

# BRAUER'S HEIGHT ZERO CONJECTURE FOR PRINCIPAL BLOCKS

GUNTER MALLE AND GABRIEL NAVARRO

ABSTRACT. We prove *the other half* of Brauer's Height Zero Conjecture in the case of principal blocks.

## 1. INTRODUCTION

Richard Brauer's famous Height Zero Conjecture [2] from 1955 proposes that if  $p$  is a prime,  $G$  is a finite group, and  $B$  is a Brauer  $p$ -block of  $G$  with defect group  $D$ , then  $D$  is abelian if and only if all the irreducible complex characters of  $G$  in  $B$  have height zero. For abelian  $D$ , the proof was finally completed in [11]. The other direction was proven to hold assuming that all simple groups satisfy the *inductive Alperin–McKay condition* (see Navarro–Späth [19]). Prior to this, the conjecture was shown for  $p = 2$  and blocks of maximal defect by Navarro–Tiep [21].

The inductive Alperin–McKay condition (Späth [23]) is a natural but difficult-to-check statement on blocks of simple groups which involves action of automorphisms and cohomology. In this paper, we take a step back and conduct a direct reduction to almost simple groups of Brauer's conjecture for principal blocks. Then we handle these groups via the classification of finite simple groups, hence proving the conjecture in this case.

We denote by  $B_0(G)$  the principal  $p$ -block of  $G$ , and by  $\text{Irr}(B_0(G))$  the set of complex irreducible characters of  $G$  in  $B_0(G)$ .

**Theorem A.** *Let  $p$  be a prime and let  $G$  be a finite group. Then the Sylow  $p$ -subgroups of  $G$  are abelian if and only if  $p$  does not divide  $\chi(1)$  for all  $\chi \in \text{Irr}(B_0(G))$ .*

In Problem 12 of his famous list [3] from 1963 Brauer asked if the character table of a finite group  $G$  detects if  $G$  has abelian Sylow  $p$ -subgroups. Although this has been previously solved in [13] (theoretically) and [20] (with an algorithm), Theorem A gives the solution to Problem 12 that surely Brauer had in mind.

A crucial ingredient in our proof is the result by Kessar–Malle [12] that Brauer's Height Zero Conjecture holds for all quasi-simple groups.

At several points of the proof we do use special properties of principal blocks, such as the Alperin–Dade theory of isomorphic blocks, which do not hold true for general blocks.

---

*Date:* April 13, 2021.

*2010 Mathematics Subject Classification.* 20C20.

*Key words and phrases.* Brauer's Height Zero Conjecture, principal blocks.

The first author gratefully acknowledges financial support by SFB TRR 195. The research of the second author is supported by Ministerio de Ciencia e Innovación PID2019-103854GB-I00 and FEDER funds. We thank Yanjun Liu, Lizhong Wang, Wolfgang Willems and Jiping Zhang for e-mails prompting our interest in the subject of this paper, and Yanjun Liu for pertinent questions on an earlier version.

## 2. REDUCTION TO ALMOST SIMPLE GROUPS

**Theorem 2.1.** *Brauer's Height Zero Conjecture is true for principal  $p$ -blocks if it is true for the principal  $p$ -block of every almost simple group  $S$  such that  $S/\text{soc}(S)$  is a  $p$ -group.*

*Proof.* Let  $G$  be a finite group, and let  $P \in \text{Syl}_p(G)$ . We assume that  $p$  does not divide the degrees of the complex irreducible characters in  $B_0(G)$ . We argue by induction on  $|G|$  that  $P$  is abelian.

Let  $1 < N < G$  be a normal subgroup of  $G$ . Since  $\text{Irr}(B_0(G/N)) \subseteq \text{Irr}(B_0(G))$ , we have that  $G/N$  has abelian Sylow  $p$ -subgroups, by induction. If  $\theta \in \text{Irr}(B_0(N))$ , then there exists  $\chi \in \text{Irr}(B_0(G))$  such that  $\theta$  is an irreducible constituent of the restriction  $\chi_N$  (by [18, Thm. 9.4]). Since  $\theta(1)$  divides  $\chi(1)$ , we have that  $N$  has abelian Sylow  $p$ -subgroups, by induction. In particular, we have that  $\mathbf{O}^{p'}(G) = G$ . If  $G$  has two different minimal normal subgroups  $N_1$  and  $N_2$ , then  $G$  is isomorphic to a subgroup of  $G/N_1 \times G/N_2$  that has abelian Sylow  $p$ -subgroups, and therefore  $G$  has abelian Sylow  $p$ -subgroups. So we may assume that  $G$  has a unique minimal normal subgroup  $N$ .

Since  $\text{Irr}(B_0(G)) = \text{Irr}(B_0(G/\mathbf{O}_{p'}(G)))$  (by [18, Thm. 6.10]), we may assume by induction that  $\mathbf{O}_{p'}(G) = 1$ .

Recall that if a finite group  $H$  has abelian Sylow  $p$ -subgroups and  $\mathbf{O}_{p'}(H) = 1$ , then  $\mathbf{O}^{p'}(H) = \mathbf{O}^{pp'}(H) \times \mathbf{O}_p(H)$ , where  $\mathbf{O}^{pp'}(H) = \mathbf{O}^p(\mathbf{O}^{p'}(H))$  is either trivial or the direct product of non-abelian simple groups of order divisible by  $p$ , with abelian Sylow  $p$ -subgroups, and  $\mathbf{O}_p(H)$  is abelian. (See for instance, [22, Thm. 2.1].) Hence, if  $M$  is any proper normal subgroup of  $G$ , then  $\mathbf{O}^{p'}(M)$  is a direct product of  $\mathbf{O}_p(M)$  and  $\mathbf{O}^{p'p}(M)$ , and this latter is either trivial or a direct product of simple groups of order divisible by  $p$ . Also, again, if  $L/N = \mathbf{O}_{p'}(G/N)$ , we have that  $G/L = X/L \times Y/L$ , where  $X/L$  is a direct product of simple groups of order divisible by  $p$  (or trivial), and  $Y/L$  is a  $p$ -group.

Suppose first that  $N$  is a central  $p$ -group. Let  $\tau \in \text{Irr}(P)$ . Write  $\tau_N = e\lambda$ , where  $\lambda \in \text{Irr}(N)$ . By [18, Thm. 9.4], let  $\gamma_0 \in \text{Irr}(B_0(\mathbf{N}_G(P)))$  over  $\tau$ . By [18, Cor. 6.4] and Brauer's Third Main theorem, there is some  $\hat{\tau} \in \text{Irr}(B_0(G))$  over  $\gamma_0$ . Thus  $\hat{\tau}$  has  $p'$ -degree. Hence,  $\hat{\tau}_P$  contains some  $p'$ -degree character  $\gamma$ , and  $\gamma_N = \lambda \in \text{Irr}(N)$  by [9, Cor. 11.29]. Since  $P/N$  is abelian by induction, by Gallagher's Corollary 6.17 of [9] we have  $\tau = \gamma\rho$  for some linear  $\rho \in \text{Irr}(P/N)$ . Hence, all irreducible characters of  $P$  are linear, and we are done in this case.

Suppose next that  $N$  is a  $p$ -group. Let  $C = \mathbf{C}_G(N)$ . Then all characters in  $\text{Irr}(G/C)$  are in the principal  $p$ -block of  $G$  (by [21, Lemma 3.1] and Brauer's third main theorem). By the Itô–Michler theorem [17] and our hypothesis, we have that  $G/C$  has a normal Sylow  $p$ -subgroup. Since  $\mathbf{O}^{p'}(G) = G$ , we have that  $G/C$  is a  $p$ -group. By the previous paragraph, we may assume that  $C < G$ . Hence,  $\mathbf{O}^{p'}(C)$  is a direct product of  $\mathbf{O}^{p'p}(C)$  and  $\mathbf{O}_p(C)$ . These are normal subgroups of  $G$ , since they are characteristic in  $C$ . Since  $G$  has a unique minimal normal subgroup and  $\mathbf{O}_p(C) > 1$ , we conclude that  $\mathbf{O}^{p'}(C)$  is a  $p$ -group. In this case,  $G$  is  $p$ -solvable and  $\text{Irr}(B_0(G)) = \text{Irr}(G)$  (by [18, Thm. 10.20], for instance). Again by the Itô–Michler theorem, we conclude that  $G$  has a normal Sylow  $p$ -subgroup. Hence  $G = P$  as  $\mathbf{O}^{p'}(G) = G$ , and again we are done.

Thus we may assume that  $\mathbf{O}_p(G) = 1$ . Hence,  $N = \mathbf{F}^*(G)$  is a direct product of non-abelian simple groups of order divisible by  $p$ . Also, if  $M$  is any proper normal subgroup of  $G$ , then  $\mathbf{O}^{p'}(M)$  is a direct product of non-abelian simple groups of order divisible by  $p$ .

Let  $Q = P \cap N \in \text{Syl}_p(N)$ . We have that the irreducible characters of  $\text{Irr}(G/N\mathbf{C}_G(Q))$  are in the principal block (again by [21, Lemma 3.1] and Brauer's third main theorem), so  $G/N\mathbf{C}_G(Q)$  is a  $p$ -group by the Itô–Michler theorem, and using that  $\mathbf{O}^{p'}(G) = G$ . Then  $L \subseteq N\mathbf{C}_G(Q)$ . Hence,  $L = N\mathbf{C}_L(Q)$ .

Since  $X$  is not a  $p'$ -group (because  $\mathbf{O}_{p'}(G) = 1$ ), we have that  $N \subseteq \mathbf{O}^{p'}(X)$ . Assume that  $X < G$ . Since  $\mathbf{O}^{p'}(X)$  is a direct product of non-abelian simple groups and  $N \trianglelefteq \mathbf{O}^{p'}(X)$ , we have  $\mathbf{O}^{p'}(X) = N \times U$ , by elementary group theory. But then  $U = 1$ , because  $N = \mathbf{F}^*(G)$  and  $\mathbf{C}_G(N) \subseteq N$ . Thus  $\mathbf{O}^{p'}(X) = N$ . Since  $X/L$  is a direct product of non-abelian simple groups of order divisible by  $p$  (or trivial) and  $N \subseteq L \subseteq X$ , we deduce that  $X = L$  and therefore  $Y = G$ . Thus  $G/N$  has a normal  $p$ -complement  $L/N$ . We claim that  $PN$  satisfies the hypotheses. Let  $\tau \in \text{Irr}(B_0(PN))$ . Let  $\gamma \in \text{Irr}(N)$  be under  $\tau$ . Then  $\gamma$  lies in the principal block of  $N$ . Since  $\tau$  lies under some character in the principal block of  $G$ , we have that  $\gamma$  has  $p'$ -degree using the hypothesis. By the Alperin–Dade theory on isomorphic blocks ([1, 6]),  $\gamma$  has a canonical extension  $\hat{\gamma} \in \text{Irr}(L)$  in the principal block of  $L$ . Now,  $\hat{\gamma}$  lies under some  $\chi \in \text{Irr}(B_0(G))$ . By hypothesis,  $\chi$  has  $p'$ -degree, and therefore  $\chi_L = \hat{\gamma}$  (by [9, Cor. 11.29]), and  $\hat{\gamma}$  is  $G$ -invariant. In particular,  $\gamma$  is  $G$ -invariant too. By the Isaacs character correspondence ([8, Cor. 4.2]) there is a bijection

$$\text{Irr}(G|\hat{\gamma}) \rightarrow \text{Irr}(PN|\gamma),$$

given by restriction. Since  $\tau$  lies over  $\gamma$ , let  $\hat{\tau} \in \text{Irr}(G)$  over  $\hat{\gamma}$  that restricts to  $\tau$ . Since  $G/L$  is a  $p$ -group and  $\hat{\tau}$  lies over  $\hat{\gamma}$ , we have that  $\hat{\tau}$  is in the principal  $p$ -block of  $G$  ([18, Cor. 9.6]), thus  $\tau$  has  $p'$ -degree. So, by induction, we may assume that  $G = NP$ . Let  $S$  be normal simple in  $N$  and write  $N = S^{x_1} \times \cdots \times S^{x_t}$ , where  $P = \mathbf{N}_P(S)x_1 \cup \cdots \cup \mathbf{N}_P(S)x_t$ . Now, let  $1 \neq \tau \in \text{Irr}(B_0(S))$ , and consider  $\theta = \tau \times 1 \times \cdots \times 1 \in \text{Irr}(B_0(N))$ . Let  $\chi \in \text{Irr}(B_0(G))$  over  $\tau$ . Then  $\chi$  has  $p'$ -degree, so  $\chi_N = \theta$ . Then  $\theta$  is invariant. But this is impossible unless  $t = 1$ . Thus we obtain that  $G = NP$  is almost simple, with  $\text{soc}(G) = N = S$ , and  $G/N$  a  $p$ -group.

We are left with the case that  $X = G$ . Hence  $G/L$  is a direct product of non-abelian simple groups of order divisible by  $p$ . Since  $L \subseteq N\mathbf{C}_G(Q)$  and  $G/N\mathbf{C}_G(Q)$  is a  $p$ -group, we have that  $G = N\mathbf{C}_G(Q)$ . Write  $N = S^{z_1} \times \cdots \times S^{z_u}$ , where  $S$  is simple non-abelian, and  $\{S^{z_1}, \dots, S^{z_u}\}$  are the different  $G$ -conjugates of  $S$ . Now,  $Q \cap S \in \text{Syl}_p(S)$ . If  $1 \neq x \in Q \cap S$  and  $c \in \mathbf{C}_G(Q)$ , then  $x^c = x \in S$ . In particular,  $c \in \mathbf{N}_G(S)$ , and therefore  $S \trianglelefteq G$ . Hence  $S = N$ . By the Schreier theorem, we have that  $G/N$  is solvable, but this is not possible.  $\square$

As pointed out by the referee, the proof of Theorem 2.1 could be shortened by using the theory of the so called  $p^*$ -groups. Indeed, if  $M = \mathbf{O}_{p^*}(G)$  is the Bender subgroup of  $G$  and  $e_0(G)$  is the principal block idempotent of  $B_0(G)$ , we have that  $e_0(G) = e_0(M)$  (by using [26, Thm. 2.1 and Lemma 1.2]). This implies that  $B_0(G)$  is the only block of  $G$  covering  $B_0(M)$ , and therefore  $G/M$  has a normal Sylow  $p$ -subgroup, by the Itô–Michler theorem. By using the second paragraph in the previous proof,  $G/M$  is then a  $p$ -group. Since it is known that  $[\mathbf{O}_p(G), M] = 1$ , we conclude that  $\mathbf{O}_p(G) \subseteq \mathbf{Z}(G)$ . Using the previous proof, we can assume that  $\mathbf{O}_p(G) = 1$ . (This avoids dealing with the case where  $\mathbf{O}_p(G)$  is non-central.) Now,  $M = N\mathbf{O}_{p'}(\mathbf{C}_G(Q))$ , where  $Q$ , and  $N$  are as in the previous proof. Using the next to the last paragraphs in the proof, one concludes Theorem 2.1 (therefore avoiding the final paragraph).

## 3. ALMOST SIMPLE GROUPS

We now deal with almost simple groups with  $p$ -automorphisms. The first observation follows from the classification of finite simple groups. We refer the reader to [7, §2.5] for a description of the outer automorphism groups of the groups in question.

**Lemma 3.1.** *Let  $S$  be non-abelian simple with abelian Sylow  $p$ -subgroups and suppose that  $S$  has a non-trivial outer automorphism  $\sigma$  of  $p$ -power order. Then  $S$  is of Lie type and either  $\sigma$  is a field automorphism and  $p$  does not divide the order of the group of outer diagonal automorphisms of  $S$ , or one of the following holds:*

- (1)  $p = 2$  and  $S = \mathrm{PSL}_2(q)$  with  $q \equiv 3, 5 \pmod{8}$ ;
- (2)  $p = 3$  and  $S = \mathrm{PSL}_3(q)$  with  $q \equiv 4, 7 \pmod{9}$ ; or
- (3)  $p = 3$  and  $S = \mathrm{PSU}_3(q)$  with  $q \equiv 2, 5 \pmod{9}$ .

*Proof.* If  $S$  is sporadic or alternating, then  $\mathrm{Out}(S)$  has 2-power order, so  $p = 2$ . The only such  $S$  with abelian Sylow 2-subgroups is  $J_1$ , but  $\mathrm{Out}(J_1) = 1$ .

Thus, by the classification of finite simple groups,  $S$  is of Lie type. Here, if  $p = 2$ , then by the well-known classification by Walter [25],  $S$  is one of  $\mathrm{SL}_2(2^f)$  with  $f \geq 2$ ,  $\mathrm{PSL}_2(q)$  with  $q \equiv 3, 5 \pmod{8}$ , or  ${}^2G_2(3^{2f+1})$  with  $f \geq 1$ . The groups not occurring in the conclusion do not have outer automorphisms that are not field automorphisms. Now assume that  $p > 2$ . Here  $\sigma$  is a product of diagonal, graph and field automorphisms. Non-trivial graph automorphisms of order  $p \geq 3$  only exist for groups of type  $D_4(q)$ , but here Sylow 3-subgroups are non-abelian (since, for example,  $\mathrm{PSL}_4(q)$  is a section). Non-trivial outer diagonal automorphisms of order  $p \geq 3$  only exist for groups of types  $A_n$  and  $E_6$ . For  $S = E_6(q)$  or  ${}^2E_6(q)$  these have order 3, but Sylow 3-subgroups of  $S$  are non-abelian. For  $S = \mathrm{PSL}_n(q)$  or  $\mathrm{PSU}_n(q)$  the outer diagonal automorphisms have order dividing  $n$ , but Sylow  $p$ -subgroups of  $S$  for  $3 \leq p|n$  are non-abelian, unless  $n = p = 3$ . The latter cases appear in our conclusion.  $\square$

We are thus reduced to considering groups of Lie type. For these, we require some notions and results from Deligne–Lusztig theory. Let  $\mathbf{G}$  be a simple algebraic group of adjoint type and  $F : \mathbf{G} \rightarrow \mathbf{G}$  a Steinberg endomorphism, with finite group of fixed points  $G = \mathbf{G}^F$ . Let  $(\mathbf{G}^*, F)$  be in duality with  $\mathbf{G}$ . Any field automorphism  $\sigma$  of  $G$  is induced by a Frobenius map  $F_1 : \mathbf{G} \rightarrow \mathbf{G}$  commuting with  $F$  and the corresponding Frobenius map  $F_1 : \mathbf{G}^* \rightarrow \mathbf{G}^*$  induces a field automorphism  $\sigma^*$  of  $G^* = \mathbf{G}^{*F}$  (see [24, §5]).

**Lemma 3.2.** *Let  $G$  be as above of adjoint Lie type with abelian Sylow  $p$ -subgroups for a non-defining prime  $p$ , and  $\sigma$  a field automorphism of  $G$  of  $p$ -power such that  $G\langle\sigma\rangle$  has non-abelian Sylow  $p$ -subgroups. Then  $\sigma^*$  does not fix all classes of  $p$ -elements in  $G^*$ .*

*Proof.* First note that our assumptions on  $G$  force  $p > 2$ . Let  $F_1 : \mathbf{G} \rightarrow \mathbf{G}$  be a Frobenius map inducing  $\sigma$  and commuting with  $F$ . Since a Sylow  $p$ -subgroup  $P$  of  $G$  is abelian, it is contained in an  $F$ -stable maximal torus  $\mathbf{T}$  of  $\mathbf{G}$  (see [16, Thm. 25.14]), which we may assume to be  $F_1$ -stable. In fact, if  $F$  is not very twisted,  $\mathbf{T}$  contains a Sylow  $d$ -torus  $\mathbf{T}_d$  of  $\mathbf{G}$ , where  $d$  is the order of  $q$  modulo  $p$  and  $q$  is the underlying field size of  $G$ , while in the case of Suzuki and Ree groups, it contains a Sylow  $\Phi$ -torus which again we denote  $\mathbf{T}_d$ , where  $\Phi$  is the minimal polynomial over  $\mathbb{Q}(\sqrt{2})$  respectively  $\mathbb{Q}(\sqrt{3})$  of a primitive  $d$ th root of unity (see [4, 3F]). Since  $G$  and  $G^*$ , as well as  $\mathbf{T}^F$  and  $\mathbf{T}^{*F}$ , have the same order,

$\mathbf{T}^{*F}$  contains a Sylow  $p$ -subgroup of  $G^*$ , which is hence also abelian. Let  $\sigma^*$  be the dual automorphism of  $G^*$  induced by  $F_1 : \mathbf{G}^* \rightarrow \mathbf{G}^*$  which we may and will assume to stabilise  $\mathbf{T}^*$ . Let  $P^*$  be the Sylow  $p$ -subgroup of  $G^*$  contained in  $\mathbf{T}^{*F}$ .

By assumption,  $\sigma$  acts non-trivially on  $P$ . Thus the centraliser  $G_1 := C_G(\sigma)$  of  $\sigma$ , a subfield subgroup of  $G$ , does not contain a Sylow  $p$ -subgroup of  $G$ . Since  $\sigma^*$  is dual to  $\sigma$ ,  $G_1^* := C_{G^*}(\sigma^*)$  has the same order as  $G_1$ , so  $\sigma^*$  does also not centralise  $P^*$ . Now  $G^*$ -fusion in  $P^*$  is controlled by the relative Weyl group  $W = N_{G^*}(\mathbf{T}_d^*)/C_{G^*}(\mathbf{T}_d^*)$  of the Sylow torus  $\mathbf{T}_d^*$ , by [14, Prop. 5.11]. Let  $q_1$  be the underlying field size of  $\mathbf{G}^{F_1}$ , so  $q = q_1^{p^a}$  for some  $a \geq 1$ . Since  $q$  and  $q_1$  have the same order  $d$  modulo  $p$ , the relative Weyl groups  $W$  in  $G^*$  and  $W_1 = N_{G_1^*}(\mathbf{T}_d^*)/C_{G_1^*}(\mathbf{T}_d^*)$  in  $G_1^*$  agree, that is,  $\sigma^*$  commutes with the action of  $W$  on  $P^*$ . Now  $W$  has order prime to  $p$  as  $\mathbf{T}_d^{*F}$  contains a Sylow  $p$ -subgroup of  $\mathbf{G}^{*F}$ , while  $\sigma^*$  has some non-trivial orbit, of length divisible by  $p$ , on  $P^*$ . So  $\sigma^*$  induces additional fusion on elements of  $P^*$  and our claim follows.  $\square$

**Theorem 3.3.** *Let  $p$  be a prime,  $S$  be a non-abelian simple group and  $S \leq A \leq \text{Aut}(S)$  such that  $A/S$  is a  $p$ -group. Assume that all characters in the principal  $p$ -block of  $A$  have degree prime to  $p$ . Then the Sylow  $p$ -subgroups of  $A$  are abelian.*

*Proof.* We will assume that  $A$  has non-abelian Sylow  $p$ -subgroups and argue by contradiction. The case  $A = S$  is the main result of [12], so we may assume that  $A \neq S$ ,  $S$  has abelian Sylow  $p$ -subgroups, and we have to exhibit a character in the principal  $p$ -block of  $S$  that is not  $A$ -invariant.

By Lemma 3.1 we may assume that  $S$  is of Lie type. We first discuss the groups showing up in Lemma 3.1(1)–(3). For  $S = \text{PSL}_2(q)$  the outer automorphism group is generated by diagonal and field automorphisms. As  $q \not\equiv 1 \pmod{8}$ ,  $S$  does not have even order field automorphisms. Now the principal 2-block of  $S$  contains two irreducible characters of degree  $(q+\epsilon)/2$ , where  $q \equiv \epsilon \pmod{4}$ ,  $\epsilon \in \{\pm 1\}$ , that are not invariant under the diagonal automorphism (see e.g. [10, Lemma 15.1]). For  $\text{PSL}_3(q)$  with  $3 \mid (q-1)$ , the diagonal automorphism of order 3 moves the three characters of degree  $(q+1)(q^2+q+1)/3$ , and similarly for  $\text{PSU}_3(q)$  with  $3 \mid (q+1)$  the three characters of degree  $(q-1)(q^2-q+1)/3$ , (see e.g. [15, Lemmas 3.5, 3.11]). For the given congruences,  $S$  does not have field automorphisms of 3-power order. Thus, we may now assume that  $S$  is not one of the exceptions in Lemma 3.1.

If  $p$  is the defining characteristic of  $S$ , then as  $S$  has abelian Sylow  $p$ -subgroups, we have  $S = \text{PSL}_2(q)$  with  $q = p^f > p$ . Here, all irreducible characters apart from the Steinberg character lie in the principal block. Let  $\chi$  be a Deligne–Lusztig character of degree  $q-1$  corresponding to a semisimple element  $s$  generating the non-split maximal torus  $T$  of order  $(q+1)/(2, q-1)$  in the dual group. A field automorphism  $\sigma$  of  $S$  centralises the corresponding group over a subfield; since no proper subfield group contains a cyclic subgroup of order  $(q+1)/(2, q-1)$ ,  $\sigma$  must act non-trivially on  $T$ . Thus,  $\chi$  is not  $\sigma$ -stable.

So  $p$  is not the defining characteristic of  $S$ . Let  $\mathbf{G}$  and  $G = \mathbf{G}^F$  be as above such that  $S = [G, G]$ . This is possible unless  $S$  is the Tits simple group, which has no non-trivial field automorphisms. Any field automorphism of  $S$  extends to  $G$ . Now note that Sylow  $p$ -subgroups of  $G$  are also abelian, since  $G$  acts by diagonal automorphisms on  $S$  and  $p$  does not divide their order. Then by Lemma 3.2 there is some conjugacy class of a  $p$ -element  $s \in G^*$  not fixed by  $\sigma^*$ . By [24, Prop. 7.2] this implies that the corresponding

Lusztig series  $\mathcal{E}(G, s)$  is not fixed by  $\sigma$ . In particular the semisimple characters in  $\mathcal{E}(G, s)$ , which lie in the principal  $p$ -block by [5, Thm.] are not fixed by  $\sigma$ . Since  $p$  does not divide the order of the group of outer diagonal automorphisms of  $G$ , the number of characters of  $S$  below  $G$  is not divisible by  $p$ , hence there is some character of  $S$  moved by  $\sigma$ , and we conclude.  $\square$

Our main theorem now follows by combining Theorems 2.1 and 3.3.

#### REFERENCES

- [1] J. L. ALPERIN, Isomorphic blocks. *J. Algebra* **43** (1976), 694–698.
- [2] R. BRAUER, Number theoretical investigations on groups of finite order. *Proceedings of the International Symposium on Algebraic Number Theory* (Tokyo and Nikko, 1955), pp. 55–62. Science Council of Japan, Tokyo, 1956.
- [3] R. BRAUER, Representations of finite groups. *Lectures on Modern Mathematics, vol. I*, pp. 133–175. John Wiley & Sons, 1963.
- [4] M. BROUÉ, G. MALLE, Théorèmes de Sylow génériques pour les groupes réductifs sur les corps finis. *Math. Ann.* **292** (1992), 241–262.
- [5] M. CABANES AND M. ENGUEHARD, On unipotent blocks and their ordinary characters. *Invent. Math.* **117** (1994), 149–164.
- [6] E. C. DADE, Remarks on isomorphic blocks. *J. Algebra* **45** (1977), 254–258.
- [7] D. GORENSTEIN, AND R. LYONS, AND R. SOLOMON, *The classification of the finite simple groups*. American Mathematical Society, 40, Number 3. Providence, RI, 1998.
- [8] I. M. ISAACS, Characters of  $\pi$ -separable groups. *J. Algebra* **86** (1984), 98–128.
- [9] I. M. ISAACS, *Character Theory of Finite Groups*. AMS Chelsea Publishing, Providence, RI, 2006.
- [10] M. ISAACS, G. MALLE, G. NAVARRO, A reduction theorem for the McKay conjecture. *Invent. Math.* **170** (2007), 33–101.
- [11] R. KESSAR, G. MALLE, Quasi-isolated blocks and Brauer’s height zero conjecture. *Ann. of Math. (2)* **178** (2013), 321–384.
- [12] R. KESSAR, G. MALLE, Brauer’s height zero conjecture for quasi-simple groups. *J. Algebra* **475** (2017), 43–60.
- [13] W. KIMMERLE, R. SANDLING, Group theoretic determination of certain Sylow and Hall subgroups and the resolution of a question of R. Brauer. *J. Algebra* **171** (1995), 329–346.
- [14] G. MALLE, Height 0 characters of finite groups of Lie type. *Represent. Theory* **11** (2007), 192–220.
- [15] G. MALLE, Extensions of unipotent characters and the inductive McKay condition. *J. Algebra* **320** (2008), 2963–2980.
- [16] G. MALLE, D. TESTERMAN, *Linear Algebraic Groups and Finite Groups of Lie Type*. Cambridge University Press, Cambridge, 2011.
- [17] G. MICHLER, A finite simple group of Lie type has  $p$ -blocks with different defects if  $p \neq 2$ . *J. Algebra* **104** (1986), 220–230.
- [18] G. NAVARRO, *Characters and Blocks of Finite Groups*. Cambridge University Press, Cambridge, 1998.
- [19] G. NAVARRO, B. SPÄTH, On Brauer’s height zero conjecture. *J. Eur. Math. Soc.* **16** (2014), 695–747.
- [20] G. NAVARRO, R. SOLOMON, P. H. TIEP, Abelian Sylow subgroups of finite groups, II. *J. Algebra* **421** (2015), 3–11.
- [21] G. NAVARRO, P. H. TIEP, Brauer’s height zero conjecture for the 2-blocks of maximal defect. *J. reine angew. Math.* **669** (2012), 225–247.
- [22] G. NAVARRO, P. H. TIEP, Characters of relative  $p'$ -degree with respect to a normal subgroup. *Ann. of Math. (2)* **178** (2013), 1135–1171.
- [23] B. SPÄTH, A reduction theorem for the Alperin–McKay conjecture. *J. reine angew. Math.* **680** (2013), 153–189.

- [24] J. TAYLOR, Action of automorphisms on irreducible characters of symplectic groups. *J. Algebra* **505** (2018), 211–246.
- [25] J. H. WALTER, The characterization of finite groups with abelian Sylow 2-subgroups. *Ann. of Math.* (2) **89** (1969), 405–514.
- [26] W. WILLEMS,  $p^*$ -Theory and modular representation theory. *J. Algebra* **104** (1986), 135–140.

FB MATHEMATIK, TU KAISERSLAUTERN, POSTFACH 3049, 67653 KAISERSLAUTERN, GERMANY.  
*Email address:* malle@mathematik.uni-kl.de

DEPARTAMENT DE MATEMÀTIQUES, UNIVERSITAT DE VALÈNCIA, 46100 BURJASSOT, VALÈNCIA, SPAIN.

*Email address:* gabriel@uv.es