)(a - \log_\ell d)/\ell \leq aw - 0.83 w \log_\ell d \quad \text{for } w_0 \geq \ell,$$

and also $w/(a - \log_\ell d)/\ell \leq aw - 0.83 w \log_\ell d$, as well as

$$k(4, 5)\, (5/2)^{(w-5)/5} \leq 5^{w - 0.83 w \log_5 2} \quad \text{for } w \geq 10$$

(for the summands with $w_0 = 5$ in the case $\ell^a = 5$, $d = 2$), so we obtain

$$k(\ell, a, d, w) \leq p_\ell(w)\, \ell^{aw - 0.83\, w \log_\ell d},$$

as claimed. When $\ell^a = w = 5$ and $d = 2$, then

$$k(5, 1, 2, 5) = k(4, 5) + k(2, 1) = 254 < 2 \cdot 5^{5(1 - 0.83 \log_5 2)}$$

by explicit calculation. □

We will also need the following lower bound.

**Lemma 5.9.** *Let $b \geq c \geq w \geq 1$. Then $k(b, w) \geq \binom{c}{w} y^w$, where $y = \lfloor b/c \rfloor$.*

*Proof.* We have $b = cy + r$ for some $0 \leq r \leq c - 1$. In the sum on the right hand side of the equation

$$k(b, w) = k(cy + r, w) = \sum_{(i_1, \ldots, i_c) \models w} k(y, i_1) \cdots k(y, i_{c-1}) k(y + r, i_c)$$

we only consider those summands indexed by compositions $(i_1,\ldots,i_c)$ of $w$ with exactly $w$ entries equal to 1 and all others 0. Since there are exactly $\binom{c}{w}$ of those, this yields the stated lower bound. □

5.3. **The conjecture for $\mathrm{GL}_n(q)$ and $\mathrm{GU}_n(q)$.** To deal with the general linear and unitary groups, the following elementary observation will allow us to estimate the number of conjugacy classes in defect groups:

**Lemma 5.10.** *Let $m\ge 1$, $i\ge 0$ and set $D_{i,m}:=C_m\wr C_\ell\wr\cdots\wr C_\ell$ the iterated wreath product with $i$ factors of the cyclic group $C_\ell$ of order $\ell$. Then*

$$k(D_{i,m})\ge m^{\ell^i}/\ell^{(\ell^i-1)/(\ell-1)}$$

*and*

$$k(D'_{i,m})\ge m^{\ell^i-1}/\ell^{(\ell^i-1)/(\ell-1)+i-1}\qquad \text{for } i\ge 1.$$

*Proof.* We argue by induction on $i$. The statement is clear when $i=0$. For $i>0$ we have $D_{i,m}=D_{i-1,m}\wr C_\ell$; by the inductive assumption the base group has at least $m^{\ell^i}/\ell^{\ell(\ell^{i-1}-1)/(\ell-1)}$ conjugacy classes, and the cyclic group on top has orbits of length at most $\ell$ on these. The first claim follows. For the second one, by [26, Lemma 1.4] we have that $D'_{i,m}$ lies in the base group $D_{i-1,m}^\ell$ of the outer wreath product, and has index $m\ell^i$ therein. Using the first assertion this yields the stated lower bound. □

**Proposition 5.11.** *Let $\ell\ge 5$ be a prime not dividing $q$. The unipotent $\ell$-blocks of $\mathrm{GL}_n(q)$ and of $\mathrm{GU}_n(q)$ do not provide counterexamples to* (C1) *or* (C2).

*Proof.* Let $d$ be the order of $q$ modulo $\ell$. Let $B$ be a unipotent $\ell$-block of $\mathrm{GL}_n(q)$. According to the reduction given in [24, Thm. 1.9] there exists $w\ge 0$ such that all relevant block theoretic invariants of $B$ are the same as those of the principal $\ell$-block of $\mathrm{GL}_{wd}(q)$; $w$ is then called the *weight* of $B$. Hence we may and will now assume that $B$ is the principal block of $G:=\mathrm{GL}_{wd}(q)$. Let $\ell^a$ be the precise power of $\ell$ dividing $q^d-1$. Write $w=\sum_{i=0}^v a_i\ell^i$ for the $\ell$-adic decomposition of $w$. The Sylow $\ell$-subgroups of $G$ are of the form $\prod_{i=0}^v D_i^{a_i}$ with $D_i:=D_{i,\ell^a}$ from Lemma 5.10, so $k(D)=\prod_i k(D_i)^{a_i}$. According to [27, Prop. 6] the invariant $k(B)$ is given by

$$k(B)=\sum_{\mathbf{w}} k(b,w_0)\prod_{i\ge 1}k(b_1,w_i)=k(\ell,a,d,w),$$

where $b=d+(\ell^a-1)/d$, $b_1=(\ell^a-\ell^{a-1})/d$, and the sum runs over the set of $\ell$-compositions $\mathbf{w}$ of $w$. Clearly, $w_0\ge a_0$, and by Lemma 5.5(b) we have $k(b,w_0)\le k(b,a_0)k(b,w_0-a_0)$. It follows that we may assume $a_0=0$ when proving (C2) by using that $k(b,a_0)\le b^{a_0}\le \ell^{aa_0}=k(D_0)^{a_0}$. Similarly, by [27, p. 46] we have $k_0(B)=\prod_{i\ge 0}k(b\ell^i,a_i)$ and thus we may also assume $a_0=0$ when proving (C1) by cancelling $k(b,a_0)$ from the upper bound for $k(B)$ and from the expression for $k_0(B)$.

So from now on, we assume $w=\sum_{i=1}^v a_i\ell^i$ is divisible by $\ell$. Then by Lemma 5.10

$$k(D)=\prod_{i=1}^v k(D_i)^{a_i}\ge\prod_{i=1}^v\left(\ell^{a\ell^i}/\ell^{(\ell^i-1)/(\ell-1)}\right)^{a_i}=\ell^{aw-\sum a_i(\ell^i-1)/(\ell-1)}\ge \ell^{aw-w/(\ell-1)}.$$

Furthermore, the unipotent characters in $B$ form a basic set for $B$ (see e.g. [12]), and they are in bijection with $d$-multipartitions of $w$ (see [2, 5]), whence $l(B) = k(d,w) \geq k(1,w)$, the number of partitions of $w$. As for $k(B)$, by Lemma 5.8

$$k(B) \leq \begin{cases} p_\ell(w)\,\ell^{aw-0.83w\ln(d)/\ln(\ell)} & \text{for } d>1,\\ p_\ell(w)\,\ell^{aw-0.9w/\ln(\ell)} & \text{if } a>1,\ d=1,\\ p_\ell(w)\,\ell^{w-0.57w/\ln(\ell)} & \text{for } a=d=1,\end{cases}$$

unless $(\ell^a, d, w) = (5,1,5)$. Now observe that $p_\ell(w) \leq k(1,w) \leq l(B)$. Combining our estimates we then indeed obtain

$$k(B) \leq p_\ell(w)\,\ell^{w(a-1/(\ell-1))} \leq l(B)\; k(D),$$

as required for (C2). (This also holds in the case $(\ell^a, d, w) = (5,1,5)$ as there $k(B) = k(5,5) + k(4,1) = 510$, $l(B) = k(1,5) = 7$ and $k(D) \geq 625$.)

Now consider (C1). As the Sylow $\ell$-subgroups of $G$ are of the form $\prod_{i=0}^v D_i^{a_i}$, we have $k(D') = \prod k(D_i')^{a_i}$. By Lemma 5.10

$$k(D') = \prod_{i=1}^{v} k(D_i')^{a_i} \geq \prod_{i=1}^{v} \left(\ell^{a(\ell^i-1)-(\ell^i-1)/(\ell-1)-(i-1)}\right)^{a_i} = \ell^{aw-w/(\ell-1)-\sum a_i(a+i-\ell/(\ell-1))}.$$

Furthermore, using Lemma 5.9, we have

$$k_0(B) = \prod_{i\geq 1} k(b\ell^i, a_i) \geq \prod_{i\geq 1} (b\ell^{i-1})^{a_i} = (b/\ell^a)^{\sum a_i}\ell^{\sum a_i(a+i-1)}.$$

Let first $d = 1$. Thus $b = \ell^a$. We also use from [21, Lemma 5.2] that $p_\ell(w) \leq \ell^{\binom{u+1}{2}}$ with $u = \lfloor \log_\ell(w) \rfloor$. Then with the estimates from Lemma 5.8 we find

$$\begin{aligned} k(D')k_0(B)/k(B) \geq& \ell^{aw-w/(\ell-1)-\sum a_i(a+i-\ell/(\ell-1))+\sum a_i(a+i-1)-aw+0.9w/\ln(\ell)}/p_\ell(w)\\ \geq& \ell^{0.9w/\ln(\ell)-u(u+1)/2-w/(\ell-1)} > 1\end{aligned}$$

for $a > 1$, and

$$k(D')k_0(B)/k(B) \geq \ell^{0.57w/\ln(\ell)-u^2-w/(\ell-1)+\sum a_i/(\ell-1)} > 1$$

for $(\ell, w) \neq (5,5)$ when $a = 1$. The case $\ell = w = 5$ can be checked directly.

Now consider the case when $d > 1$, where with $b \geq \ell^a/d$ we have

$$k_0(B) \geq (b/\ell^a)^{\sum a_i}\ell^{\sum a_i(a+i-1)} \geq \ell^{\sum a_i(a+i-1)}d^{-\sum a_i} = \ell^{\sum a_i(a+i-1)-\sum a_i \log_\ell d}.$$

Combining the above estimates we find

$$k(D')k_0(B)/k(B) \geq \ell^{(0.83w-\sum a_i)\log_\ell d-w/(\ell-1)+\sum a_i/(\ell-1)-u(u+1)/2} \geq 2$$

unless $d = 2$, and either $\ell = 5$, $w \in \{5, 10, 25, 30, 35\}$, or $\ell = w = 7$. (The stronger bound 2 will be needed in the proof of Theorem 5.18.) For $w = \ell = 7$ we get $k(B) = 2996$, $k_0(B) = 35$ and $k(D') \geq 7^5$; for $\ell = 5$ and $w \in \{10, 25, 30\}$, replacing our bound for $p_5(w)$ by the actual values 3, 7 and 9, respectively shows the claim, and finally for $w = 5$ we have $k(5,1,2,5) = 254$, $k_0(B) = 20$ and $k(D') \geq 5^3$.

Now let $B$ be a unipotent $\ell$-block of $\mathrm{GU}_n(q)$, and write $d$ for the order of $-q$ modulo $\ell$. Then the block theoretic invariants of $B$ are the same as for the principal $\ell$-block of $\mathrm{GL}_{wd}(q')$, where $w$ is the weight of $B$ and $q'$ is any prime (power) such that $q'$ has order

$d$ modulo $\ell$ (see [27]). Thus the claim for $B$ follows from the result for blocks of $\mathrm{GL}_n(q)$ proved above. □

5.4. **Block invariants of special linear and unitary groups.** We now turn to the quasi-simple groups $\mathrm{SL}_n(q)$ and $\mathrm{SU}_n(q)$. For this we need to recall in some detail Olsson's description [27, p. 45] of the set of characters in the principal $\ell$-block of $\mathrm{GL}_n(\epsilon q)$, $\epsilon = \pm 1$. (As customary we write $\mathrm{GL}_n(-q) := \mathrm{GU}_n(q)$, and similarly $\mathrm{SL}_n(-q) := \mathrm{SU}_n(q)$ and so on.) If $\ell$ divides $q-\epsilon$, the principal $\ell$-block is the unique unipotent block, so by the result of Broué–Michel [6, Thm. 9.12(i)] mentioned earlier it consists of the union of the Lusztig series indexed by conjugacy classes of $\ell$-elements in $\mathrm{GL}_n(\epsilon q)$. A conjugacy class of $\mathrm{GL}_n(q)$ is uniquely determined by the characteristic polynomial of its elements, which is a product of minimal polynomials over $\mathbb{F}_q$ of elements of $\ell$-power order in $\overline{\mathbb{F}}_q^\times$. Let $\mathcal{F}^i$ denote the set of such polynomials of degree $\ell^i$, where $i \geq 0$. Then $|\mathcal{F}^0| = \ell^a$ and $|\mathcal{F}^i| = \ell^a - \ell^{a-1}$ for $i > 0$, where we let $\ell^a$ denote the precise power of $\ell$ dividing $q-1$. The conjugacy classes of $\ell$-elements in $\mathrm{GL}_n(q)$ are thus in bijection with maps

$$m : \mathcal{F} := \bigcup \mathcal{F}^i \to \mathbb{Z}_{\geq 0}, \quad f \mapsto m_f, \qquad \text{with} \quad \sum_i \sum_{f \in \mathcal{F}^i} m_f \ell^i = n,$$

in such a way that $m$ labels the class of $\ell$-elements with characteristic polynomial $\prod_f f^{m_f}$. If $t \in \mathrm{GL}_n(q)$ corresponds to $m$, then the Lusztig series $\mathcal{E}(G,t)$ contains $\prod_f k(1, m_f)$ characters (all lying in the principal $\ell$-block). It is shown from this in [27, Prop. 6] that $k(B) = k(\ell, a, 1, n)$.

We now determine the number of characters of height 0 in the principal $\ell$-block of $\mathrm{SL}_n(\epsilon q)$ for $\ell | (q-\epsilon)$; the formula for the number of modular characters is an immediate consequence of the main result of Kleshchev and Tiep [19]:

**Theorem 5.12.** *Let $\mathrm{SL}_n(\epsilon q) \leq G \leq \mathrm{GL}_n(\epsilon q)$ with $\epsilon \in \{\pm 1\}$, and $\ell > 2$ be a prime dividing $q - \epsilon$. Set $\ell^a := (q-\epsilon)_\ell$, $\ell^g := |\mathrm{GL}_n(\epsilon q) : G|_\ell$ and $\ell^u = \gcd(\ell^g, n, q-\epsilon)$. Let $B$ denote the principal $\ell$-block of $G$, and $\tilde{B}$ the principal $\ell$-block of $\mathrm{GL}_n(\epsilon q)$. Then*

$$k_0(B) = k_0(\tilde{B})/\ell^g + \begin{cases} \ell^{a+f-g} & \text{if } n = \ell^f \text{ and } g > 0, \\ 0 & \text{else,} \end{cases}$$

*and*

$$l(B) = k(1,n) + \sum_{i=1}^{u} (\ell^i - \ell^{i-1})\, k(1, n/\ell^i).$$

*Proof.* Let first $\epsilon = 1$, so $\mathrm{SL}_n(q) \leq G \leq \mathrm{GL}_n(q) =: \tilde{G}$. The characters in the principal $\ell$-block $B$ of $G$ are precisely the constituents of the restrictions to $G$ of the characters in the principal $\ell$-block $\tilde{B}$ of $\tilde{G}$. If $\chi \in \mathrm{Irr}_0(\tilde{B})$ then, being of $\ell'$-degree, by Clifford theory it cannot split upon restriction to a normal subgroup of $\ell$-power index, so the restrictions of characters in $\mathrm{Irr}_0(\tilde{B})$ contribute $k_0(\tilde{B})/\ell^g$ to $k_0(B)$. Any further height 0 character of $B$ must be the constituent of a splitting character in $\tilde{B}$, of degree divisible by $\ell$. Assume $\chi \in \mathrm{Irr}(\tilde{B})$ lies in the Lusztig series $\mathcal{E}(\tilde{G}, t)$ of the $\ell$-element $t \in \mathrm{GL}_n(q)$. As argued in the proof of [21, Thm. 5.1] its restriction to $\mathrm{SL}_n(q)$ splits into $A_t := |C_{\mathrm{PGL}_n}(\bar{t})^F : C^\circ_{\mathrm{PGL}_n}(\bar{t})^F|$ distinct constituents, where $\bar{t}$ is the image of $t$ in $\mathrm{PGL}_n$ and $F$ is the standard Frobenius endomorphism on $\mathrm{PGL}_n$. By Clifford theory $\chi$ then splits into $\ell^b = \min\{A_t, \ell^g\}$ distinct constituents upon restriction to $G$. Thus $\chi|_G$ contributes to $\mathrm{Irr}_0(B)$ if and only if $\chi(1)_\ell = \ell^b$. Now $\ell^i|A_t$ if the set of eigenvalues of $t$ is invariant under multiplication by an $\ell^i$th root of unity $\zeta \in \mathbb{F}_q^\times$. So $\ell$-elements $t$ with $\ell^i|A_t$ are parametrised by maps $m : \mathcal{F} \to \mathbb{Z}_{\geq 0}$ as above that are constant on $\zeta$-orbits. For $m$ such a map, $C_{\tilde{G}}(t) \cong \prod_{f\in\mathcal{F}} \mathrm{GL}_{m_f}(q^{\deg(f)})$. Thus, if $m_f > 0$ for some non-linear polynomial $f \in \mathcal{F}$ (and hence with $\deg(f) \geq \ell$), then $|\tilde{G} : C_{\tilde{G}}(t)|_\ell \geq \ell^{a(\ell-1)} > \ell^a$, whence the Lusztig series of $t$ cannot contribute to $\mathrm{Irr}_0(B)$. To count characters of height 0 we thus only need to consider $\ell$-elements $t$ that are diagonalisable over $\mathbb{F}_q$.

Let $m$ be the map corresponding to such an element, with support on linear polynomials in $\mathcal{F}$ and consider the $\ell$-adic decompositions $m_f = \sum_{i\geq 0} m_{f,i}\ell^i$ for $f \in \mathcal{F}$ and $n = \sum_i n_i\ell^i$. Then as discussed in [24, proof of Cor. 2.6] every index $i$ for which $\sum_f m_{f,i} - n_i = k > 0$ contributes a factor of at least $(k!)_\ell$ to the index of $C_{\tilde{G}}(t)$ in $\mathrm{GL}_n(q)$ and thus to the degree of any $\chi \in \mathcal{E}(\tilde{G}, t)$. If $\chi$ splits into $\ell^i$ factors, then as seen before all multiplicities $m_f$ occur a multiple of $\ell^i$ times, and by the preceding discussion each such will contribute a factor $(\ell^i!)_\ell$ to $\chi(1)_\ell$. This is bigger than $\ell^i$ if $i > 1$ or if there are at least two such orbits of polynomials. Thus contributions to $\mathrm{Irr}_0(B)$ can occur in this way only when first $n = \ell^f$ is a power of $\ell$, and second, $g \geq 1$ and $i = 1$. In this case, there are exactly $\ell^{a-1}$ orbits of linear polynomials over $\mathbb{F}_q$, and hence the same number of maps $m$. The centraliser of a corresponding $\ell$-element has the structure $C_{\tilde{G}}(t) \cong \mathrm{GL}_{n/\ell}(q)^\ell$. Now $\mathrm{GL}_{n/\ell}(q)$ has precisely $n/\ell = n^{f-1}$ unipotent characters of height 0, each leading to $\ell$ height 0 characters. Any of these necessarily restricts irreducibly to $G$. Thus we obtain $\ell^{a-1}\ell^{f-1}\ell/\ell^{g-1} = \ell^{a+f-g}$ further characters in $\mathrm{Irr}_0(B)$, as claimed.

Let us now consider the number $l(B)$. Kleshchev and Tiep [19, Thm. 1.1] describe the number of irreducible constituents of the restriction to $\mathrm{SL}_n(q)$ of an irreducible $\ell$-modular Brauer character of $\mathrm{GL}_n(q)$, as follows: The Brauer characters in the principal block are indexed by partitions $\lambda \vdash n$, and the character labelled by $\lambda$ splits into $\gcd(q - 1, \lambda'_1, \lambda'_2, \ldots)_\ell$ constituents for $\mathrm{SL}_n(q)$, where $\lambda' = (\lambda'_1, \lambda'_2, \ldots)$ is the partition conjugate to $\lambda$. Now clearly for any divisor $\ell^i$ of $n$, the partitions of $n$ with all parts divisible by $\ell^i$, are in natural bijection with partitions of $n/\ell^i$, whence their number is $k(1, n/\ell^i)$. By Clifford theory, the splitting occurs 'at the top' of the chain of normal subgroups of $\ell$-power index, whence the claimed formula for $l(B)$ by induction.

Now let $\epsilon = -1$. As pointed out before, the block theoretic invariants for the principal $\ell$-block of $\mathrm{GU}_n(q)$ coincide with those of the principal $\ell$-block of $\mathrm{GL}_n(q^2)$, and a Sylow $\ell$-subgroup of $\mathrm{GU}_n(q)$ becomes a Sylow $\ell$-subgroup of $\mathrm{GL}_n(q^2)$ under the natural embedding $\mathrm{GU}_n(q) \leq \mathrm{GL}_n(q^2)$. The proof for $\mathrm{GU}_n(q)$ is then entirely similar to the above, where for the second part we replace the reference to [19] by the corresponding result [7, Prop. 4.9] of Denoncin for the groups $\mathrm{SU}_n(q)$. □

The situation for the projective general linear and unitary groups is somewhat easier:

**Proposition 5.13.** *Let $\ell > 2$ be a prime dividing $q - \epsilon$. Set $\ell^a = (q - \epsilon)_\ell$ and $\ell^m = \gcd(n, q - \epsilon)_\ell$. Then the principal $\ell$-block $\bar{B}$ of $\mathrm{PGL}_n(\epsilon q)$ satisfies $k_0(\bar{B}) = k_0(\tilde{B})/\ell^{a-m}$, where $\tilde{B}$ is the principal $\ell$-block of $\mathrm{GL}_n(\epsilon q)$.*

*Proof.* We present the argument in the case $\epsilon = 1$, the case $\epsilon = -1$ again being entirely analogous due to the agreement of block theoretic invariants between $\mathrm{GU}_n(q)$ and $\mathrm{GL}_n(q^2)$.

As $\mathrm{PGL}_n(q) = \mathrm{GL}_n(q)/Z(\mathrm{GL}_n(q))$ the characters in $\mathrm{Irr}_0(\bar{B})$ are the characters in $\mathrm{Irr}_0(\tilde{B})$ having $Z(\tilde{G})$ in their kernel, with $\tilde{B}$ the principal $\ell$-block of $\tilde{G} = \mathrm{GL}_n(q)$. Let $\chi \in \mathrm{Irr}_0(\tilde{B})$. Then it lies above a character of $\mathrm{SL}_n(q)$ having $Z := Z(\mathrm{SL}_n(q))_\ell$ in its kernel (as all characters above non-trivial central characters of $\ell$-power order have degree divisible by $\ell$). Furthermore, $\delta \otimes \chi \in \mathrm{Irr}_0(\tilde{B})$ for all linear characters $\delta$ of $\tilde{G}$ of $\ell$-power order. Assume that $\delta \otimes \chi = \chi$ for some $\delta \neq 1$. Then $\chi$ is induced from some subgroup of $\tilde{G}$ of index divisible by $\ell$, which is not possible as $\chi(1)$ is prime to $\ell$. Tensoring by the $\ell^a$ linear characters of $\ell$-power order permutes the various central characters modulo $Z$, hence exactly $|Z|_\ell = \ell^m$ out of each orbit on $\mathrm{Irr}_0(\tilde{B})$ lie above the trivial character of $Z(\tilde{G})$, whence $k_0(\bar{B}) = k_0(\tilde{B})/\ell^{a-m}$. □

*Example* 5.14. (a) Assume that $n$ is not divisible by $\ell$. Then in the situation and notation of Theorem 5.12 we have $g = u = 0$, and hence

$$k_0(B) = k_0(\tilde{B})/\ell^g \qquad \text{and} \qquad l(B) = l(\tilde{B}),$$

as expected (as restrictions of characters in $\ell$-element Lusztig series are irreducible in this case), and $k_0(\bar{B}) = k_0(\tilde{B})/\ell^a$ in the situation of Proposition 5.13.

(b) Next assume that $n = \ell$; in this case the only proper subgroup $G$ of $\mathrm{GL}_n(\epsilon q)$ allowed in Theorem 5.12 is $G = \mathrm{SL}_\ell(\epsilon q)$. Here, $g = u = 1$, and thus

$$k(B) = \big(k(\ell, a, 1, \ell) + (\ell^2 - 1)\, k(\ell, a, 1, 1)\big)/\ell^a = \big(k(\ell^a, \ell) + \ell^{a+2} - \ell^{a-1}\big)/\ell^a$$

by [21, Thm. 5.1],

$$k_0(\tilde{B}) = k(\ell^{a+1}, 1) = \ell^{a+1},\ k_0(B) = k_0(\tilde{B})/\ell^a + \ell = 2\ell,\ k_0(\bar{B}) = k_0(\tilde{B})/\ell^{a-1} = \ell^2,$$

by Theorem 5.12 and Proposition 5.13, and

$$l(B) = k(1, \ell) + \ell - 1.$$

5.5. **The conjecture for** $\mathrm{SL}_n(q)$ **and** $\mathrm{SU}_n(q)$**.** To show our inequalities for blocks of $\mathrm{SL}_n(\epsilon q)$ we will need to control the derived subgroup of a Sylow subgroup:

**Proposition 5.15.** *For $\ell > 2$ with $\ell|(q-1)$, let $\tilde{D}$ be a Sylow $\ell$-subgroup of $\mathrm{GL}_n(q)$ and $D = \tilde{D} \cap \mathrm{SL}_n(q)$, a Sylow $\ell$-subgroup of $\mathrm{SL}_n(q)$. Then*

$$|\tilde{D}' : D'| = \begin{cases} \ell & \text{if } n = \ell^f \text{ for some } f \geq 1, \\ 1 & \text{else.} \end{cases}$$

*Proof.* Let $n = \sum_{i \geq 0} a_i \ell^i$ be the $\ell$-adic decomposition of $n$, and $\ell^a$ the precise power of $\ell$ dividing $q-1$. Then $\tilde{D} = \prod_{i \geq 0} D_i^{a_i}$, with $D_i = D_{i,\ell^a}$, and $D$ is the subdirect product of these factors defined by the determinant condition. Now if that product has more than one factor, it is straightforward to see that the derived subgroup of the subdirect product $D$ agrees with that of $\tilde{D}$. Hence now assume that $n = \ell^i$ is a power of $\ell$. So $\tilde{D} = D_{i-1} \wr C_\ell$. Clearly, $\tilde{D}'$ and thus also $D'$ lies in the base group. We claim that $D'$ consists of all $\ell$-tuples $(x_1, \ldots, x_l)$ of elements of $D_{i-1}$ such that $x_1 \cdots x_\ell \in D'_{i-1}$ and $x_1 x_2^2 \cdots x_{\ell-1}^{\ell-1}$ is an $\ell$th power. Indeed, $D'$ contains any tuple with entries in $D'_{i-1}$, as well as every tuple of the form $(x^\ell, x^{-\ell}, 1, \ldots, 1)$ (being the commutator of $(x^{\ell-1}, x^{-1}, \ldots, x^{-1})$ with an $\ell$-cycle), as well as all tuples of the form $(x, x^{-1}, x^{-1}, x, 1, \ldots, 1)$ (being the commutator of

$(x, x^{-1}, 1, \ldots, 1)$ with an $\ell$-cycle). According to [26, Lemma 1.4], $\tilde{D}'$ consists of all $\ell$-tuples with $x_1 \cdots x_\ell \in D'_{i-1}$, and the group described above has index $\ell$ therein. □

**Theorem 5.16.** *Let $H = G/Z$ with $G \in \{\mathrm{SL}_n(q), \mathrm{SU}_n(q)\}$ and $Z \leq Z(G)$. Assume that $\ell \geq 5$. Then the unipotent $\ell$-blocks of $H$ are not counterexamples to* (C1) *or* (C2).

*Proof.* We consider $G = \mathrm{SL}_n(q)$ as a normal subgroup of $\tilde{G} := \mathrm{GL}_n(q)$. As $\tilde{G}/G$ is cyclic, restriction of characters from $\tilde{G}$ to $G$ is multiplicity-free. Moreover, as pointed out in the proof of Theorem 5.12 all characters in $\ell$-element Lusztig series restrict irreducibly unless $\ell$ divides $\gcd(n, q-1)$.

Let $B$ be a unipotent $\ell$-block of $G$ with defect group $D$. Then there is a unipotent $\ell$-block $\tilde{B}$ of $\tilde{G}$ covering $B$, with defect group $\tilde{D} \geq D$, and by Proposition 5.11 both inequalities are satisfied for $\tilde{B}$.

First assume that $\ell$ does not divide $q-1$, thus $\tilde{D} = D \leq G$. As $\mathrm{Irr}(\tilde{B}) \subseteq \mathcal{E}_\ell(\tilde{G}, 1)$ the preceding discussion shows that all $\chi \in \mathrm{Irr}(\tilde{B})$ restrict irreducibly to $G$. The various characters of $\tilde{G}$ with the same restriction to $G$ lie in the Lusztig series of the $q-1$ central $\ell'$-elements of $\tilde{G}^* = \mathrm{GL}_n(q)$ and thus in pairwise distinct $\ell$-blocks of $\tilde{G}$. So we obtain $k(B) = k(\tilde{B})$, $k_0(B) = k_0(\tilde{B})$, and also $l(B) = l(\tilde{B})$ since by [12] the unipotent characters form a basic set for the unipotent blocks of $G$. In particular the conjecture holds for the block $B$ of $\mathrm{SL}_n(q)$. Furthermore, $|Z(G)|$ is a divisor of $q-1$ and thus prime to $\ell$ in this case, so all characters in $\mathrm{Irr}(B)$ have $Z(G)$ in their kernel, and the claim also follows for $H = G/Z$ for any $Z \leq Z(G)$.

So now assume that $\ell | (q-1)$. Then all unipotent characters of $G$ lie in the principal $\ell$-block (see e.g. [5, Thm.]), so $B$ has weight $w = n$ and any defect group $D$ of $B$ is a Sylow $\ell$-subgroup of $G$. Let us set $\ell^m := \gcd(n, q-1)_\ell = \min\{w_\ell, \ell^a\} = |Z(G)|_\ell$. Now note that $k(D) \geq k(\tilde{D})/\ell^a \geq \ell^{a(w-1)-w/(\ell-1)}$ and $k(\bar{D}) \geq k(D)/\ell^m \geq \ell^{a(w-1)-w/(\ell-1)-m}$, where $\bar{D}$ is the image of $D = \tilde{D} \cap \mathrm{SL}_n(q)$ in $\mathrm{PSL}_n(q)$. Furthermore, we have that $l(B) \geq l(\tilde{B}) = k(1, w)$ since the restrictions of unipotent characters of $\mathrm{SL}_n(q)$ to $\ell$-regular classes are linearly independent (or by Theorem 5.12). To estimate $k(B)$, first observe that for any $x \geq 1$

$$k(\ell, a, 1, x) = \sum_{\mathbf{w}} k(\ell^a, w_0) \prod_{i \geq 1} k(\ell^a - \ell^{a-1}, w_i) \leq \sum_{\mathbf{w}} \ell^{aw_0} \prod_{i \geq 1} \ell^{aw_i} \leq p_\ell(x) \ell^{ax}$$

by Lemma 5.5(a), where $\mathbf{w}$ runs over $\ell$-compositions of $x$. Thus from [21, Thm. 5.1] we find

$$k(B) \leq \Big( k(\ell, a, 1, w) + \sum_{j=1}^{m} p_\ell(w/\ell^j) \ell^{2j + aw/\ell^j} \Big) / \ell^a.$$

By [21, Lemma 5.2], $p_\ell(w/\ell^j) \leq \ell^{\binom{u+1-j}{2}}$ with $u = \lfloor \log_\ell w \rfloor$, so that with the estimates in Lemma 5.8 we obtain

$$k(B) \leq \Big( \ell^{aw - cw/\ln(\ell) + \binom{u+1}{2}} + \sum_{j=1}^{m} \ell^{2j + aw/\ell^j + \binom{u+1-j}{2}} \Big) / \ell^a$$

where $c = 0.9$ for $a > 1$ and $c = 0.57$ for $a = 1$. Now

$$2j + aw/\ell^j + \binom{u+1-j}{2} \leq aw - cw/\ln(\ell) + \binom{u+1}{2} - 2 - j$$

for $j \geq 1$ and all relevant $\ell, w$, unless $\ell^a = w \in \{5, 7\}$, so

$$k(B) \leq \ell^{a(w-1)-cw/\ln(\ell)+\binom{u+1}{2}}\big(1 + \sum_{j\geq 1} \ell^{-2-j}\big)$$
$$\leq \ell^{a(w-1)-cw/\ln(\ell)+\binom{u+1}{2}}\big(1 + \frac{1}{\ell^2(\ell-1)}\big) \leq \ell^{a(w-1)-cw/\ln(\ell)+\binom{u+1}{2}+d_\ell}$$

where $d_\ell := 1/(\ell^2(\ell-1)\ln(\ell))$, with the same exceptions. Now let $\bar{B}$ denote the principal $\ell$-block of $\mathrm{PSL}_n(q)$. Then we certainly have $k(\bar{B}) \leq k(B)$ and $l(\bar{B}) = l(B)$, and thus we find

$$k(\bar{D})l(\bar{B})/k(\bar{B}) \geq \ell^{cw/\ln(\ell)-w/(\ell-1)-m-d_\ell}$$

which is bigger than 1 unless $\ell^a = w \in \{5, 7\}$. (Observe that $m \leq a$ always.) In the excluded cases, by Example 5.14(b) we have $k(B) = (k(5,5) + 5^3 - 1)/5 = 126$, while $l(\bar{B}) = k(1,5) = 7$, $k(\bar{D}) \geq k(D)/5 = 149/5$ (respectively $k(B) = (k(7,7) + 7^3 - 1)/7 = 1821$, $l(\bar{B}) = k(1,7) = 15$, $k(\bar{D}) \geq k(\tilde{D})/49 = 117697/49$).

As for (C1), first we have by Proposition 5.15

$$k(\bar{D}') \geq k(D')/\ell^m \geq k(\tilde{D}')/\ell^{m+1} \geq \ell^{aw-w/(\ell-1)-\sum a_i(a+i-\ell/(\ell-1))-m-1}$$

and

$$k_0(\bar{B}) = k_0(B) \geq k_0(\tilde{B})/\ell^a \geq \ell^{\sum a_i(a+i-1)-a}$$

by Theorem 5.12, so

$$k(\bar{D}')k_0(\bar{B})/k(\bar{B}) \geq \ell^{cw/\ln(\ell)+\sum a_i/(\ell-1)-w/(\ell-1)-m-1-\binom{u+1}{2}-d_\ell}.$$

When $a > 1$ this is at least 1 unless $w = \ell \in \{5, 7\}$. Now for $w = \ell$ it is easily seen that $\bar{D}'$ is abelian of order $\ell^{a\ell-a-2}$, a factor of $\ell$ bigger than our general estimate, and the desired inequality follows.

Finally, when $a = 1$ then we conclude with the general estimate unless $\ell = w \leq 19$, or $\ell = 7$, $w \leq 14$, or $\ell = 5$, $w \leq 35$. The case $\ell = w > 7$ is handled as when $a > 1$, and this clearly also applies when $w = k\ell$ with $k \leq \ell - 1$, so we are left with $w = \ell = 7$ and $\ell = 5$, $w \in \{5, 25\}$. These three cases are easily checked directly.

Again, the proof for $\mathrm{SU}_n(q)$ is entirely similar, with $d$ now denoting the order of $-q$ modulo $\ell$. □

5.6. **Groups of classical type.** We start off by determining the number of height 0 characters in unipotent $\ell$-blocks of classical type groups. Let $q$ be a prime power and $\ell \neq 2$ an odd prime not dividing $q$. We write $d = d_\ell(q)$ for the order of $q$ modulo $\ell$ and set $d' := d/\gcd(d, 2)$. First let $G_n(q)$ be one of $\mathrm{Sp}_{2n}(q)$ or $\mathrm{SO}_{2n+1}(q)$ with $n \geq 2$, and $\mathbf{G}_n$ the corresponding simple algebraic group over $\overline{\mathbb{F}}_q$. The unipotent $\ell$-blocks of $G_n(q)$ are parametrised by $d$-cuspidal pairs in $G_n(q)$ (see [5, Thm.]), that is, by pairs $(\mathbf{L}, \lambda)$ (up to conjugation) where $\mathbf{L}$ is a $d$-split Levi subgroup of $\mathbf{G}_n$ with $L = \mathbf{L}^F \cong G_{n-wd'}(q) \times T_d^w$ for some $w \geq 0$, and $\lambda$ is a $d$-cuspidal unipotent character of $L$, and hence of $G_{n-wd'}(q)$. Here, $T_d$ denotes a torus $T_d \cong \mathrm{GL}_1(q^d)$ if $d$ is odd, and $T_d \cong \mathrm{GU}_1(q^{d'})$ where $d = 2d'$ if $d$ is even.

Now let $G_n^\epsilon(q) = \mathrm{SO}_{2n}^\epsilon(q)$, with $\epsilon \in \{\pm\}$, $n \geq 4$, be an even-dimensional orthogonal group. (Here, as customary, we write $\mathrm{SO}_{2n}$ for the connected component of the identity in the general orthogonal group $\mathrm{GO}_{2n}$.) The unipotent $\ell$-blocks of $G_n^\epsilon(q)$ are again parametrised by $d$-cuspidal pairs $(\mathbf{L},\lambda)$, where $L = G^{\delta}_{n-wd'}(q)\times T_d^w$, with $T_d$ as above, and $\delta=\epsilon$ if $d$ is odd or $w$ is even, and $\delta=-\epsilon$ else, and $\lambda$ is a $d$-cuspidal unipotent character of $L$. In either case we write $b(\mathbf{L},\lambda)$ for the corresponding block and call $w$ its *weight*. The number $k(b(\mathbf{L},\lambda))$ of characters was determined in [21, Prop. 5.4].

By [5, Thm. 4.4(ii)] the defect groups of $b(\mathbf{L},\lambda)$ are isomorphic to Sylow $\ell$-subgroups of $C_G([\mathbf{L},\mathbf{L}])$. For $G=\mathrm{Sp}_{2n}(q)$ we have $[\mathbf{L},\mathbf{L}]=\mathrm{Sp}_{2(n-wd')}$ has centraliser $\mathrm{Sp}_{2wd'}(q)$ in $G$; for $G=\mathrm{SO}_{2n+1}(q)$ we have $[\mathbf{L},\mathbf{L}]=\mathrm{SO}_{2(n-wd')+1}$ with centraliser $\mathrm{GO}^{\pm}_{2wd'}(q)$ in $G$ where the "+" sign occurs if and only if $d$ is odd; and for $G=\mathrm{SO}^{\epsilon}_{2n}(q)$, $C_G([\mathbf{L},\mathbf{L}])=\mathrm{GO}^{\epsilon\delta}_{2wd'}(q)$. Observe that by the parity condition on the sign $\epsilon\delta$, a Sylow $\ell$-subgroup of $\mathrm{GO}^{\epsilon\delta}_{2wd'}(q)$ is also a Sylow $\ell$-subgroup of $\mathrm{SO}_{2wd'+1}(q)$. In any case, a Sylow $\ell$-subgroup $P$ of $C_G([\mathbf{L},\mathbf{L}])$ is isomorphic to the wreath product $C_{\ell^a}\wr S$, with $\ell^a$ the precise power of $\ell$ dividing $q^d-1$ and $S$ a Sylow $\ell$-subgroup of the complex reflection group $G(2d',1,w)$. Now by its definition $d'$ is not divisible by $\ell$, and thus a Sylow $\ell$-subgroup of the wreath product $G(2d',1,w)\cong C_{2d'}\wr\mathfrak{S}_w$ is isomorphic to a Sylow $\ell$-subgroup of $\mathfrak{S}_w$. So $P$ is isomorphic to a Sylow $\ell$-subgroup of $\mathrm{GL}_{dw}(q)$ (if $d$ is odd) or $\mathrm{GU}_{d'w}(q)$ (if $d$ is even).

Let $\tilde B$ be a block of $\mathrm{GO}^{\epsilon}_{2n}(q)$ lying above the unipotent $\ell$-block $B=b(\mathbf{L},\lambda)$ of $\mathrm{SO}^{\epsilon}_{2n}(q)$. Then either $\tilde B$ lies above a unique unipotent block of $\mathrm{SO}^{\epsilon}_{2n}(q)$, in which case the tensor product of $\tilde B$ with the non-trivial linear character of $\mathrm{GO}^{\epsilon}_{2n}(q)$ is another block above $B$, or else the cuspidal pair $(\mathbf{L},\lambda)$ is such that $\lambda$ is labelled by a degenerate symbol and $\mathbf{L}=\mathbf{G}$, in which case $\tilde B$ lies above the two blocks parametrised by the two unipotent characters labelled by this degenerate symbol. In either case, the unipotent characters in $\tilde B$ are in bijection with the irreducible characters of $G(2d',1,w)$.

**Theorem 5.17.** *Let $G$ be one of $\mathrm{Sp}_{2n}(q)$ ($n\ge 2$), $\mathrm{SO}_{2n+1}(q)$ ($n\ge 3$), or $\mathrm{GO}^{\pm}_{2n}(q)$ ($n\ge 4$), let $\ell\ne 2$ be a prime not dividing $q$ and let $B$ be a unipotent $\ell$-block of $G$ of weight $w$. Let $d=d_\ell(q)$, $d':=d/\gcd(d,2)$ and write $\ell^a$ for the precise power of $\ell$ dividing $q^d-1$. Then*

$$k_0(B)=\prod_{i\ge 0} k\big((2d'+(\ell^a-1)/2d')\ell^i, a_i\big),$$

*where $w=\sum_{i\ge 0} a_i\ell^i$ is the $\ell$-adic decomposition of $w$, and*

$$l(B)=k(2d',w).$$

*Proof.* The characters of $G$ in a unipotent $\ell$-block $B=b(\mathbf{L},\lambda)$ are parametrised in [21, Prop. 5.4 and 5.5]: Let $\chi\in\mathrm{Irr}(B)$ and $t\in G^*$ an $\ell$-element such that $\chi\in\mathcal{E}(G,t)$. Then, up to conjugation we must have that $\mathbf{L}$ is a $d$-split Levi subgroup of the dual of $\mathbf{C}=C_{\mathbf{G}^*}(t)$, and the Jordan correspondent $\psi\in\mathcal{E}(C,1)$ of $\chi$ also lies in the unipotent block $B_C$ of $C$ parametrised by $(\mathbf{L},\lambda)$. For $\chi$ to be of height 0, it then follows from the degree formula for Jordan decomposition that a Sylow $\ell$-subgroup of $C^*$ has to be a Sylow $\ell$-subgroup of $C_G([\mathbf{L},\mathbf{L}])$, and furthermore, $\psi$ has to be of height 0 in $B_C$. This imposes exactly the same conditions on $t,\psi$ as in the proof of the corresponding result [24, Prop. 2.13] for blocks of $\mathrm{GL}_{2d'w}(q)$, and thus we can conclude as there.

The statement about $l(B)$ follows as the unipotent characters form a basic set for the unipotent blocks [12], and the unipotent characters in $B$ are in bijection with the irreducible characters of $G(2d',1,w)$ (see [2, 5]), of which there are precisely $k(2d',w)$. □

**Theorem 5.18.** *Let $H$ be quasi-simple of classical Lie type in characteristic $p$ and assume that $\ell \neq 2, p$. Then the unipotent $\ell$-blocks of $H$ are not counterexamples to* (C1) *or* (C2).

*Proof.* By Proposition 2.5 we need not concern ourselves with exceptional covering groups. The special linear and unitary groups were handled in Theorem 5.16. For the other groups of classical type the order of the centre of any non-exceptional cover $H$ is a 2-power, and all unipotent $\ell$-blocks have $Z(H)$ in their kernel, so we can restrict to the case when $H$ is simple.

Let us first consider the orthogonal groups $H = \mathrm{O}_{2n+1}(q)$ with $n \geq 2$. Let $d = d_\ell(q)$, and let $B = b(\mathbf{L}, \lambda)$ be the unipotent $\ell$-block of $G = \mathrm{SO}_{2n+1}(q)$ parametrised by the $d$-cuspidal pair $(\mathbf{L}, \lambda)$, with $\mathbf{L}$ of semisimple rank $n - wd'$, where $d' = d/\gcd(d, 2)$. As recalled above, the Sylow $\ell$-subgroups of $C_G([\mathbf{L}, \mathbf{L}])$ are defect groups of $B$. In our case, $C_G([\mathbf{L}, \mathbf{L}]) = \mathrm{GO}^{\pm}_{2wd'}(q)$, with the "+"-sign occurring when $d$ is odd. Now observe that Sylow $\ell$-subgroups of $\mathrm{GO}^{\pm}_{2wd'}(q)$ are contained in a subgroup $\mathrm{GL}_{wd'}(q)$ if $d$ is odd, and $\mathrm{GU}_{wd'}(q)$ if $d$ is even. They are hence isomorphic to a direct product $\prod_{i\geq 0} D_i^{a_i}$, where $\sum_{i\geq 0} a_i \ell^i$ is the $\ell$-adic decomposition of $w$ and $D_i = D_{i,\ell^a}$ with $\ell^a = (q^d - 1)_\ell$. Assume that $d$ is odd, so $d' = d$. Choose $q'$ a prime such that $q'$ has order $2d$ modulo $\ell$ (which is possible as $d$ and $\ell$ both are odd). Then the formulas in [21, Prop. 5.4] and in Theorem 5.17 show that the principal $\ell$-block $B'$ of $\mathrm{GL}_{2dw}(q')$ has the same invariants $k(B')$, $k_0(B')$, $l(B')$ and $D$ as the block $B$. Similarly, if $d = 2d'$ is even, then the principal $\ell$-block of $\mathrm{GL}_{dw}(q)$ has the same invariants as $B$. Thus both (C1) and (C2) for $B$ follow from the corresponding result for the general linear group in Proposition 5.11. The derived subgroup $H = [G, G]$ has index 2 in $G$, and by Lusztig's result [6, Thm. 15.11] characters in $\ell$-series restrict irreducibly to $H$, so $k(B') = k(B)$ for the unipotent $\ell$-block $B'$ of $H$ covered by $B$, and the other invariants do not change either. This gives the claim for $H = \mathrm{O}_{2n+1}(q)$.

The situation for $H = \mathrm{S}_{2n}(q)$ is entirely analogous. Finally assume that $G = \mathrm{SO}^{\pm}_{2n}(q)$ is an even-dimensional orthogonal group, with $n \geq 4$. Let $B$ be a unipotent $\ell$-block of $G$ of weight $w$, and $\tilde{B}$ a block of $\mathrm{GO}^{\pm}_{2n}(q)$ covering $B$. It is shown in [21, Cor. 5.7] that $k(B) \leq k(\tilde{B})$. Now according to our formulas for block invariants, the block $\tilde{B}$ has the same invariants as the principal block $B'$ of $\mathrm{GL}_{2d'w}(q')$, with $q'$ of multiplicative order $2d'$ modulo $\ell$. As $|\mathrm{GO}^{\pm}_{2n}(q) : \mathrm{SO}^{\pm}_{2n}(q)| = 2$ we certainly have $k_0(B) \geq k_0(\tilde{B})/2$ and $l(B) \geq l(\tilde{B})/2$. So (C1) and (C2) hold for $B$ if we can show that these inequalities hold for $B'$ with $2k(B')$ in place of $k(B')$. That is, we need to study the proof of Proposition 5.11, for the case when $d > 1$. For (C1) we have already obtained this better estimate in the proof. On the other hand, for (C2) this follows directly, as in fact $k(d, w) \geq k(2, w) \geq 2k(1, w) \geq 2p_\ell(w)$ for all $\ell, w$ when $d \geq 2$. Thus the claim holds for the unipotent $\ell$-blocks of $G$.

The restrictions of characters in $\ell$-element series to the derived subgroup $[G, G]$ are again irreducible, and all of them have the centre in their kernel, so we reach the desired conclusion for the simple group $[G/Z(G), G/Z(G)]$ as well. □

We have now shown the assertion of Theorem 1 for groups of classical Lie type:

**Corollary 5.19.** *Let $B$ be an $\ell$-block of a quasi-simple group of classical Lie type, with $\ell \geq 5$. Then $B$ is not a minimal counterexample to* (C1) *or* (C2).

*Proof.* By Corollary 4.3 there are no counterexamples when $\ell$ is the defining characteristic. By Lemma 5.1 and Proposition 5.3 the non-unipotent blocks do not give rise to minimal counterexamples of either (C1) or (C2). By Theorems 5.16 and 5.18 neither do the unipotent blocks. □

## 6. Groups of exceptional Lie type in non-defining characteristic

To complete the proof of Theorem 1 we need to deal with the blocks of exceptional groups of Lie type in non-defining characteristic $\ell$.

6.1. **Exceptional groups of small rank.** We first consider the five series of exceptional groups of small rank. The following result has been communicated to us by Frank Himstedt as a consequence of his investigation of blocks of the Steinberg triality groups [13, 14]:

**Proposition 6.1.** *Let $B$ denote the principal $\ell$-block of ${}^3D_4(q)$. Then:*
(a) $k(B) = 9 + 2^{a+1} + (4^{a-1} - 1)/3$ *for* $\ell = 2$,
(b) $k(B) = 7 + 3^{a+1} + (9^a - 1)/4$ *for* $\ell = 3$.

(We remark that these values are not given correctly in [8, p. 68].)

**Proposition 6.2.** *Let $G$ be one of ${}^2B_2(q^2)$, ${}^2G_2(q^2)$, ${}^2F_4(q^2)'$, $G_2(q)$ or ${}^3D_4(q)$. Then all $\ell$-blocks of $G$ satisfy inequalities* (C1) *and* (C2) *for all primes $\ell$.*

*Proof.* By Theorem 4.2 we may assume that $\ell$ does not divide $q^2$. By Theorem 2.1, we can furthermore discard the cases when Sylow $\ell$-subgroups of $G$ are abelian. Hence by [23, Thm. 25.14], the prime $\ell$ has to divide the order of the Weyl group of $\mathbf{G}$.

For the Suzuki groups ${}^2B_2(q^2)$ as well as for the Ree groups ${}^2G_2(q^2)$ all Sylow $\ell$-subgroups for non-defining primes $\ell$ are abelian. The group ${}^2F_4(2)'$ was already treated in Proposition 2.4. For the Ree groups $G = {}^2F_4(q^2)$, $q^2 = 2^{2f+1}$ with $f \geq 1$, by [20, Bem. 2], only the principal 3-block $B$ has non-abelian defect groups. Here, $k(B) = (3^{2a} + 36 \cdot 3^a + 555)/48$, where $3^a = (q^2 + 1)_3$, while a Sylow 3-subgroup $D$ of $G$ is an extension of a homocyclic group $C_{3^a} \times C_{3^a}$ with the cyclic group of order 3. In particular $k(D) \geq 3^{2a-1}$, and (C2) follows when $a > 1$. For $a = 1$ we have $k(B) = 14$, $k(D) = 11$ and $l(B) \geq 2$, so again we are done. It is easy to see from the character degrees given in [20] that furthermore $k_0(B) = 9$, and as we have $k(D') = 3^{2a-1}$ this also shows (C1).

The only relevant primes for the groups $G = G_2(q)$ are $\ell = 2, 3$. The blocks and their invariants have been determined by Hiss and Shamash [15, 16]. When $\ell = 2$, the principal 2-block has $k(B) = 9 + 2^{a+1} + (4^{a-1} - 1)/3$, $k_0(B) = 8$ and $l(B) = 7$, where $2^a$ is the precise power of 2 dividing $q - \epsilon$, with $\epsilon \in \{\pm 1\}$ such that $q \equiv \epsilon \pmod 4$. A Sylow 2-subgroup $D$ of $G$ is contained in the normaliser of a maximally split torus or of a Sylow 2-torus and thus is an extension of a homocyclic group $C_{2^a} \times C_{2^a}$ with a Klein four group. So $k(D) \geq 2^{2a-2}$. Inequality (C2) follows by using that necessarily $2^a \geq 4$. Moreover, we have $D'$ is abelian of order $2^{2a-1}$, so (C1) is also satisfied.

For $q \equiv \pm 1 \pmod{12}$ there is a further 2-block $B$ in the Lusztig series of an isolated element $s$ of order 3 whose defect groups are isomorphic to Sylow 2-subgroups of $C_G(s) \cong \mathrm{SL}_3(\epsilon q)$, where $q \equiv \epsilon \pmod 3$ (see [16]). Thus they are central products of a semidihedral group with a cyclic group and thus covered by Proposition 2.2. All other blocks have abelian or semidihedral defect groups.

When $\ell = 3$ only the principal 3-block $B$ of $G_2(q)$, $3 \not| q$, has non-abelian defect groups, with $k(B) = 8 + 2 \cdot 3^a + (3^a - 3)^2/12$, $k_0(B) = 9$ and $l(B) = 7$, where $3^a$ is the precise power of 3 dividing $q - \epsilon$, with $q \equiv \epsilon \pmod 3$. A Sylow 3-subgroup $D$ of $G_2(q)$ is contained in a subgroup $\mathrm{SL}_3(\epsilon q)$. Thus, it is an extension of a homocyclic group $C_{3^a} \times C_{3^a}$ with the cyclic group of order 3. So clearly $k(D) \geq 3^{2a-1}$ and the derived subgroup is abelian of index 9, so $k(D') = 3^{2a-1}$. The inequalities follow from this.

The blocks of groups ${}^3D_4(q)$ were determined by Deriziotis–Michler [8]. In particular, the defect groups of any 2-block of non-maximal defect are either abelian, semidihedral or semidihedral central product with abelian, so by Proposition 2.2 we need not consider these further. Any Sylow 2-subgroup $D$ of ${}^3D_4(q)$ is contained in a subgroup $G_2(q)$ and thus, as we saw above, $k(D) \geq 2^{2a-2}$ and $k(D') = 2^{2a-1}$, with $2^a = (q-\epsilon)_2$ where $q \equiv \epsilon \pmod 4$. Moreover, the invariants $k(B)$ (given in Proposition 6.1), $k_0(B) = 8$ and $l(B) = 7$ are the same as for the principal 2-block of $G_2(q)$. The inequalities are thus satisfied for $a \geq 2$ (which by the definition of $\epsilon$ is always the case here).

For $\ell = 3$ any block with non-abelian defect has the Sylow 3-subgroups as defect groups by [8, Prop. 5.4]. The only 3-block of maximal defect is in fact the principal block, with $k(B) = 7 + 3^{a+1} + (9^a - 1)/4$ (see Proposition 6.1), $k_0(B) = 9$ and $l(B) = 7$. Depending on the congruence of $q$ modulo 3, a Sylow 3-subgroup $D$ of $G$ is contained in the normaliser of a maximally split torus of $G$ or of its Ennola dual. Hence it is an extension of an abelian group $C_{3^a} \times C_{3^{a+1}}$ by the cyclic group of order 3, and $k(D) \geq 3^{2a}$. It contains a Sylow 3-subgroup of $G_2(q)$, from which we deduce that $k(D') \geq 3^{2a-1}$. The claim follows. □

6.2. **Exceptional groups of large rank.** To deal with the exceptional groups of large rank, we need some preparations. We continue to use the setup from Section 4 with $\mathbf{G}$ a simply connected simple algebraic group with Frobenius map $F$.

**Lemma 6.3.** *Let* $\mathbf{G}$ *be simple of one of the types given in Table 2. Then the number of unipotent characters* $|\mathcal{E}(\mathbf{L}^F, 1)|$ *for any proper* $F$*-stable Levi subgroup* $\mathbf{L} < \mathbf{G}$ *is bounded above as shown.*

TABLE 2. Numbers of unipotent characters and upper bounds

| $G$ | $A_1$ | $A_2$ | $B_2$ | $A_3$ | $B_3$ | $A_4$ | $B_4$ | $D_4$ | ${}^2D_4$ | $F_4$ |
|---|---|---|---|---|---|---|---|---|---|---|
| $\lvert\mathcal{E}(G,1)\rvert$ | 2 | 3 | 6 | 5 | 12 | 7 | 25 | 14 | 10 | 37 |
| $\lvert\mathcal{E}(L,1)\rvert \leq$ | 0 | 2 | 2 | 4 | 6 | 6 | 12 | 8 | 5 | 12 |

| $G$ | $A_5$ | ${}^{(2)}D_5$ | $A_6$ | $D_6$ | ${}^2D_6$ | ${}^{(2)}E_6$ | $A_7$ | ${}^{(2)}D_7$ | $E_7$ | $E_8$ |
|---|---|---|---|---|---|---|---|---|---|---|
| $\lvert\mathcal{E}(G,1)\rvert$ | 11 | 20 | 15 | 42 | 36 | 30 | 22 | 65 | 76 | 166 |
| $\lvert\mathcal{E}(L,1)\rvert \leq$ | 10 | 14 | 15 | 28 | 20 | 20 | 25 | 42 | 42 | 76 |

*Proof.* By Lusztig's results the number of unipotent characters of the $F$-fixed points of a connected reductive group $\mathbf{H}$ only depends on the root system of $\mathbf{H}$ and the action of $F$ on it (see e.g. [2]). Thus, inductively we are done if we can show that for any indecomposable parabolic subsystem $\Phi$ of the root system of $\mathbf{G}$, and for every maximal parabolic subsystem $\Psi$ of $\Phi$ the number of unipotent characters of any connected reductive group with root system $\Psi$ is smaller than the stated bound. Now the maximal parabolic subsystems of $\Phi$ are obtained by removing one node in the Dynkin diagram of $\Phi$. From this together with the list of numbers of unipotent characters reproduced in Table 2 it is now straightforward to conclude. As an example, when $\Phi$ has type $F_4$, the maximal parabolic subsystems are of types $B_3$, $A_2A_1$ and $C_3$, with $12, 6, 12$ unipotent characters respectively, consistent with the claimed bound. $\square$

**Proposition 6.4.** *Let $B$ be a unipotent $\ell$-block of a quasi-simple group of exceptional Lie type, with $\ell \geq 5$. Then $B$ is a minimal counterexample to neither* (C1) *nor* (C2).

*Proof.* Let $G$ be a quasi-simple group of exceptional Lie type. By Theorem 4.2 we may assume that $\ell$ is not the defining characteristic of $G$. By Proposition 6.2 we may also assume that $G$ is of type $F_4$, $E_6$, ${}^2E_6$, $E_7$ or $E_8$. Furthermore, by Proposition 2.5, $G$ is not an exceptional covering group, so we may further assume that $G$ is a central quotient of a group $\mathbf{G}^F$ as above.

If Sylow $\ell$-subgroups of $G$ are non-abelian, then $\ell$ divides the order of the Weyl group of $\mathbf{G}$ (see [23, Thm. 25.14]). In particular $\ell \leq 7$. Now it is easily seen that Sylow 7-subgroups can be non-abelian only when $G = E_7(q)$ or $E_8(q)$ and $7|(q^2-1)$, and Sylow 5-subgroups are only non-abelian when $G$ is of type $E$, and moreover $5|(q^2-1)$ when $G \neq E_8(q)$. The unipotent $\ell$-blocks have been classified in [5] for good primes, and for bad primes in [9]. It ensues that for $\ell \geq 5$ these are in bijection with $d$-Harish-Chandra series of unipotent characters of $G$, where $d$ is the order of $q$ modulo $\ell$. But then the explicit knowledge of these series [5, Thm. (ii)] shows that defect groups of non-principal unipotent $\ell$-blocks are always abelian in our cases.

Hence we only need to consider the principal $\ell$-block $B_0$. Here

$$\mathrm{Irr}(B_0) \subseteq \mathcal{E}_\ell(G,1) = \coprod_t \mathcal{E}(G,t)$$

where the union runs over $\ell$-elements $t \in G^*$ up to conjugation. In particular the union certainly has at most $k(D)$ terms, where $D$ is a Sylow $\ell$-subgroup of $G$. In order to prove our inequalities, we will do two things: first bound $|\mathcal{E}(G,t)|$ suitably, and second relate the number of conjugacy classes of $\ell$-elements in $G$ to $k(D)$.

For the first step, observe that by Lusztig's Jordan decomposition $\mathcal{E}(G,t)$ is in bijection with $\mathcal{E}(C_{G^*}(t),1)$, hence we need to control the number of unipotent characters of the centralisers of $\ell$-elements in $G$. The candidates for these centralisers can easily be enumerated by the algorithm of Borel–de Siebenthal from the extended Dynkin diagram [23, Thm. B.18] (note that in types $E_n$ all maximal subsystems are necessarily closed). Since $\ell$ is prime to $|Z(\mathbf{G})|$ all such centralisers are connected. Moreover, unless $G$ is of type $E_8$ and $\ell = 5$, the prime $\ell$ is good for $G$ and thus the centraliser of any non-trivial $\ell$-element of $G^*$ is even a Levi subgroup. Here, an upper bound for $|\mathcal{E}(C_{G^*}(t),1)|$ is given in Table 2. When $\ell = 5$ in $G = E_8(q)$, the only isolated centraliser of a 5-element is of type $A_4A_4$, which has at most 49 unipotent characters (depending on its rational type). Note that when $q^2 \equiv 4 \pmod 5$ then there is no 5-element in $E_8(q)$ with centraliser of type $E_7$, $D_6$ or $D_7$, so in that case the number of unipotent characters of any proper centraliser is bounded above by 36.

For the comparison of conjugacy classes in $G$ and in $D$ let's start with the case of $G = E_6(q)$ with $\ell = 5|(q-1)$. Then a Sylow 5-subgroup $D$ of $G$ is contained in the normaliser of a maximally split torus $T$ of $G$. If $1 \neq t \in D \cap T$ then $t$ is conjugate to at least 27 elements of $D$ (the index of the largest proper subgroup of the Weyl group $W$ of $G$), while in $D$ it is conjugate to at most 5 elements. So the number of conjugacy classes of such elements in $G$ is at most one fifth of their number in $D$. On the other hand, if $t \in D \setminus T$ then it centralises a Sylow 5-torus of $G$, hence its centraliser lies in a subgroup of type $A_1(q).(q^5-1)$. In particular $|\mathcal{E}(C_{G^*}(t),1)| \leq 2$. Taking together our results we find that $k(B_0) \leq ck(D)$ with $c = 30/5 = 6$.

Table 3. $l(B_0)$, $c$ and $k_0(B_0)$ in exceptional types

| $G$ | ${}^{\epsilon}E_6(q)$ | $E_7(q)$ | $E_7(q)$ | $E_8(q)$ | $E_8(q)$ | $E_8(q)$ |
|---|---|---|---|---|---|---|
| $\ell$ | $q \equiv \epsilon 1\,(5)$ | $q^2 \equiv 1\,(5)$ | $q^2 \equiv 1\,(7)$ | $q^2 \equiv 1\,(5)$ | $q^2 \equiv 4\,(5)$ | $q^2 \equiv 1\,(7)$ |
| $l(B_0)$ | 25 | 60 | 60 | 112 | 59 | 112 |
| $c$ | 6 | 1 | 15/4 | 25/4 | 3/4 | 38/17 |
| $k_0(B_0) \geq$ | $5+5^{a+1}$ | 14 | | 40 | 20 | |

With exactly the same arguments we obtain the constants $c$ listed in Table 3 in all cases except for $E_7(q)$ with $\ell = 5$. In the latter case we need to be a bit more careful. The centre of $D$ has order $5^{3a}$, all other elements in $D$ lie in orbits of length at least 2520 under the action of $W$, and their centraliser in $G$ is of type $A_3 + A_3 + A_1$ or smaller, so a corresponding Lusztig series has at most 50 elements. Thus we obtain $k(B) \leq 65 \cdot 5^{3a} + 50(5^{7a+1} - 5^{3a})/2520$. With $k(D) = 5^{2a}((5^{5a} - 5^a)/5 + 5^{a+1})$ this yields the bound $c = 1$.

The number $l(B_0)$ is given in Table 3. (When $\ell$ is good for $G$ then by [12] this is just $|\mathcal{E}(G,1) \cap \mathrm{Irr}(B_0)|$.) As visibly $c \leq l(B_0)$ in all cases we have shown (C2).

We now turn to (C1). Here it suffices to see that

$$c\,k(D) \leq k_0(B_0)\,k(D').$$

First consider $G = E_6(q)$, with $\ell = 5$ dividing $q-1$. A Sylow 5-subgroup $D$ of $G$ is contained in a Levi subgroup of type $A_4$ and thus is isomorphic to a Sylow 5-subgroup of $\mathrm{GL}_5(q).(q-1)$. Thus $k(D) = 5^a((5^{5a} - 5^a)/5 + 5^{a+1})$ and $k(D') = 5^{4a}$, where $5^a$ is the precise power of 5 dividing $q-1$ (see Lemma 5.10). Furthermore, there are 10 unipotent characters of height 0 in $B_0$. The centre of a Levi subgroup of $G^*$ of type $A_4$ contains an abelian subgroup $A$ of order $5^{2a}$ any element of which is $G^*$-conjugate to at most one further element of $A$. The corresponding Lusztig series each contain 5 characters in $B_0$ of height 0, so $k_0(B_0) \geq 10 + 5(5^{2a}-1)/2$. We obtain that $k_0(B_0)\,k(D')/k(D) \geq 25/2 > c = 6$. Exactly the same arguments apply for $G = {}^2E_6(q)$ with $5|(q+1)$.

For $G = E_7(q)$ with $\ell = 5$ dividing $q-1$ a Sylow 5-subgroup $D$ of $G$ is again contained in a Levi subgroup of type $A_4$, so is isomorphic to a Sylow 5-subgroup of $\mathrm{GL}_5(q).(q-1)^2$. Hence we have $k(D) = 5^{2a}((5^{5a} - 5^a)/5 + 5^{a+1})$ and $k(D') = 5^{4a}$. Besides the 30 unipotent characters of height 0 in $B_0$ there are at least $5(5^{3a}-1)/12$ further such characters in the Lusztig series of 5-elements in the centre of a Levi subgroup of $G^*$ of type $A_4$. Thus $k_0(B_0)\,k(D')/k(D) \geq 25/12 > c = 1$. The case when $5|(q+1)$ is completely analogous.

For $G = E_7(q)$ with $\ell = 7$ dividing $q-1$ a Sylow 7-subgroup $D$ of $G$ is contained in a Levi subgroup of type $A_6$, so isomorphic to a Sylow 7-subgroup of $\mathrm{GL}_7(q)$. Hence we have $k(D) = (7^{7a} - 7^a)/7 + 7^{a+1}$ and $k(D') = 7^{6a}$. Besides the 14 unipotent characters of height 0 in $B_0$ there are further $7(7^a - 1)/2$ such characters in the $(7^a - 1)/2$ Lusztig series of 7-elements in $G^*$ with centraliser containing $A_6(q)$. Again $k_0(B_0)\,k(D')/k(D) \geq 24 > c = 15/4$.

For $E_8(q)$ with $\ell = 5|(q-1)$, a Sylow 5-subgroup $D$ of $G$ is contained in a maximal rank subgroup of type $A_4(q)^2$, so it is a homocyclic group of order $5^{8a}$ extended by an elementary abelian group of order 25, where $5^a$ is the precise power of 5 dividing $q-1$. Here $k(D) = (5^{4a-1} + 24)^2$ and $k(D') = 5^{8a-2}$, and furthermore $k_0(B_0) = 40$. So we have $k_0(B_0)\,k(D')/k(D) > 28 > c$.

For $E_8(q)$ with $\ell = 5|(q^2+1)$, a Sylow 5-subgroup $D$ of $G$ is contained in a maximal rank subgroup ${}^2A_4(q^2)$, hence isomorphic to a Sylow 5-subgroup of $\mathrm{SU}_5(q^2)$. So we have $k(D) = 5^{4a-1} + 24$ and $k(D') = 5^{4a-1}$, where $5^a = (q^2+1)_5$. Since $k_0(B_0) = 20$ we get $k_0(B_0)\,k(D')/k(D) > 16 > c$.

For $G = E_8(q)$ with $\ell = 7$ dividing $q-1$ a Sylow 7-subgroup $D$ of $G$ is contained inside a Levi subgroup of type $A_6$, so isomorphic to a Sylow 7-subgroup of $\mathrm{GL}_7(q).(q-1)$. Hence we have $k(D) = 7^a((7^{7a} - 7^a)/7 + 7^{a+1})$ and $k(D') = 7^{6a}$. Besides the 28 unipotent characters of height 0 in $B_0$ there are 7 height 0 characters in at least $(7^{2a} - 1)/4$ further Lusztig series of 7-elements in $G^*$ in the centre of a Levi subgroup of type $A_6$. Thus $k_0(B_0)\,k(D')/k(D) > 12 > c$, which completes the proof. □

**Theorem 6.5.** *Let $B$ be an $\ell$-block of a quasi-simple group of exceptional Lie type, with $\ell \geq 5$. Then $B$ is not a minimal counterexample to* (C1) *or* (C2).

*Proof.* By Lemma 5.1 and Proposition 5.3 we need only to consider isolated $\ell$-blocks $B$. The unipotent blocks have been dealt with in Proposition 6.4. We are left with blocks labelled by isolated $\ell'$-elements $1 \neq s \in G^*$. By Proposition 5.3, $\ell$ is bad for $G$, so $G = E_8(q)$ and $\ell = 5$. The isolated 5-blocks with non-abelian defect were determined in [18, Prop. 6.10]: they are those corresponding to semisimple $5'$-elements $s$ with centraliser of type

$$A_8,\ A_7 + A_1,\ A_5 + A_2 + A_1,\ D_8,\ E_7 + A_1,\ D_5 + A_3,\ E_6 + A_2,$$

when $q \equiv 1 \pmod 5$, their Ennola duals when $q \equiv -1 \pmod 5$, while there are none when $q^2 \equiv -1 \pmod 5$. In the first three cases, the corresponding Broué–Michel union $\mathcal{E}_5(G, s)$ is a single 5-block $B$, and $l(B)$ agrees with the corresponding number in the principal 5-block of $C_{G^*}(s)^*$, so we are done by Lemma 5.4.

Next consider the case of centraliser $D_8(q)$. Here, $\mathcal{E}_5(G, s)$ is the union of two blocks, one with trivial Harish-Chandra source, the other above a cuspidal character of $D_4(q)$. The second block has abelian defect, and for the first we have

$$k(B) \leq |\mathcal{E}_5(G, s)| = k(5, a, 1, 8) + k(5, a, 1, 4) \leq 2k(5^a, 3) \cdot 5^{5a-2.7} + 5^{4a}$$

by Lemmas 5.8 and 5.5, while defect groups are of the form $C_{5^a}^3 \times C_{5^a} \wr C_5$, so

$$k(D) \geq 5^{8a-1}, \quad k(D') \geq 5^{4a}, \quad \text{and } k_0(B) \geq k(5^a, 3)k(5^{a+1}, 1) = 5^{a+1}k(5^a, 3),$$

from which we may conclude.

The remaining three cases are settled analogously. □

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FB Mathematik, TU Kaiserslautern, Postfach 3049, 67653 Kaiserslautern, Germany.
*E-mail address*: malle@mathematik.uni-kl.de