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Products and commutators of classes in algebraic groups

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Dedicated to the memory of Tonny Springer

Abstract. We classify pairs of conjugacy classes in almost simple algebraic groups whose product consists of finitely many classes. This leads to several interesting families of examples which are related to a generalization of the Baer–Suzuki theorem for finite groups. We also answer a question of Pavel Shumyatsky on commutators of pairs of conjugacy classes in simple algebraic groups. It turns out that the resulting examples are exactly those for which the product also consists of only finitely many classes.

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1. Introduction

The Baer–Suzuki theorem asserts that in a finite group G, if $x \in G$ is such that $\langle x, x^g \rangle$ is a p-group for all $g \in G$, then $\langle x^G \rangle$ is a normal p-subgroup of G.

We were recently informed by Bernd Fischer that Reinhold Baer had asked what one can say if, given $x, y \in G$, we have that $\langle x, y^g \rangle$ is a p-group for all $g \in G$. Examples in Guralnick–Malle–Tiep [7] show that there is not too much to say in general. However, with some extra hypothesis [7, 6], there are generalizations along these lines.

Here we consider the corresponding question for almost simple algebraic groups. Our main result is the following characterization, part of which is an analogue of the corresponding result in the connected case [7, Cor. 5.14]:

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Theorem 1. *Let G be an almost simple algebraic group over an algebraically closed field, and let C, D be G°-classes. The following are equivalent:*

(i) *CD is a finite union of G°-conjugacy classes.*
(ii) *$C_{G^\circ}(x_1)\backslash G^\circ/C_{G^\circ}(x_2)$ is finite for all $(x_1, x_2) \in C \times D$.*
(iii) *$\langle x_1, x_2\rangle$ normalizes some Borel subgroup of G° for every $(x_1, x_2) \in C \times D$.*

Moreover, any of the above conditions implies:

(iv) *$[C, D] := \{[x, y] \mid (x, y) \in C \times D\}$ is a finite union of G°-conjugacy classes.*

Here, an *almost simple algebraic group* is a possibly disconnected linear algebraic group whose connected component G° is simple and such that $C_G(G^\circ) = Z(G^\circ)$. See Theorem 11 for further equivalent conditions. Note that the result holds for G-classes as well as long as we avoid the case $G = D_4.\mathfrak{S}_3$ (because in the disconnected cases one of the classes is outer and so the centralizer intersects all cosets except in the excluded case).

While some implications in this statement have general proofs, others rely on our explicit classification of all pairs of classes with the stated properties.

Thus, we first classify in Theorem 7 all pairs C, D of unipotent classes in (disconnected) almost simple algebraic groups such that every pair in $C \times D$ generates a unipotent group (the connected case was done in [7]). In fact, it turns out to be equivalent to the condition that CD consists of unipotent elements.

We also extend the result of [7, Thm. 1.1] to the disconnected case by classifying pairs C, D of conjugacy classes in (disconnected) almost simple groups such that CD is a finite union of conjugacy classes.

It should be noted that for connected groups examples occur if and only if the group is of non-simply laced type, while for disconnected groups examples exist if and (clearly) only if the type is simply laced. See Proposition 2 for some relation between these two types of examples. Let's point out the following consequence of our classification: whenever CD is a finite union of conjugacy classes, then at least one of C, D contains a quasi-central class in its closure (recall that a quasi-central class is the smallest conjugacy class in a given coset of G°). It would be nice to have an a priori proof of this.

Two corollaries of the results and proofs are (extending the results of [7] to the disconnected case):

Corollary 1. *Let G be an almost simple algebraic group over an algebraically closed field. Suppose that C and D are conjugacy classes of G.*

Then either CD is the union of at most 3 *conjugacy classes or CD contains infinitely many conjugacy classes.*

In the previous result, if we replace G-classes by G°-classes, then 5 classes suffice.

Corollary 2. *Let G be an almost simple algebraic group over an algebraically closed field k. If $x \in G \backslash Z(G^\circ)$, then* $\dim(C_G(x)yC_G(x)) < \dim G$ *for all $y \in G$, unless $G = D_4(k).\mathfrak{S}_3$ and x is an outer involution with $C_G(x) = B_3(k)$ and y is not in $\langle G^\circ, x\rangle$.*

Note that in the excluded case, we really get counter-examples. This last corollary in the connected case was used by Prasad [12] in his study of quasi-reductive groups.

We also answer in the case of algebraic groups a question that Pavel Shumyatsky had asked in the finite case (the result is much stronger for algebraic groups):

Theorem 2. *Let G be an almost simple algebraic group over an algebraically closed field of characteristic $p \geq 0$. Let C be a G°-conjugacy class of G outside $Z(G^\circ)$. Then*

$$[C, C] := \{[x, y] \mid x, y \in C\}.$$

is the union of infinitely many conjugacy classes.

We observe that an old result of the first author [4] answers a question of Nick Katz [8, 2.16.7] about pairs of elements whose commutator is a transvection. This is useful in the proof of the previous result.

Combining Theorem 10 and Proposition 8 gives the following extension of Theorem 2.

Theorem 3. *Let G be an almost simple algebraic group over an algebraically closed field k of characteristic $p \geq 0$. Let C, D be G°-conjugacy classes of G outside $Z(G^\circ)$. Suppose that*

$$[C, D] := \{[x, y] \mid x \in C,\ y \in D\}$$

is the union of finitely many conjugacy classes. Then one of the following holds:

(1) *C, D are as given in Theorem 7 or as in Theorem 8(a)–(j); or*
(2) *$G = D_4(k).\mathfrak{S}_3$, C and D are classes of involutions in distinct outer cosets of G° which correspond to reflections.*

Finally, we observe that it is quite easy to extend our results to almost simple algebraic groups over infinite fields. In particular, we can answer the following question of Diaconis.

Theorem 4. *Let G be a simple compact Lie group. If C and D are non-central conjugacy classes of G, then CD consists of infinitely many conjugacy classes.*

The paper is organized as follows. In Section 2 we collect some auxiliary results on closed conjugacy classes in not necessarily connected algebraic groups. We then consider products of unipotent classes in disconnected algebraic groups in Section 3 and classify all cases where the product consists of finitely many classes in Theorem 7. In Section 4 we classify arbitrary products of classes meeting only finitely many conjugacy classes and thus prove Theorem 8. In Section 5 we show that these are precisely the pairs of conjugacy classes whose commutator consists just of finitely many classes, see Theorem 10. In Section 6, we prove the common characterization in Theorem 1 of various finiteness properties on products, commutators and double cosets and prove Corollary 1. In Section 7, we fully investigate the products of two non-trivial cosets of order 2 in the disconnected groups of type $D_4(k).\mathfrak{S}_3$. In the final section, we point out how we may extend our results to the case of infinite fields.

Similar questions for finite groups are considerably harder and are dealt with in [6].

We thank Pavel Shumyatsky for communicating his question to us, George McNinch for pointing out the relevance of the results of Richardson [13], and Sebastian Herpel for some comments on a preliminary version. We also thank the referee for his or her careful reading of the manuscript and useful suggestions.

2. Closed conjugacy classes in disconnected algebraic groups

We'll need some information on closed conjugacy classes. In connected reductive algebraic groups, the closed classes are precisely the semisimple ones. In the disconnected case, there can be non-semisimple elements whose class is closed. Following Steinberg [17, §9] we call an automorphism of a connected reductive algebraic group *quasi-semisimple* if it normalizes a Borel subgroup and a maximal torus thereof. We also use the following result of Spaltenstein [15, II.2.21]:

Theorem 5 (Spaltenstein). *Let G be a reductive algebraic group over an algebraically closed field. Then any coset of G° in G contains at most* *one closed unipotent class. In particular, this class is contained in the closure of all unipotent classes in that coset.*

Lemma 1. *Let H be a simple algebraic group over an algebraically closed field of characteristic $p > 0$. Assume that u is a quasi-semisimple automorphism of H of order p and $s \in H$ is a semisimple element commuting with u. Then us is also quasi-semisimple.*

Proof. By assumption u normalizes a Borel subgroup B of H and a maximal torus $T \leq B$. Let $D := C_H(u)$. By [2, Thm. 1.8(iii)], $T \cap D$ is a maximal torus of D, $B \cap D$ is a Borel subgroup of D and D is connected (see also Steinberg [17, 8.2] for the simply connected case). (Alternatively, this can be seen by inspection of the various cases.) Thus, s is conjugate in D to an element of $T \cap D$ and so we may assume that $s \in T \cap D$. Then us normalizes T and B.

Theorem 6. *Let G be a reductive algebraic group over an algebraically closed field k of characteristic p and let $C \subset G$ be a conjugacy class, $g \in C$. Then the following are equivalent:*

(i) *C is closed;*
(ii) *$C_G(g)$ is reductive; and*
(iii) *g is quasi-semisimple.*

Proof. The equivalence of (i) and (iii) is shown in [15, Lemme II.1.15] (see also [15, Cor. II.2.22]).

Now assume (i), so C is closed. Then C is an affine variety (since G is affine). It follows by [13, Lemma 10.1.3] that $C_G(g)$ is reductive.

Finally let's show that (ii) implies (iii). Note that all the conclusions concern only $\langle g, G^\circ \rangle$ and so we assume that $G = \langle g, G^\circ \rangle$. There is no loss in assuming that G° is semisimple with $C_G(G^\circ) = 1$. Moreover, we may assume that g acts transitively on the simple components of G°. If there are m components, then $C_{G^\circ}(g) \cong C_L(g^m)$ where the components are isomorphic to L. Moreover, g is quasi-semisimple if and only if g^m is. Thus, it suffices to assume that G° is simple and so G/G° embeds in $\mathfrak{S}_3$.

Assume that g is not quasi-semisimple. We show that $C_G(g)$ is not reductive.

Write $g = su = us$ where s is semisimple and u is unipotent. Let v be a generator of $\langle u \rangle \cap G^\circ$. If $v \neq 1$, then by the Borel–Tits theorem, the unipotent radical U of $C_{G^\circ}(sv)$ is nontrivial, whence since u acts as a p-element on U, $C_U(u)^\circ$ is a positive dimensional subgroup contained in the unipotent radical of $C_G(g)$. So $v = 1$, and hence $u^p = 1$.

If $p \neq [G : G^\circ]$, then $g = s$ is semisimple and in particular quasi-semisimple. Thus we may assume that $p = [G : G^\circ]$ (in particular, $p = 2$ or 3).

By the discussion above, we have that u is an automorphism of order p and $s \in G^\circ$. Note that u does not lie in the unique class of outer unipotent elements which are quasi-semisimple, since otherwise by the previous lemma, g would also be quasi-semisimple.

We consider the various cases. For the moment, exclude the case that $p = 2$ and $G^\circ = D_n(k)$, $n \geq 4$. If $p = 2$, it follows by [1, 19.7–19.9] that $u = vt$ where v is a quasi-semisimple graph automorphism and t is a long root element in $C_{G^\circ}(v)$. But then, $C_{G^\circ}(u) < C_{G^\circ}(v)$ has a nontrivial unipotent radical. The same argument applies for $p = 3$ and $G = D_4(k)$ [3, 4.9.2].

Finally, consider the case that $G^\circ = D_n(k)$, $n \geq 4$ and $p = 2$. Let V be the natural $2n$-dimensional orthogonal module for G.

Suppose that $s = 1$. Note that u is not a transvection on V (which is the unique quasi-semisimple unipotent class in uG°). Thus, the fixed space W of u has dimension at most $2n - 2$. Let R be the radical of W (with respect to the alternating form preserved by G). Note that $R \neq 0$ for otherwise, W is nonsingular and so $V = W \oplus W^\perp$, a contradiction since u will have fixed points on $W^\perp$. If R is totally singular with respect to the quadratic form preserved by G, then the centralizer of u is contained in the parabolic subgroup stabilizing R, with radical Q say. So $C_Q(u)$ is a positive dimensional subgroup of the unipotent radical of $C_{G^\circ}(u)$, which thus is not reductive. Indeed the same argument applies if $\dim R > 1$, for then replace R by R', the radical with respect to the quadratic form. So R is 1-dimensional and is generated by a non-isotropic vector. The stabilizer of this 1-dimensional space is $\langle x\rangle \times \mathrm{Sp}_{2n-2}(k)$ with x a transvection. If $u \notin \langle x\rangle$, then $C_{G^\circ}(u) \cong C_{\mathrm{Sp}_{2n-2}(k)}(y)$ for some nontrivial unipotent element y and so is not reductive.

If $s \neq 1$, write $V = W \perp W'$ where $W = [s, V]$. Note that $W \neq 0$ since $s \neq 1$. Also, (since $p = 2$) the centralizer of s in the stabilizer of W is contained in $\mathrm{SO}(W)$. In particular, if u acts nontrivially on W, it follows (by inspection or Borel–Tits) that the centralizer of G has nontrivial unipotent radical on W and so on V. Thus, u is trivial on W. Since the centralizer of u on W' is reductive, it follows by the previous case that u induces a transvection on W', whence u is a transvection. This contradiction completes the proof.

Note that the proof actually also shows that:

Corollary 3. *Let G be a reductive algebraic group over an algebraically closed field. Let $g \in G$ with semisimple part s. There exists a unique closed conjugacy class in gG° whose elements have semisimple part conjugate to s.*

We also have the obvious corollary.

Corollary 4. *Let G be a reductive algebraic group over an algebraically closed field k. Assume that the order of g in G/G° is prime to the characteristic of k. Then the class of g is closed if and only if g is semisimple.*

Proof. In this situation, quasi-semisimple elements are semisimple (see [17, §9]), and the result follows.

3. Pairs of unipotent classes in almost simple algebraic groups

In this section we classify pairs of unipotent conjugacy classes in almost simple algebraic groups such that all pairs of elements generate a unipotent subgroup (see Theorem 7). Note that the connected case was already treated in [7, Thm. 1.1]. First note the following:

Lemma 2. *Let G be a reductive algebraic group and $C_1, C_2, D \subset G$ three G°-orbits under conjugation. Assume that* $\dim C_1 + \dim C_2 = \dim D$ *and* $D \subseteq C_1C_2$. *Then:*

(a) *D is open dense in the product C_1C_2;*
(b) $\{(c_1, c_2) \in C_1 \times C_2 \mid c_1c_2 \in D\}$ *is a dense open subset of $C_1 \times C_2$;*
(c) *C_1C_2 consists of finitely many classes.*

Proof. Since G° is connected, the G°-orbits C_1, C_2, D are irreducible and then so is C_1C_2. Now $\dim C_1C_2 \leq \dim C_1 + \dim C_2 = \dim D$ and $D \subset C_1C_2$ imply that $\dim C_1C_2 = \dim D$. Thus, their closures agree. Since any two dense subsets intersect non-trivially, D is the unique dense class in C_1C_2, so it must be open. The second statement now follows. As the closure of any class consists of finitely many classes, we have (c).

We now collect some actual examples:

Example 1. Let $S = \mathrm{SL}_{2n}(k)$, $n \geq 2$, with k an algebraically closed field. Let $x \in S$ be a transvection, $y \in \mathrm{Aut}(S)$ a graph automorphism with centralizer $\mathrm{Sp}_{2n}(k)$ and $G = \langle S, y\rangle$. Let U be a root subgroup containing x. Let $N = N_S(U)$. Since $\mathrm{Sp}_{2n}(k)$ has two orbits on pairs consisting of a 1-space and a hyperplane containing it we see that $|\mathrm{Sp}_{2n}(k)\backslash S/N| = 2$. Thus, $x^S \times y^S$ consists of two S-orbits. One orbit is the set of commuting pairs. When k has characteristic $p > 0$, reducing to the case of $n = 2$ shows that in the other orbit the product will have order $2p$. In particular, if $p = 2$, $\langle x, y^g\rangle$ is either elementary abelian of order 4 or a dihedral group of order 8 for all $g \in S$.

Example 2. Let $G = \mathrm{GO}_{2n}(k)$, $n \geq 4$, with k an algebraically closed field of characteristic 2. Then G fixes a non-degenerate alternating form (,) on $V = k^{2n}$, so we may view G as a subgroup of the isometry group $\mathrm{Sp}_{2n}(k)$ of this form. Let $x \in G$ be a transvection, and y an involution with $(yv, v) = 0$ for all $v \in V$. Then by [7, Ex. 6.6], the order of xy is either 2 or 4, so x, y always generate a 2-group.

Example 3. Let k be an algebraically closed field of characteristic 3 and $G = \mathrm{SO}_8(k).3$, the extension of the simple algebraic group $G^\circ = \mathrm{SO}_8(k)$ by a graph automorphism of order 3. Let C_1 be the class of long root elements in G°, with centralizer of type $3A_1$, and C_2 the class of the graph automorphism σ with centralizer $G_2(k)$ in G°. We claim that C_1C_2 consists of 3-elements.

Indeed, let $x_i(1)$, $1 \le i \le 4$, denote long root elements for the four simple roots, labelled so that the node in the centre of the Dynkin diagram has label 2, and σ permutes $x_1(1), x_3(1), x_4(1)$ cyclically. So $x := x_2(1) \in C_1$, and it's easily seen that $y := x_1(1)x_3(1)x_4(1)\sigma$ is conjugate to σ, so lies in C_2. The product $xy = x_2(1)x_1(1)x_3(1)x_4(1)\sigma$ lies in a class D, denoted C_4 in [15, I.3], of dimension 24 (see also the representative u_4 in [10, Tab. 8]). Now C_1 has dimension 10 and C_2 has dimension 14. Since C_1 is a single class under G°-conjugation, and C_2, D, being outer classes, necessarily are, Lemma 2 shows that D is dense in C_1C_2.

Since σ normalizes the maximal unipotent subgroup generated by the standard positive root subgroups, $\langle x, y\rangle$ is unipotent for our choice of elements $x \in C_1$, $y \in C_2$ above, hence for all choices. Our claim follows.

Note that $x_2(1)$ and $x_1(1)x_3(1)x_4(1)$ are long, respectively short root elements in the centralizer $C_{G^\circ}(\sigma) = G_2(k)$ of σ, with product a regular unipotent element. This thus leads to the example in $G_2(k)$ from [7, Ex. 6.1], and we see that in both cases, the generated groups in the generic case agree (see also Proposition 2).

Example 4. Let k be an algebraically closed field of characteristic 2 and $G = E_6(k).2$, the extension of a simple algebraic group $G^\circ = E_6(k)$ of type E_6 by a graph automorphism of order 2. Let C_1 be the class of long root elements in G°, with centralizer $U.A_5(k)$, where U is unipotent of dimension 21, and C_2 the class of the graph automorphism σ with centralizer $F_4(k)$ in G°. We claim that again C_1C_2 only contains unipotent elements.

Indeed, let $x_i(1)$, $1 \le i \le 6$, denote long root elements for the six simple roots, labelled in the standard way (so that node 4 is at the centre of the Dynkin diagram, and σ interchanges $x_3(1), x_5(1)$). So $x := x_4(1) \in C_1$, and it's easily seen that $y := x_3(1)x_5(1)\sigma$ is conjugate to σ, so lies in C_2. The product $xy = x_4(1)x_3(1)x_5(1)\sigma$ lies in a class D, with representative u_{15} in [15, p. 160], of dimension 48 (see also [11, Tab. 10]). As C_1 has dimension 22 and C_2 has dimension 26, D is dense in C_1C_2 by Lemma 2.

Again, σ normalizes the maximal unipotent subgroup generated by the positive root subgroups, so $\langle x, y\rangle$ is unipotent, and hence by density this holds for all pairs from $C_1 \times C_2$. (Note that both classes have representatives in a Levi subgroup of type $A_3.2$, where they correspond to Example 1.)

As in the case of $\mathrm{SO}_8(k)$, we also recover the example in the connected group $F_4(k)$ in characteristic 2 from [7, Ex. 6.3], by considering the centralizer $C_{G^\circ}(\sigma) = F_4(k)$ of σ. There, $x_4(1)$ is a long root element, and $x_3(1)x_5(1)$ is a short root element. We see that the group generated by them is isomorphic to the one in G.

We are now ready to prove the main result of this section. Throughout our proofs the following result will make inductive arguments work (see [5, Thm. 1.2]):

Proposition 1. *Let G be reductive, $H \leq G$ a closed subgroup, and C, D conjugacy classes of G.*

(a) *If $(C\cap H)(D\cap H)$ hits infinitely many semisimple H-classes, then CD hits infinitely many semisimple G-classes.*
(b) *If H is reductive and $(C \cap H)(D \cap H)$ hits infinitely many H-classes, then CD hits infinitely many G-classes.*

Theorem 7. *Let G be an almost simple algebraic group of adjoint type over an algebraically closed field k of characteristic $p \geq 0$ with connected component G°. Let C_1, C_2 be non-trivial unipotent G°-classes of $G = \langle C_1, C_2\rangle$ such that xy is unipotent for all $x \in C_1$, $y \in C_2$. Then $p \in \{2, 3\}$ and (up to order of C_1, C_2) either:*

(1) *$G = G^\circ$ is connected, C_1 contains long root elements and one of:*
  (a) *$p = 2$, $G = \mathrm{Sp}_{2n}(k) = \mathrm{Sp}(V)$ with $n \geq 2$, and C_2 contains involutions y with $(yv, v) = 0$ for all v in V; or*
  (b) *$(G, p) = (F_4, 2)$ or $(G_2, 3)$, and C_2 contains short root element; or*
(2) *G is the extension of G° by a graph automorphism of the Dynkin diagram of order p, C_2 is the class of this quasi-central automorphism, and one of:*
  (c) *$p = 2$, $G = A_{2n-1}(k).2$ with $n \geq 2$, C_1 consists of long root elements and C_2 is the class of the graph automorphism with centralizer $C_n(k)$;*
  (d) *$p = 2$, $G = D_n(k).2$, C_1 contains involutions as given in (1)(a) for $C_n(k)$ and C_2 consists of transvections (the class of graph automorphisms with centralizer $B_{n-1}(k)$);*
  (e) *$p = 2$, $G = E_6(k).2$, C_1 is the class of long root elements and C_2 is the class of the graph automorphism with centralizer $F_4(k)$; or*

(f) *$p = 3$, $G = D_4(k).3$, C_1 is the class of long root elements and C_2 is the class of the graph automorphism with centralizer $G_2(k)$.*

Moreover, in all these cases, $\langle x, y\rangle$ is unipotent for all $x \in C_1$, $y \in C_2$.

Proof. By [7, Thm. 1.1] we may assume that G is disconnected and C_2, say, lies in $G \setminus G^\circ$. So either $p = 2$ and G° has type A_n ($n \geq 2$), D_n ($n \geq 4$), or E_6, or $p = 3$ and G° has type D_4. By Examples 1–4 above, all the cases described in (2)(c)–(2)(f) do give examples.

Suppose that C_1 also consists of outer elements. By Corollary 3 there is a unique outer unipotent class lying in the closure of all other unipotent classes in a fixed coset. By taking closures we may assume that C_1, C_2 are these. If $p = 3$ and $C_1 = C_2$ then a computation in $\mathrm{SO}_8^+(3).3$ shows that the product contains non-unipotent classes. Else, C_2 contains the inverses of C_1, and since outer quasi-semisimple elements normalize, but do not centralize, a maximal torus of G°, we obtain non-trivial semisimple products.

So we may assume that $C_1 \subset G^\circ$. Suppose for the moment that $p = 2$.

First consider the case that $G^\circ = A_n(k)$, $n \geq 2$. Suppose that C_1 does not consist of transvections. Then by taking closures, we may assume that C_1 either consists of elements with one Jordan block of size 3 or two Jordan blocks of size 2, and C_2 consists of quasi-semisimple elements. Thus, it suffices to work in $A_2(k)$ or $A_3(k)$. Note that all inner unipotent classes and all classes of involutory graph automorphisms intersect $A_2(2).2 \cong \mathrm{PGL}_2(7)$ resp. $A_3(2).2 \cong \mathfrak{S}_8$ and we obtain a contradiction. So C_1 consists of transvections. If n is even, we may assume that C_2 contains quasi-semisimple elements and reduce to the case of $A_2(k).2$. Computing in $A_2(2).2$, we see that there are no examples. If $n \geq 3$ is odd, by [15] all outer unipotent classes except for the quasi-semisimple one, contain the transpose inverse graph automorphism in their closure. So we may reduce to the case of $A_3(2).2$ to see that C_2 must correspond to transpositions in the symmetric group and so must be the class of the graph automorphisms with centralizer $C_n(k)$, the quasi-semisimple class. We thus arrive at case (2)(c).

Next consider the case that $G^\circ = D_n(k)$. Suppose that C_2 does not consist of transvections in the natural $2n$-dimensional representation of $D_n(k).2$. By passing to closures, we may assume that C_2 contains elements with exactly three nontrivial Jordan blocks of size 2 in the natural $2n$-dimensional representation. In particular, we see that elements of C_2 leave invariant a nondegenerate 6-dimensional space. Passing to the closure, we may assume that C_1 consists of long root elements. In particular, we can reduce to the case of $D_3(k) = A_3(k)$ for which we already saw that there are no such examples.

So we may assume that C_2 consists of transvections. We claim that C_1 contains involutions x so that $(xv, v) = 0$ for all $v \in V$, the natural $2n$-dimensional module for $C_n(k) \geq D_n(k).2$ as in (2)(d). If C_1 consists of involutions which do not have that property, then any $x \in C_1$ leaves invariant a nondegenerate 2-dimensional space on which it acts as a transvection. Thus, xy ($y \in C_2$) can have odd order on that 2-dimensional space.

If C_1 does not consist of involutions, then going to the closure allows us to assume that C_1 contains elements of order 4. It follows that any $x \in C_1$ leaves invariant a nondegenerate subspace of dimension 6 or 8 (acting as an element of order 4). Thus, we can reduce to either $D_3(k) = A_3(k)$ or $D_4(k)$. In the first case, we are done by the result for type A. In the second case, we note that every class of elements of order 4 in $\mathrm{SO}_8(k)$ intersects $\mathrm{SO}_8^+(2)$ and so we may compute structure constants in $\mathrm{GO}_8^+(2)$ to conclude that there are no such examples.

Now suppose that $G^\circ = E_6(k)$. First suppose that elements of C_2 do not have centralizer $F_4(k)$. By taking closures, we may assume that C_1 consists of long root elements in $E_6(k)$. By [15, p. 250] all outer unipotent classes except for the quasi-semisimple one contain the graph automorphism with centralizer $C_4(k)$ in their closure. Let τ be an involution with centralizer $F_4(k)$. Let $x \in F_4(k)$ be a long root element in $E_6(k)$ such that $\tau x \in C_2$. Then by the result for the connected group $F_4(k)$ there is $y \in C_1 \cap F_4(k)$ with xy of odd order.

So we may assume that elements of C_2 have centralizer $F_4(k)$. If C_1 consists of long root elements then we are in case (2)(e). Else, by taking closures, we may assume that C_1 is the class $2A_1$ [15, p. 247]. This class has representatives in the Levi subgroup $A_5(k)$ and so the result follows by that case.

Finally, consider the case $p = 3$ with $G^\circ = D_4(k)$. Suppose that elements of C_2 do not have centralizer $G_2(k)$. By taking closures, we may assume that C_1 consists of long root elements. Let τ be the graph automorphism with centralizer $G_2(k)$. Choose $x, y \in G_2(k)$ which are long root elements of $D_4(k)$ such that xy is not a 3-element (by the result for connected groups) and $\tau x \in C_2$. Then τxy is not a 3-element either. So we may assume that elements of C_2 have centralizer $G_2(k)$. Note that the graph automorphism fuses the three non-trivial unipotent classes of G° of second smallest dimension (see [15, p. 239]), so if C_1 is not as in case (2)(f), then by closure we may assume that C_1 is one of these classes. But computation of structure constants in the finite subgroup $\mathrm{SO}_8^+(3).3$ shows that this does not give an example.

Note that in all the exceptions in the statement, the group generated by $x \in C_1$, $y \in C_2$ is always unipotent, by Examples 1–4, resp. by the examples in [7]. This completes the proof of Theorem 7.

4. Products of conjugacy classes in disconnected algebraic groups

In [7, Thm. 1.1] we classified pairs of conjugacy classes C, D in a simple algebraic group such that the product CD is a finite union of conjugacy classes (equivalently, the semisimple parts of elements in CD form a single conjugacy class). We build on the results from the previous section to extend this to the disconnected case.

First we give some examples.

Example 5. Let $S = \mathrm{SL}_{2n}(k)$, $n \geq 2$, with k algebraically closed. Let $x \in S$ be any element that is (up to scalar) a pseudoreflection, y a graph automorphism with centralizer $\mathrm{Sp}_{2n}(k)$ and $G = \langle S, y\rangle$. Then by [7, Ex. 7.2], there is a single G-orbit on $x^S \times y^S$, whence $x^S y^S$ is a single conjugacy class. In particular, if k is not of characteristic 2, and x is a 2-element, $\langle x, y\rangle$ is a 2-group.

Example 6. Let $G = \mathrm{GO}_{2n}(k)$, $n \geq 4$, with k algebraically closed. Suppose that $x \in G$ is an element whose centralizer is $\mathrm{GL}_n(k)$ and $y \in G$ is a reflection. As $\mathrm{GL}_n(k)$ acts transitively on non-degenerate 1-spaces of the natural module for G, we have $G = \mathrm{GO}_{2n-1}(k)\mathrm{GL}_n(k)$ and so G has a single orbit on $x^G \times y^G$. If k is not of characteristic 2 and x has finite order m, then xy^g will have order $2m/(2, m)$.

Example 7. Let $G = \mathrm{GO}_{2n}(k)$, $n \geq 4$, with k algebraically closed of characteristic not 2. Suppose that x is unipotent with all Jordan blocks of size at most 2 and y is a reflection in G. Then x and y are both trivial on a totally singular subspace of dimension $n - 1$ of the natural module. Let P denote the parabolic subgroup stabilizing this space. Then x lies in the unipotent radical of P, whence xy has constant semisimple part y. Thus we get an example with $C_1 = x^G$, $C_2 = y^G$.

Example 8. Let $G = \mathrm{SO}_8(k).3$ with k algebraically closed of characteristic not 3. Let C_1 be the class of root elements of G°, and $y \in C_2$, the class of graph automorphisms with centralizer $H = G_2(k)$ in G°. Let $s \in H$ be an element of order 3 with $C_H(s) = \mathrm{SL}_3(k)$. By looking at the centralizer in the adjoint action on the Lie algebra of G° one sees that sy is G-conjugate to y. Note that $C_1 \cap H$ are long root elements in H. Thus by [7, Ex. 6.2] the product $(C_1 \cap H)s$ contains an element us with u regular unipotent in $C_H(s) = \mathrm{SL}_3(k)$, with centralizer dimension 4 in H. Thus $xsy \sim usy$ for some $x \in C_1$, with $[u, sy] = 1$ and of coprime order. It follows that $\dim C_G(xsy) = \dim C_G(usy) = \dim C_H(u) = 4$, so that C_1C_2 contains a class of dimension $\dim G - 4 = 24 = \dim C_1 + \dim C_2$. An application of Lemma 2 shows that C_1C_2 is a union of finitely many classes.

Similarly, let $G = E_6(k).2$ with k algebraically closed of characteristic not 2. Let C_1 be the class of root elements of G° and $y \in C_2$, the class of graph automorphisms with centralizer $H = F_4(k)$ in G. Let $s \in H$ be an involution with $C_H(s) = B_4(k)$. By looking at the adjoint action one sees that sy is G-conjugate to y. Again, $C_1 \cap H$ consists of long root elements in H. Thus by [7, Ex. 6.4] the product $(C_1 \cap H)s$ contains elements $u \in B_4(k)$ whose square is a short root element, with centralizer dimension 30 in H. Arguing as before, we see that C_1C_2 contains a class of dimension $\dim G - 30 = 48 = \dim C_1 + \dim C_2$, and we conclude by Lemma 2 that C_1C_2 is a finite union of classes.

Example 9. Let $G = \mathrm{SO}_8(k).\mathfrak{S}_3$ with k algebraically closed. Let C_1, C_2 be G°-classes of graph automorphisms with centralizer $H = B_3(k)$ in G° lying in two different G°-cosets. Then C_1, C_2 both have dimension 7, while any class in the coset of order 3 has dimension at least 14, which is attained for the quasi-central class of graph automorphisms of order 3. So the product consists of that single class by Lemma 2. See Section 7 for more on this.

Theorem 8. *Let G be an almost simple algebraic group of adjoint type over an algebraically closed field k of characteristic $p \geq 0$ with connected component G°. Let C_1 and C_2 be nontrivial G°-conjugacy classes in $G = \langle C_1, C_2 \rangle$. Suppose that C_1C_2 consists of finitely many conjugacy classes. Then one of the following holds:*

(1) *C_1, C_2 are unipotent classes and we are in one of the cases of Theorem 7; or*

(2) *$G = G^\circ$ is connected, C_2 is unipotent, and one of*
  (a) *$G = \mathrm{Sp}_{2n}(k)$, $n \geq 2$, $p \neq 2$, C_1 contains involutions and C_2 contains long root elements;*
  (b) *$G = \mathrm{SO}_{2n+1}(k)$, $n \geq 2$, $p \neq 2$, C_1 contains reflections (modulo scalars) and C_2 contains unipotent elements with all Jordan blocks of size at most 2;*
  (c) *$G = F_4$, $p \neq 2$, C_1 contains involutions with centralizer $B_4(k)$ and C_2 contains long root elements;*
  (d) *$G = G_2$, $p \neq 3$, C_1 contains elements of order 3 with centralizer $\mathrm{SL}_3(k)$ and C_2 contains long root elements; or*

(3) *G is disconnected, C_2 contains quasi-semisimple graph automorphisms, and one of*

(e) *$[G:G^\circ]=2$, $G^\circ=A_{2m-1}(k)$, $m>1$, C_1 contains (semisimple) pseudo-reflections and C_2 contains graph automorphisms with centralizer $C_m(k)$;*

(f) *$[G:G^\circ]=2$, $G^\circ=A_{2m-1}(k)$ with $p\neq 2$, $m>1$, C_1 contains transvections and C_2 contains graph automorphisms with centralizer $C_m(k)$;*

(g) *$[G:G^\circ]=2$, $G^\circ=D_n(k)$, $n\geq 4$, C_1 contains semisimple elements with centralizer $A_{n-1}(k)$ and C_2 contains graph automorphisms with centralizer $B_{n-1}(k)$;*

(h) *$[G:G^\circ]=2$, $G^\circ=D_n(k)$ with $p\neq 2$, $n\geq 4$, C_1 contains unipotent elements with all Jordan blocks of size at most 2 and C_2 contains graph automorphisms with centralizer $B_{n-1}(k)$;*

(i) *$[G:G^\circ]=2$, $G^\circ=E_6(k)$ with $p\neq 2$, C_1 contains long root elements and C_2 contains graph automorphisms with centralizer $F_4(k)$;*

(j) *$[G:G^\circ]=3$, $G^\circ=D_4(k)$ with $p\neq 3$, C_1 contains long root elements and C_2 contains graph automorphisms with centralizer $G_2(k)$; or*

(k) *$[G:G^\circ]=6$, $G^\circ=D_4(k)$, C_1 and C_2 are classes in different cosets modulo G° of graph automorphisms of order 2.*

Note that cases (f), (h), (i) and (j) are direct analogues of the unipotent pairs in Theorem 7. We analyze the case 3(k) completely in Section 7, see Proposition 7. If we consider G-classes instead of G°-classes, we get the same examples except for the ones in 3(k).

The proof of Theorem 8 will be given in a series of five lemmas. First note that if G is connected, the result was shown in [7, Thm. 1.1]. So we may and will assume that G° is one of $A_n(k)$ $(n\geq 2)$, $D_n(k)$ $(n\geq 4)$, or $E_6(k)$. Let $r=[G:G^\circ]$. We may assume that C_2 consists of outer automorphisms. We consider various cases.

Lemma 3. *The claim in Theorem 8 holds when neither of the classes C_1, C_2 is contained in G°.*

Proof. Passing to closures, we may first assume by Theorem 6 that C_1, C_2 are both quasi-semisimple. Then there exist $(x,y)\in C_1\times C_2$ normalizing a common Borel subgroup B and a maximal torus $T\leq B$. Let $S:=[y,T]$. Then $yS\subseteq C_2$ and S is infinite as $y\notin C_G(T)=C_{G^\circ}(T)$. Let m denote the order of $z=xy$ in G/G°. If $m=1$ then $xyS\subset T\cap C_1C_2$ is infinite, so contains infinitely many classes.

If $m>1$ then necessarily $G^\circ=D_4(k)$. If $m=2$ then $(xyt)^2=z^2t^zt$, for $t\in S$, is finite only if z acts as -1 on S. Note that this configuration can only occur when C_1,C_2 lie in cosets of G° of order 2, 3 respectively. Explicit computation shows that here z does not act as -1 on S.

Thus we have that $m = 3$. If C_1, C_2 lie in the same coset of order 3, then $(xyt)^3 = \{z^3 t^{z^2} t^z t \mid t \in S\}$. Again, explicit computation shows that this is not finite for suitable $N_G(T)$-conjugates of x, y. The only remaining possibility is that $G/G^\circ \cong \mathfrak{S}_3$, x, y lie in two different cosets of order 2, as in (3)(k) of the conclusion.

Lemma 4. *Let $G = A_1(k) \wr Z_r$, the wreath product of $A_1(k)$ with the cyclic group of prime order r. Let $C_1 \subset G^\circ$ be a conjugacy class whose projection to at least two factors is non-central, and $C_2 \subset G \setminus G^\circ$ any conjugacy class. Then:*

(a) *C_1C_2 consists of infinitely many conjugacy classes.*
(b) *$[C_1, C_2]$ consists of infinitely many conjugacy classes.*

Proof. Let σ denote a generator of the cyclic subgroup of G permuting the $A_1(k)$-factors. Any outer element is conjugate to one of the form $\sigma(1, \ldots, 1, x)$ for some $x \in A_1(k)$. An easy computation with such elements shows (a) and (b), using that the product of any two non-central classes of $A_1(k)$ meets infinitely many classes.

Lemma 5. *The claim in Theorem 8 holds when $G^\circ = A_n(k)$.*

Proof. By our initial reductions and Lemma 3 we may assume that $r = 2$, $C_1 \subset G^\circ$ and C_2 is outer.

Case 1. n is even.
In this case, we claim there are no examples. By taking closures, we may assume that C_2 contains quasi-semisimple elements. Thus $x_2 \in C_2$ normalizes a maximal torus, hence any Levi subgroup L of G° invariant under the graph automorphism. Again by closure, C_1 consists of semisimple elements or of transvections. In either case, we see that one can find $x_i \in C_i$ normalizing a subgroup $A_2(k)$ and acting nontrivially. Thus, it suffices to take $n = 2$. Now any quasi-semisimple element of $A_2(k)$ normalizes a maximal torus, so is of the form xt, $t \in T$, for some fixed quasi-semisimple element x. It is then a straightforward matrix computation to see that C_1C_2 always meets elements with distinct characteristic polynomial, so with distinct semisimple part.

Case 2. $n = 2m - 1 \geq 3$ is odd.
Arguing as above we can reduce to x_1, x_2 lying in a Levi subgroup of type A_3. Inspection shows that then x_2 has to be an involution. We claim that C_2 consists of involutions with centralizer $C_m(k)$. Suppose not. Then by [3, Tab. 4.5.1], elements in C_2 have centralizer $D_m(k)$. We may assume that $x_1 \in C_1$ is either semisimple or a transvection, and then after conjugation we have x_1, x_2 in a disconnected subgroup with connected component a Levi subgroup of type A_{n-1}, where no example exists by the first case.

If x_1 is semisimple, there is some root of G which is non-trivial on x_1. Since G° has just one root length, after conjugation we may assume that x_1 lies in an x_2-stable Levi subgroup L of type D_2 but not in its center, with x_2 swapping the two factors. Moreover, if no eigenspace of $x_1 \in C_1$ has dimension $n-1$, then we may arrange that the image of x_1 in both factors is non-trivial. In that case, by Lemma 4 the product C_1C_2 meets infinitely many semisimple classes inside the wreath product $A_1(k) \wr 2$. On the other hand, if x_1 is a pseudo-reflection (modulo scalars), we get case (3)(e) of the assertion by Example 5.

If x_1 is unipotent, we may assume that $p > 2$ since otherwise we are in the situation of Theorem 7. If x_1 is a transvection, then we get case (f) by Example 1. Else, by closure we may assume that x_1 has two Jordan blocks of size 2 or one of size 3. In the first case, we may reduce to A_3, in the second to A_2, and no examples arise.

If x_1 is neither unipotent nor semisimple, then by the previous arguments it must be the commuting product of a pseudo-reflection with a transvection. In this case, we may again reduce to the wreath product $A_1(k) \wr 2$ and apply Lemma 4.

Lemma 6. *The claim in Theorem 8 holds when* $G^\circ = D_n(k)$, $n \geq 4$.

Proof. As in Lemma 5 we may assume that $C_1 \subset G^\circ$ and C_2 is outer, so $r \in \{2, 3\}$.

Case 1. $r = 2$.
First suppose that x_1 is semisimple but not as in the conclusion (g). By taking closures we may assume x_2 is quasi-semisimple. Then we can reduce to $\mathrm{GO}_4(k) = A_1(k) \wr 2$. Moreover x_1 projects to elements which are nontrivial in each factor $A_1(k)$ whence C_1C_2 contains infinitely many classes by Lemma 4. Next suppose that x_1 has centralizer $\mathrm{GL}_n(k)$. If x_2 is quasi-semisimple but not a reflection, we can reduce to $D_3(k) = A_3(k)$ and invoke Lemma 5. So we may assume that the semisimple part of x_2 is a reflection. If its unipotent part is nontrivial, then by taking closures we may assume it is a root element in $\mathrm{SO}_{2n-1}(k)$ and so either has two Jordan blocks of size 2 or one Jordan block of size 3. Again, we can reduce to $D_3(k)$ where this cannot happen. When x_2 is a reflection, we arrive at (3)(g) by Example 6.

If x_1 is unipotent with all Jordan blocks of size at most 2 and x_2 is a reflection, then this gives (3)(h) by Example 7. If x_2 does not consist of reflections, we may again reduce to $D_3(k)$ to obtain a contradiction. If x_1 has a Jordan block of size at least 3, by taking closures we may assume that it has just one Jordan block and x_2 is semisimple, and then we may reduce to a subgroup $D_3(k)$ for which there are no such examples.

Finally, suppose that x_1 is a mixed element. The above argument shows that x_2 is a reflection, the semisimple part of x_1 has centralizer $\mathrm{GL}_n(k)$ and its unipotent part has Jordan blocks of size at most 2. Again, we can reduce to $D_3(k)$ to rule out this case.

Case 2. $r = 3$, so $n = 4$.
First assume that $x_2 \in C_2$ is quasi-semisimple. If x_1 is unipotent, then x_1, x_2 lie in the normalizer of a maximal parabolic subgroup P of type A_1^3. If x_1 is not a long root element, the image of x_1 in the Levi subgroup L of P is non-trivial in at least two of the factors, whence the product C_1C_2 meets infinitely many classes in L by Lemma 4. If x_1 is a long root element, we may conjugate x_1, x_2 into the graph automorphism centralizer $G_2(k)$. By [7, Thm. 5.11] the only example there is for x_2 to be the 3-element with centralizer $\mathrm{SL}_3(k)$. But by the argument in Example 8, all such elements fuse into the class of graph automorphisms with centralizer $G_2(k)$, so we arrive at (3)(j).

Any non-central semisimple element has a conjugate which is non-trivial in at least two factors of the Levi subgroup L of type A_1^3 and thus gives infinitely many classes there. Finally, if x_1 is a mixed element, we may assume that its unipotent part is a long root element, and then again its image in L is non-trivial in at least two factors.

Finally, if x_2 is not quasi-semisimple, by the previous argument and taking closures, we may assume that $x_2 = xu = ux$ where u is a long root element and x is quasi-semisimple of order 3, and x_1 is a long root element. In particular if we take x_1 centralizing x, we see that x_1 and u are conjugate in $G_2(k)$ and since $u^{G_2(k)}u^{G_2(k)}$ hits infinitely many classes, we are done.

Lemma 7. *The claim in Theorem 8 holds when $G^\circ = E_6(k)$.*

Proof. Here we have $r = 2$. By closure we may assume that $x_1 \in C_1$ is semisimple or unipotent of type A_2, or with unipotent part a long root element, and $x_2 \in C_2$ is quasi-semisimple. Hence we may arrange so that x_1, x_2 are contained in a Levi subgroup of type A_5, which is normalized by the graph automorphism. By Lemma 5 this forces x_2 to be an involution. There are two classes of outer involutions in G, with centralizers $F_4(k)$ resp. $C_4(k)$. The outer involutions of $A_5(k).2$ with centralizer of type D_3 fuses into the outer class with centralizer of type C_4 (since $A_5(k)$ has composition factors of dimensions $6, 6, 15$ on the 27-dimensional module for $E_6(k)$, and so has $D_3(k)$, while $F_4(k)$ has a 1-dimensional constituent), so by our considerations in Lemma 5, x_2 must have centralizer $F_4(k)$, and x_1 must be a long root element. This occurs by Example 8.

We have now discussed all possibilities for G and thus completed the proof of Theorem 8.

We note the following relation between the examples in disconnected groups in Theorem 8 and the ones in connected groups in [7, Thm. 1.1]:

Proposition 2. *Let G be a simple algebraic group, τ a graph automorphism of G with reductive centralizer $H = C_G(\tau)$. Suppose moreover that for some $1 \neq g \in H$, τ and $g\tau$ are G-conjugate. If $C := \tau^G$ and $D \subset G$ are classes such that CD consists of only finitely many classes in $G\tau$, then the same holds for $C_1 = g^H$, $D_1 = D \cap H$ in H.*

Proof. Clearly, $C_1 D_1 \tau = g^H \tau D_1 = (g\tau)^H D_1 \subset CD$, and if the latter meets only finitely many G-classes, then the former only meets finitely many H-classes, see Proposition 1.

This applies to cases (e), (f), (h), (i) and (j) in Theorem 8.

5. Commutators

In this section we turn to commutators of pairs of conjugacy classes. We first investigate the commutator of a single class and thus prove Theorem 2. The proof follows the ideas in [7, §5].

It is convenient to recall the following result [4] (which answered a question of Katz before it was asked — see [8, 2.16.7]).

Theorem 9. *Let G be a simple algebraic group of type A or C. Suppose that $x, y \in G$ with $[x, y]$ a long root element (i.e., a transvection). Then $\langle x, y\rangle$ is contained in a Borel subgroup.*

Proof. We may as well work in SL_n. Suppose that $z := x^{-1}y^{-1}xy$ is a long root element. Write $z = I + N$ where N is a nilpotent rank one matrix. Then $xy - yx = yxN$ is a rank one matrix. Now apply the main result of [4].

Proof (Proof of Theorem 2). Let G be an almost simple algebraic group over k and C a G°-conjugacy class of G outside $Z(G^\circ)$. Let $w : C \times C \to G$ be the map defined by $w(x, y) = [x, y]$.

Since C is an irreducible variety, if the result fails, the semisimple part of $[x, y]$ for $(x, y) \in C \times C$ would be constant. Of course, $1 = [x, x]$ and therefore the image of w must be contained in the set of unipotent elements of G. If $[C, C]$ consists of finitely many classes, the same is true for its closure, so we may also replace C by any class in its closure and so assume that C is semisimple or unipotent.

First suppose that $G \cong A_1(k)$. We claim that it suffices to show the result for $\mathrm{SL}_2(k)$. Indeed, lift C to a class of $\mathrm{SL}_2(k)$; the image of w is irreducible and contains 1, so still all commutators are unipotent (not just unipotent modulo the center).

Choose $(x, y) \in C \times C$ so that x and y are not contained in a common Borel subgroup (since any non-central element lies in at most two Borel subgroups, this clearly can be done). By Theorem 9, $[x, y]$ is not unipotent.

Next suppose that $G = G^\circ$ is connected. If C contains semisimple elements, we can choose $x \in C \cap T$ where T is a maximal torus and so x does not centralize some root subgroup U. Let U^- be the corresponding negative root subgroup. Then $\langle T, U, U^- \rangle$ is a product of a torus with an $A_1(k)$, whence the result follows.

If C is unipotent, by closure we may assume that it consists of root elements. Then we just work in $\langle U, U^- \rangle$ where U is a root subgroup generated by elements of C and again the result follows by the $A_1(k)$ result.

Finally suppose that G is disconnected. We may assume that C consists of outer automorphisms, and by closure that it is closed. In particular, elements of C are quasi-semisimple by Theorem 6. By definition, any such element normalizes, but does not centralize, a maximal torus of G°, and thus there exist non-trivial semisimple commutators of elements in C.

The next result now follows easily. This is essentially the question that Shumyatsky asked for finite groups (and indeed this result does follow from [6, Thm. 1.2]).

Corollary 5. *Let G be an almost simple algebraic group over an algebraically closed field k of characteristic p. Let ℓ be a prime and let C be a noncentral G°-conjugacy class in G. Then $[C, C]$ does not consist of ℓ-elements.*

Proof. Assume the result is false. By [17, §7], any element of $x \in C$ normalizes a Borel subgroup B. Let U be the unipotent radical of B. If x does not commute with B/U, then since B/U is abelian, we see that the set of commutators $[x, x^b]$ contains a nontrivial torus modulo U and in particular does not consist of ℓ-elements. So we may assume that $[x, B] \leq U$. Thus, either x commutes with B or $\ell = p$. In the latter case, there are only finitely many conjugacy classes of ℓ-elements, contradicting Theorem 2. However, the centralizer of a Borel subgroup in the automorphism group of G is trivial, whence x centralizes G, a contradiction.

If we take distinct classes, there are examples in [7] showing that the commutators could always be unipotent. Arguing as in [7], we can actually classify all such pairs.

Theorem 10. *Let G be an almost simple algebraic group over an algebraically closed field k of characteristic $p \geq 0$. Let C, D be non-central G°-conjugacy classes in G with $G = \langle C, D\rangle$. Assume that C, D do not lie in distinct cosets of order 2 if $G = D_4(k).\mathfrak{S}_3$. Then*

$$[C, D] := \{[x, y] \mid x \in C,\ y \in D\}$$

is the union of finitely many conjugacy classes if and only if C, D are as given in Theorem 7 or as in Theorem 8(a)–(j). In particular, $C \neq D$ in this case.

Proof. Let $w : C \times D \to G$ be as in the previous proof. By irreducibility the semisimple part of elements in $[C, D]$ has to be constant.

We can now essentially follow the proof in [7], respectively the proofs of Theorems 7 and 8. First assume that G is connected. For $G = \mathrm{SL}_2(k)$, choosing $(x, y) \in C \times D$ in the same Borel subgroup, $[x, y]$ is clearly unipotent. On the other hand, choose x and y so that they are not contained in a common Borel subgroup. Then by Theorem 9, $[x, y]$ is not unipotent. So there's no example for $\mathrm{SL}_2(k)$.

If $[C, D]$ consists of finitely many classes, the same is true for its closure, so we may also replace C and D by any classes in their closures and so assume that each is semisimple or unipotent. Moreover, if one of them is a unipotent class, we may assume that it consists of root elements. We can now argue exactly as in the proof of [7, Thm. 5.11] to rule out every configuration apart from those in [7, Thm. 5.11(2)–(6)], except for the case that $p \neq 2$, $G = \mathrm{Sp}_4(k)$, and the semisimple part of c is an involution. But again by working inside the subsystem subgroup $A_1(k)^2$ we see that $[C, D]$ can have infinitely many distinct semisimple parts by Lemma 4.

In all the exceptions of [7, Thm. 5.11(2)–(6)], any pair $(x, y) \in C \times D$ lies in some common Borel subgroup of G by [7, Cor. 5.14], whence the commutator $[x, y]$ is unipotent and so $[C, D]$ consists of finitely many (unipotent) classes.

Now assume that G is not connected, but C, D are unipotent. Then the same would have to hold for all classes in $[C, D]$, since all unipotent elements have conjugates in a maximal unipotent subgroup of the normalizer of a Borel subgroup. Following the arguments in the proof of Theorem 7 we see that the assertion follows once we show that the only pairs of 2-power classes in $\mathfrak{S}_8$, $\mathrm{PGL}_2(7)$, $\mathrm{GO}_8^+(2)$ with commutator consisting of 2-elements are those in the conclusion. This can be checked by direct computation.

Finally assume that G is disconnected and not both C, D are unipotent. Here, we go through the proof of Theorem 8. If both classes are outer (and quasi-semisimple), then we may find representatives x, y normalizing a maximal torus T, as in the proof of Lemma 3. First assume that x, y lie in the same cyclic subgroup of G/G°. Then with $S = [T, y^{-1}]$ and $R = [x, S]$ we have $[x, Sy] = \{x^{-1}y^{-1}xry \mid r \in R\} \subseteq [C_1, C_2] \cap T$, which is always infinite, since x may be chosen not to centralize the commutator space of y^{-1}. The only remaining case is when $G = D_4.\mathfrak{S}_3$ with x, y in non-trivial cosets of different order. Here, the cube of the commutator $[x, Sy]$ hits infinitely many classes in T since $\{r \in R \mid r^{z^2}r^z r\}$ is not finite, for $z = [y^{-1}, x]$.

Thus, C is inner, say. As in the proof of Lemma 5 we are done for $A_n(k)$ once we have shown the claim for $n = 2$ and $n = 3$, where it is a direct matrix calculation. The arguments for all other types go through unchanged.

In the case of commutators between G°-classes in two different cosets of order 2 in $D_4(k).\mathfrak{S}_3$, there are many further examples, see Example 13 below.

6. Double cosets, products and commutators

The aim of this section is the Characterization Theorem 1 for various finiteness properties. For this we need to investigate some of the examples from the previous three sections a bit more closely.

Proposition 3. *Let $G = \mathrm{GO}_{2n}(k)$, $n \geq 3$, with k algebraically closed of characteristic p. Let $x \in G^\circ$ be unipotent such that either*

(1) *$p \neq 2$ and all Jordan blocks of x have size at most 2; or*

(2) *$p = 2$, $x^2 = 1$ and (xv, v)=0 for all v in the natural module V for G (under the associated alternating form).*

Then $C_G(x)$ has at most two orbits on nondegenerate 1-spaces in V.

Proof. We claim that any nondegenerate 1-space is $C_G(x)$-conjugate to a 1-space in a nondegenerate 4-space or 6-space that is x-invariant.

Write $V = V_1 \perp V_2$ where V_1 and V_2 are x-invariant and x is trivial on V_1 and has all Jordan blocks of size exactly 2 on V_2.

It suffices to deal with each space separately. In the first case the centralizer contains $\mathrm{GO}(V_1)$ and we can move any 1-space into any nondegenerate 2-space. So we need to consider V_2. We claim that any vector is $C_G(x)$-conjugate to a 1-space of some x-invariant nondegenerate 4-space.

In both cases the fixed space of x contains a maximal totally singular isotropic space. Thus we can write $V_2 = U \oplus U'$ where U and U' are maximal complementary totally isotropic spaces, with U the fixed space of x. Thus x is trivial on V_2/U as well and so x is in the unipotent radical Q of the stabilizer of U. Then x corresponds to a full rank skew symmetric matrix in Q. Thus, the centralizer of x in $\mathrm{GO}(V_2)$ is $Q.\mathrm{Sp}(U)$ with $\mathrm{Sp}(U)$ acting isomorphically on both U and U'. Since $\mathrm{Sp}(U)$ is transitive on non-zero vectors of U, it is easy to see that given $v = u + u'$ with $u \in U$ and $u' \in U'$, we can move u, u' into a nondegenerate x-invariant 4-space.

Thus we are reduced to $\mathrm{GO}_4(k)$ and $\mathrm{GO}_6(k)$. It is straightforward to check that in the first case there is a unique orbit, while in the second case there are two orbits (as can be seen for example by checking that over finite fields $\mathbb{F}_q$, there are two orbits, of lengths $q^3(q^2-1), q^2(q-1)$).

In the following proof as well as later on we will use the well-known trick to count orbits which uses some easy consequences of Lang's theorem. Let G be a connected algebraic group acting on a variety V with the action defined over $\mathbb{F}_q$. Let $F = F_q$ be the q-Frobenius map. Let $\mathcal{O}$ be a G-orbit in V. Then $\mathcal{O}$ contains $\mathbb{F}_q$-points if and only if $\mathcal{O}$ is stable under F. If $v \in \mathcal{O}(q)$ has stabilizer $H \le G$ and every coset of H/H° is F-invariant, then the number of $G(q)$-orbits on $\mathcal{O}(q)$ is the number of conjugacy classes of H/H°. Now if by counting we can show that for all q^a, we have that $V(q^a)$ is the union of finitely many (fixed) orbits $\mathcal{O}_i(q^a)$, then V is the union of the $\mathcal{O}_i$ (over the algebraic closure of $\mathbb{F}_q$). By a standard compactness argument (see e.g. [7, Ex. 6.2]), the same then holds over any algebraically closed field.

Proposition 4.

(a) *Let $G = D_4(k).3$, the extension by a graph automorphism, and $x \in G^\circ$ a long root element. Then*

$$|G_2(k)\backslash G^\circ/C_{G^\circ}(x)| = 5 \quad \textit{and} \quad |G_2(k)\backslash G/C_G(x)| = 3.$$

(b) *Let $G = E_6(k).2$, the extension by a graph automorphism, and $x \in G^\circ$ a long root element. Then $|F_4(k)\backslash G/C_G(x)| = 2$.*

Proof. First consider $G = D_4(k).3$, the extension by the graph automorphism σ of order 3. Let B be a σ-stable Borel subgroup of G°. Let P be the normalizer in G° of a σ-stable root subgroup, a maximal parabolic subgroup of type A_1^3. Its derived subgroup P' is the centralizer of a long root element in G°. In order to show that $G_2(k)\backslash G/P'$ is finite, it suffices to see that $G_2(k)$ has finitely many orbits on the set of long root elements of G°. We claim that there are precisely five orbits. Clearly, there's one orbit of long root elements (in B) centralized by σ; these are long root elements in $H = C_{G^\circ}(\sigma) \cong G_2(k)$ and thus have centralizer Q' in H, where Q is a maximal parabolic subgroup of H. Furthermore, there are three orbits of long root elements fused under σ, such that the product over any

σ-orbit is a short root element in H. These have centralizer $U_3.A_1(k)$, with U_3 unipotent of dimension 3. Finally, let $x \in U = R_u(B)$ be the product of a root element with support 2 (on the simple roots of G° with respect to B) with a commuting root element with support 3 . Then the centralizer of u in $U \cap H$ has dimension 4. The product $uu^\sigma u^{\sigma^2}$ is a unipotent element $(x_{a+b}(1)x_{2a+b}(1)$ with respect to the simple roots a, b of H) with 4-dimensional unipotent centralizer in H. Thus $\dim(C_H(u)) = 4$. (This is the element constructed in Example 8 when $p \neq 3$, respectively in Example 3 when $p = 3$.)

For $p > 0$, over the field with $q = p^a$ elements the corresponding $G_2(q)$-orbits in $D_4(q).3$ have lengths $q^6 - 1$, $q^2(q^6 - 1)$ (3 times), and $q^2(q^2 - 1)(q^6 - 1)$, which adds up to the number of long root elements, so we've accounted for all orbits in this case. By arguing as in [7, Ex. 6.2] we conclude that the claim holds over any algebraically closed field. Note that since σ permutes the three orbits of dimension 8, we obtain three orbits under $C_G(\sigma)$.

Now consider $G = E_6(k).2$, the extension by the graph automorphism σ of order 2. The centralizer of a long root element in G° is the derived subgroup P' of a parabolic subgroup P of G° of type A_5. The group $H = C_{G^\circ}(\sigma) \cong F_4(k)$ has one orbit on long root elements contained in H, so centralized by σ, with centralizer Q', where Q is an end-node parabolic subgroup of H. Now let x be a long root element not fixed by σ, corresponding to the root $\alpha = \sum_{i=1}^{6} a_i\alpha_i$, with $\alpha_1, \ldots, \alpha_6$ the simple roots of E_6 in the standard numbering and $(a_1, \ldots, a_6) = (1, 1, 1, 2, 2, 1)$. This root subgroup is normalized by a maximal torus of H, and also centralized by all but three positive root subgroups in H. Moreover, α is centralized by a subgroup of type $W(A_3)$ in the Weyl group of H. Thus $C_H^\circ(x) = U.A_3(k)$ with U unipotent of dimension 15, and the H-orbit of x has dimension $22 = \dim E_6(k) - \dim P'$, so it is the dense orbit. Note that the σ-conjugate root element x^σ (in the root subgroup corresponding to (112211)) has exactly the same connected centralizer in H, and for $p \neq 2$, $C_H(xx^\sigma) = U.A_3(k).2$ is an extension of $C_H^\circ(x)$ of degree 2 (see [14]). Since $W(F_4)$ contains an element interchanging the root subgroups of x and x^σ, we deduce that $C_H(x) = U.A_3(k)$ is connected for $p \neq 2$. A direct computation in $E_6(2)$ shows that $C_H(x)$ is also connected in characteristic 2.

In the finite group over $\mathbb{F}_q$, the lengths $(q^4 + 1)(q^{12} - 1)$ and $q^3(q^4 + 1)(q^{12} - 1)$ of these two orbits add up to the total number of long root elements. We conclude as in the previous case.

We now turn to the proof of Theorem 1 from the introduction. One step is given by the following result:

Lemma 8. *Let G be an almost simple algebraic group over an algebraically closed field, and let C, D be G°-classes. If every $(x_1, x_2) \in C \times D$ normalizes some Borel subgroup of G° then CD is a finite union of G°-conjugacy classes.*

Proof. Let $(x_1, x_2) \in C \times D$ normalize the Borel subgroup B of G°. If x_1, say, is inner, then conjugates of x_1 inside B have only finitely many distinct semisimple parts, so the possible products with x_2 have only finitely many quasi-semisimple classes in their closures, whence we get (i) by [7, Lemma 5.1] (note the argument shows that the number of possible quasi-semisimple classes is independent of the Borel subgroup).

So now let's assume that C and D are both outer. We claim that this cannot occur aside from the case that C and D are in distinct cosets of outer involutions (whence $G = D_4(k).\mathfrak{S}_3$).

Note that under the assumption that every pair in $C \times D$ normalizes a Borel subgroup, the conclusion holds if and only if it holds for $C' \times D'$ where C' and D' are the closed quasi-semisimple classes in $\bar{C}$ and $\bar{D}$ respectively. So it suffices to assume that C and D are both quasi-semisimple. Note that our condition is also true for powers of C and D.

By choosing a parabolic subgroup that is invariant under the graph automorphism and such that it still induces a graph automorphism on its Levi subgroup, we can reduce to the cases $A_1(k)^2$, $A_2(k)$ or $A_1(k)^3$ (the latter in $D_4(k)$). It is straightforward in each of these cases to check that there is not always a common Borel subgroup unless possibly C and D are in distinct cosets of graph automorphisms of order 2. Moreover, in the latter case, if neither C nor D have order 2, then one sees (as in the proof of Proposition 7), there will not always be a common invariant Borel subgroup. So we may assume that C consists of graph automorphisms of order 2. Similarly, if D is not as given in Proposition 7(1), there will not always be a common invariant Borel subgroup. If D does satisfy Proposition 7(1), then in fact there are only finitely many classes in CD.

We are now ready to prove Theorem 1, which is part of the following statement:

Theorem 11. *Let G be an almost simple algebraic group over an algebraically closed field, and let C, D be G°-classes in G. The following are equivalent:*

- (i) *CD is a finite union of G°-conjugacy classes.*
- (ii) *The closure of CD contains a unique quasi-semisimple conjugacy class.*
- (iii) *$C_{G^\circ}(x_1)\backslash G^\circ / C_{G^\circ}(x_2)$ is finite for all $(x_1, x_2) \in C \times D$.*
- (iv) *G° has finitely many orbits on $C \times D$.*

(v) $\langle x_1, x_2\rangle$ *normalizes some Borel subgroup of* G° *for every* $(x_1, x_2) \in C \times D$.

(vi) $C_{G^\circ}(x_1)C_{G^\circ}(x_2)$ *is dense in* G° *for some* $(x_1, x_2) \in C \times D$.

Moreover, these conditions imply, and if $G \neq D_4.\mathfrak{S}_3$ *are all equivalent to*

(vii) $[C, D]$ *is a finite union of* G° *conjugacy classes.*

Proof. For the equivalence of (i) and (ii) note that CD consists of finitely many classes if and only if its closure $\overline{CD}$ does. Since $C \times D$ is an irreducible variety, it follows that $\overline{CD}$ is also irreducible, whence it contains a unique open, dense G° orbit. Hence it contains exactly one quasi-semisimple class by Corollary 3. If $\overline{CD}$ consists of infinitely many classes, then infinitely many of them are quasi-semisimple (since there are only finitely many classes having a given quasi-semisimple element in the closure). So statements (i) and (ii) are equivalent. Clearly (iii) and (iv) are equivalent, and furthermore, (iii) implies (vi). By [7, Lemma 5.3] we have that (vi) implies (ii).

It is shown in Lemma 8 that (v) implies (i). The equivalence of (i) and (vii) is the statement of Theorem 10.

Thus it remains to show that (i) implies (iii) and (v), which we do by going through the cases in Theorem 8. For the cases in (1), this is [7, Cor. 5.14]. The unipotent examples in (2) are those from Theorem 7(2)–(5). Here, it is shown in Examples 1, 3, 4, and in [7, Ex. 6.6] that $\langle x_1, x_2^g\rangle$ is a unipotent subgroup and thus (v) holds. Furthermore, we have (iii) by Example 1 and Propositions 3 and 4.

Now let's turn to the non-unipotent examples in Theorem 8(e)–(k). For case (e), there is just one orbit of G on $x_1^G \times x_2^G$ by [7, Ex. 7.2], and clearly we can find a pair in the normalizer of some Borel subgroup, whence (iii) and (v) hold. In case (f), we have just two orbits by Example 1. Since clearly the graph automorphism σ cannot centralize all transvections in a σ-stable Borel subgroup of G°, we get representatives of both orbits in the normalizer.

In case (g), by Example 6 there's again just a single orbit.

In case (h), by Example 7 both elements always lie in the normalizer of some parabolic subgroup, and x_1 in its unipotent radical, so we get (v), while (iii) is in Proposition 3. Finally, in cases (i) and (j), by Example 8 there is a dense pair, which lies inside a Borel subgroup, and (iii) was shown in Proposition 4. If $G = D_4.\mathfrak{S}_3$, we apply Proposition 7 and Lemma 9.

The same result holds for G-classes as well (but we can include (vii) without any restrictions as all the cosets of outer involutions are conjugate).

Note that all in cases we see from the above arguments that G either has infinitely many orbits or at most 3 orbits on $C_1 \times C_2$. This was shown in the connected case in [7] aside from two cases where a bound of 4 was given. We show that 3 is an upper bound in these cases as well in the next two results.

Proposition 5. *Let k be an algebraically closed field of characteristic 2. Let $G = \mathrm{Sp}_{2n}(k) = \mathrm{Sp}(V)$. Let $x \in G$ be an involution with $(xv, v) = 0$ for all $v \in V$. Then $C_G(x)$ has at most 3 orbits on the nonzero vectors of V.*

Proof. Note that $W := [x, V]$ is a totally singular m-space for some $m = 2s \leq n$. Let $X = [x, V]^{\perp}$. Let P be the stabilizer of W. So $P = QL$ is a maximal parabolic subgroup of G with unipotent radical Q and Levi subgroup $L \cong \mathrm{GL}(W) \times \mathrm{Sp}(X/W)$. Note that $x \in Z(Q)$ which can be identified with symmetric $m \times m$ matrices over k. Then x corresponds to a skew symmetric matrix of rank m. Thus, $C_G(x) = QJ$ where $J \cong \mathrm{Sp}(W) \times \mathrm{Sp}(X/W)$ (where $\mathrm{Sp}(W)$ preserves the alternating form defined by x). The result now follows easily by noting that J acts transitively on the nonzero elements of W, X/W and V/X and that if $v \in X \setminus W$, then $Qv = v + W$ and if $v \in V \setminus X$, then $Qv = v + X$ (if $W \neq X$, the three orbits are $W \setminus \{0\}$, $X \setminus W$ and $V \setminus X$ while if $W = X$, there are two orbits).

Proposition 6. *Let k be an algebraically closed field of characteristic $p \neq 2$. Let $G = \mathrm{SO}_n(k) = \mathrm{SO}(V)$. Let $x \in G$ be a unipotent element with all Jordan blocks of size at most 2. Then $C = C_G(x)$ has at most 3 orbits on nondegenerate 1-spaces of V.*

Proof. First suppose that n is a multiple of 4 and all Jordan blocks of x have size 2. Then $W := [x, V] = C_V(x)$ is a maximal totally singular subspace and $C = Q\mathrm{Sp}(W)$ where Q is the unipotent radical of the stabilizer of W. Then C is transitive on all cosets $v + W$ with v outside W and Q is transitive on all vectors in $v + W$ of a given norm, whence we see that C has a single orbit on nondegenerate 1-spaces (this is also clear from the proof of Proposition 3). Thus, we see that C has two orbits on nonzero singular vectors (they are either in W or not).

In the general case, we can write $V = V_0 \perp W$ where V_0 is a nondegenerate space with x trivial on V_0 and all Jordan blocks of x have size 2 on W. Let v be a nonsingular vector in V and write $v = v_0 + w$ with $v_0 \in V_0$ and $w \in W$. By the argument in the first paragraph, we can find x-invariant nondegenerate subspaces $V_1 \subset V_0$ and $W_1 \subset W$ with $\dim V_1 \leq 2$ and $\dim W_1 \leq 4$ and $c \in C$ with $cv \in V_1 + W_1$. Thus, we may assume that $n \leq 6$ (and clearly $n \geq 4$). It is straightforward to check the result in these cases.

This completes the proof of Corollary 1.

7. Cosets of order 2 in $D_4(k).\mathfrak{S}_3$

Here, we discuss the situation left open in conclusion (k) of Theorem 8.

Let k be an algebraically closed field of characteristic $p \geq 0$ and $G = D_4(k).\mathfrak{S}_3$, the extension of a simple algebraic group (of adjoint type) of type D_4 by the full group of graph automorphisms $\mathfrak{S}_3$. Let $x, y \in G$ be outer involutions in two different cosets of G°. Moreover, we assume that x and y correspond to reflections in some projective 8-dimensional representation of $\langle G^\circ, x\rangle$, resp. $\langle G^\circ, y\rangle$. Note that $G^\circ = C_{G^\circ}(x)C_{G^\circ}(y)$ whence G° is transitive on such pairs. Set $z = xy$. Thus z is a graph automorphism of G° of order 3 with centralizer $C_{G^\circ}(z) \cong G_2(k)$.

We start with some examples. In all of these C denotes the G°-class of x and D will be a class in the coset $G^\circ y$. For convenience we recall the outer unipotent conjugacy classes of $\mathrm{GO}_8(k)$, with k algebraically closed of characteristic 2 (see [15, p. 236] and [10, Tab. 10], and the unipotent classes of $G_2(k)$ (see [9, Tab. B, Tab. 1]). Here, $A(x) := C_G(x)/C_G(x)^\circ$.

Table 1. Outer unipotent classes of $\mathrm{GO}_8(k)$, $\mathrm{char}(k) = 2$

x	$C(x)/R_u(C(x))$	$\dim R_u(C(x))$	$\dim C(x)$	$A(x)$
2.1^6	C_3	0	21	1
$2^3.1^2$	A_1^2	7	13	1
4.1^4	A_1^2	5	11	Z_2
$3^2.2$	A_1	6	9	1
4.2_0^2	A_1	6	9	1
4.2^2	1	7	7	1
6.1^2	T_1	4	5	Z_2
8	1	3	3	1

Table 2. Unipotent classes of $G_2(k)$, $\mathrm{char}(k) \neq 3$

x	$C(x)/R_u(C(x))$	$\dim R_u(C(x))$	$\dim C(x)$	$A(x)$	in $D_4(k)$	
					$(p > 3)$	$(p = 2)$
1	G_2	0	14	1	1^8	1^8
A_1	A_1	5	8	1	$2^2.1^4$	$2_0^2.1^4$
$\tilde{A}_1$	A_1	3	6	1	$3.2^2.1$	2^4
$G_2(a_1)$	1	4	4	$\mathfrak{S}_3$	$3^2.1^2$	$3^2.1^2$
G_2	1	2	2	$Z_{(2,p)}$	7.1	6.2

Table 3. Unipotent classes of $G_2(k)$, $\mathrm{char}(k) = 3$

x	$C(x)/R_u(C(x))$	$\dim R_u(C(x))$	$\dim C(x)$	$A(x)$	in $D_4(k)$
1	G_2	0	14	1	1^8
A_1	A_1	5	8	1	$2^2.1^4$
$\tilde{A}_1$	A_1	5	8	1	$3.2^2.1$
$\tilde{A}_1^{(3)}$	1	6	6	1	$3.2^2.1$
$G_2(a_1)$	1	4	4	Z_2	$3^2.1^2$
G_2	1	2	2	Z_3	7.1

Example 10. Let D consist of elements conjugate to $ys = sy$ where $s \in C_{G^\circ}(y)$ is a semisimple element with at most 2 non-trivial eigenvalues. First assume that s has a non-trivial eigenvalue a of order prime to 6. Note that $\dim C = 7$ and that $\dim D = 17$. Also note that we can arrange that xy is conjugate to zt where $t \in C_{G^\circ}(z)$ with t semisimple and of order prime to 6. Thus $\dim C_{G^\circ}(zt) = \dim C_{G_2(k)}(t^3) \leq 4$ and it follows that $\dim(xy)^{G^\circ} \geq 24$, whence we have equality. Thus, $CD = (xy)^{G^\circ}$ (since the right hand side is closed). Indeed the same argument shows that for any $(x', y') \in C \times D$, $\dim(C_{G^\circ}(x') \cap C_{G^\circ}(y')) \leq \dim C_{G^\circ}(xy) = 4$, so $G^\circ = C_{G^\circ}(x)C_{G^\circ}(y)$. If $a \in k^\times$ is arbitrary, the centralizer will only get bigger whence we still have the same factorization.

Example 11. Let $p \neq 2$ and D consist of elements conjugate to $yu = uy$ where $u \in C_{G^\circ}(y)$ is unipotent with Jordan block lengths $2^2.1^4$ or 3.1^5.

First consider the case that u is a long root element. Note that every G°-orbit in $C \times D$ contains a representative (x, yu) where u is a long root element commuting with y (since G° is transitive on $x^{G^\circ} \times y^{G^\circ}$). By Proposition 4, $C_{G^\circ}(z)$ has five orbits on long root elements in G. It is easy to see that only 2 of these orbits are contained in $C_G(y)$. Thus, G° has exactly 2 orbits on $C \times D$ (depending upon whether u centralizes z or not).

There is a subgroup $A_1(k)^4$ of G° normalized by x, y. We may assume that y permutes the last two copies and x permutes the middle two copies (so both centralize the first copy). If we take $u = (a_1, a_2, 1, 1)$ where a_i are nontrivial unipotent elements of $A_1(k)$, then u has a single Jordan block of size 3 but zu is conjugate to $zv = vz$ where v has two Jordan blocks of size 3 (and two trivial Jordan blocks). It follows that v lies in the conjugacy class $G_2(a_1)$ in $C_{G^\circ}(z)$. Thus, $\dim(zv)^{G^\circ} = 24$ and so G° has a dense orbit on $C \times D$ (consisting of those pairs whose product is in the class of zv).

Example 12. Let $p = 2$ and D consist of 2-elements with Jordan block lengths $2^3.1^2$, 4.1^4 or $3^2.2$ in the natural 8-dimensional representation of $\mathrm{GO}_8(k)$. An explicit calculation in the permutation representation of $\mathrm{SO}_8^+(2).\mathfrak{S}_3$ of degree 3510 (as maximal subgroup of Fi_{22}) shows that the respective products contain elements of the form zu, with $z = xy$ a graph automorphism of order 3 and $u \in C_{G^\circ}(z) \cong G_2(k)$ unipotent in the $G_2(k)$-class $\tilde{A}_1, G_2(a_1), G_2$ respectively (see Table 2). Thus the corresponding classes $E = [zu]$ have dimensions $22, 24, 26$ respectively. As $\dim C + \dim D$ equals $7 + 15, 7 + 17, 7 + 19$ in the three cases, we conclude by Lemma 2 that E is dense in CD and that CD consists of finitely many G° classes.

We now show that the above are the only examples:

Proposition 7. *Let C, D be G°-orbits in xG° and yG°. Then CD is a finite union of G°-orbits if and only if (after reordering if necessary) C consists of involutions conjugate (in G) to x and one of*

(1) *D consists of elements conjugate to $yt = ty$ where $t \in G^\circ$ is a semisimple element with at most 2 non-trivial eigenvalues;*
(2) *$p \neq 2$, D consists of elements conjugate to $yu = uy$ where u is unipotent with Jordan block lengths $2^2.1^4$ or 3.1^5; or*
(3) *$p = 2$ and D consists of 2-elements with Jordan block lengths $2^3.1^2$, 4.1^4 or $3^2.2$ in the natural 8-dimensional representation of $\mathrm{GO}_8(k)$.*

Proof. We have already seen in Examples 10–12 that all cases (1)–(3) really occur. So we need to show that these are the only possibilities with CD containing only finitely many G°-orbits. So assume that CD consists of finitely many G°-orbits or equivalently that all quasi-semisimple elements in the closure of CD are conjugate.

First consider the case that $p \neq 2$ and C and D are both involutions but neither are reflections. Note that this prescribes C and D uniquely. Thus, for any pair of involutions $s, t \in C_{G^\circ}(z)$, $(xs, yt) \in C \times D$ and $xsyt = z(st)$. As s, t run over all such involutions, st intersects infinitely many $G_2(k)$ classes (since we can take s, t to be involutions in $G_2(k)$), whence CD consists of infinitely many classes.

If $p = 2$ and C and D both consist of 2-elements, then every unipotent class is represented in $\mathrm{Aut}(D_4(2))$ and one checks directly using GAP that the answer is as stated in (3) of the assertion.

So next assume that D does not consist of involutions and that C and D are both quasi-semisimple classes. Then we choose $(x', y') \in C \times D$ which normalize a parabolic subgroup $P = QL$ with P the normalizer of a long root subgroup, L is a Levi subgroup and Q the unipotent radical of P. Note that $L = TA_1(k)^3$ where T is a central 1-dimensional torus of L. We can assume that x permutes the first two copies of $A_1(k)$ and y the last two.

Since C and D consist of quasi-semisimple elements, we can choose x', y' actually normalizing L. It is straightforward to see that (aside from the cases allowed in the conclusion) $(C \cap L)(D \cap L)$ intersects infinitely many classes of $L.\mathfrak{S}_3$ whence CD contains infinitely many G°-orbits by Proposition 1.

Thus, we may assume that all quasi-semisimple elements in the closure of x' are conjugate to x and that the quasi-semisimple elements in the closure of y' are as given in (1) or (2). Moreover, at least one of x', y' is not quasi-semisimple.

Case 1. x' is quasi-semisimple and y does not lie in the closure of y'. By passing to closures, we may assume that $y' = y_1 y = y y_1$ where $y_1 = ut$ with t semisimple having precisely two nontrivial eigenvalues and u is a long root element. Note that such a class has a representative in L and one easily checks that there are infinitely many classes represented there.

Case 2. x' is quasi-semisimple and y lies in the closure of y'.
In this case, we can assume that $p \neq 2$ (because of the remarks above about pairs of 2-elements). By passing to closures (and noting that we may assume the unipotent part of y' is not a long root element nor has just one single Jordan block of size 3), we may assume that $y' = yu$ where u has a single Jordan block of size 3 and two Jordan blocks of size 2. Again, we can see such an element in L and obtain a contradiction.

Case 3. Neither x' nor y' are quasi-semisimple.
Passing to closures, we may assume that the unipotent parts of x' and y' are long root elements and again we can see such elements in L and derive a contradiction.

By symmetry, we have covered all cases and proved the result.

Lemma 9. *In the situation of Proposition 7, G° acts transitively on $C \times D$ in case (1); it has 2 orbits in both situations of case (2); and there are 2, 3, respectively 2 orbits in case (3). Moreover, every $(c, d) \in C \times D$ normalizes a Borel subgroup of G°.*

Proof. The case (1) was already argued in Example 10, and similarly for the first pair of classes in case (2), in Example 11. If u has Jordan blocks of lengths 3.1^5 in case (2), we need to count orbits of $G_2(k)$ on the class of u in $\mathrm{SO}_7(k)$.

Note that if $u \in \mathrm{SO}_7(k)$ is such a unipotent element, it has a unique 1-dimensional invariant (singular) subspace (namely the image of $(u-1)^2$) and then u lies in the unipotent radical of the stabilizer P_1 of that subspace (corresponding to nonsingular vectors in the unipotent radical).

Note that $G_2(k)$ acts transitively on singular 1-spaces (this is well known, but note that in the 7-dimensional representation, the stabilizer of a highest weight space is a maximal parabolic subgroup P of codimension 5, thus G_2/P is a 5-dimensional projective variety contained in $\mathrm{SO}_7(k)/P_1$ which is also projective of dimension 5, so they are equal). So we can just count orbits of P on the unipotent radical Q of P_1. Now Q is 5-dimensional and abelian and P_1/Q acts on this as $T\mathrm{SO}_5(k)$, and $P/(Q \cap P) = TA_1U_3$ acts on this so that $Q = k^2|k|k^2$ as a module for P. Because of the torus it is enough to count orbits on nondegenerate 1-spaces, whence it is easy to see that P' has two such orbits acting on Q (namely those spaces in the orthogonal complement of the k^2 and those outside).

Finally, in case (3), first assume that D contains unipotent elements with Jordan type $2^3.1^2$. As we saw in Example 12, one orbit consists of pairs (c, d) such that cd lies in class $\tilde{A}_1$ of a subgroup $G_2(k)$, with connected centralizer of dimension 6. A second orbit is given by pairs (c, d) with product in class A_1 in $G_2(k)$, and connected centralizer of dimension 8. Adding up the lengths of these two orbits over $\mathbb{F}_{2^f}$ shows that there no other orbits.

Next, assume that D contains unipotent elements with Jordan type 4.1^4. Any $d \in D$ can be written as yw where y is a reflection and w is a long root element in G° (not centralizing y for sure). This is not unique, but given any $(c, d) \in C \times D$, we can conjugate and assume $c = x$ and $d = yw$ as above. Now $G_2(k)$ centralizes x and y and so the number of orbits of G on $C \times D$ is at most the number of orbits of $G_2(k)$ on the set of long root elements w such that yw has Jordan type 4.1^4.

Since $G_2(k)$ has 5 orbits on long root elements and at least two of them do not have the property that yw has order 4 (namely the ones in $G_2(k)$ and also the ones in $\mathrm{Sp}_6(k)$), there are at most 3 orbits. In fact, we claim that there are 3 orbits. Two of the orbits are interchanged by y and we can see such a w in $A_1(k) \times A_1(k)$ where y is interchanging the 2 factors and w is trivial in one of the factors. Clearly yw has order 4 and has a 5-dimensional fixed point space, so yw has the correct Jordan type. Since y interchanges those two orbits, we see that there are at least 2 orbits. However, neither of these is the dense orbit (e.g., they have the same dimension) and so there must be a third orbit (alternatively, one can compute over $\mathbb{F}_2$).

Finally consider the case that D consists of unipotent elements with Jordan type $3^2.2$.

Computing over $\mathbb{F}_2$, we see that CD contains the conjugacy classes of elements $zu = uz$ where z is a graph automorphism of order 3 with centralizer $G_2(k)$ and u is either a regular unipotent element of $G_2(k)$ or is in the conjugacy class $G_2(a_1)$.

First consider a pair $(c,d) \in C \times D$ with $cd = zu = uz$ where u is a regular unipotent element in $G_2(k)$. It follows that J, the centralizer of $\langle c,d\rangle$ is contained in the centralizer of zu which has connected component a 2-dimensional unipotent group and has two components. Since $\dim C + \dim D = 26$, it follows that $\dim J \geq 2$, whence J° is a 2-dimensional unipotent group. Moreover, this must correspond to the dense orbit of G° and so it is unique. Computing such triples over $\mathbb{F}_2$, we see that there are two orbits each of size $|G^\circ(2)|/4/2$. It follows by Lang's theorem that $[J : J^\circ] = 2$ and so there is a single G°-orbit splitting into two $G(q)$-orbits each of size $(1/2)|G^\circ(2)|/4$.

It follows by counting that the dimension of the complement of the dense orbit in $C \times D$ is 24 and moreover, there is at most 1 orbit of that dimension. Let H be the centralizer of a $(c,d) \in C \times D$ with product $zu = uz$ where $u \in G_2(a_1)$. The centralizer H of zu in G° has connected component a 4-dimensional unipotent group with $H/H^\circ \cong \mathfrak{S}_3$ (see Table 2). We see that over $\mathbb{F}_2$ this orbit breaks up into three orbits of sizes $e|G^\circ(2)|/2^4$ withe $e = 1/2, 1/3$ and $1/6$. It follows that H is the centralizer of $\langle c,d\rangle$ and this orbit breaks up into 3 orbits over $\mathbb{F}_q$ of sizes $e|G^\circ(q)|/q^4$ with $e = 1/2, 1/3$ and $1/6$. Thus, the number of $\mathbb{F}_q$ points in this orbit of the algebraic group is $|G^\circ(q)|/q^4$.

Since $|C(q)||D(q)| = |G^\circ(q)|/q^4 + |G^\circ(q)|/q^2$, it follows that there are no further orbits and so G° has exactly 2 orbits on $C \times D$.

The last assertion follows by noting that in the examples we produce $(c,d) \in C \times D$ in the dense orbit which normalize a Borel subgroup. Since normalizing a Borel subgroup is a closed condition (as G/B is a projective variety), this implies the result.

We next consider commutators of G°-orbits in our situation. Here, we have the following curious situation:

Example 13. As above let $x, y \in G = D_4(k).\mathfrak{S}_3$ be quasi-central in distinct cosets of order 2 modulo $H := G^\circ$. Then for any $g \in H$ we have

$$[x, yg] = x(yg)^{-1}xyg = xg^{-1}(yxy)g \in x^H t^H = (xt)^H$$

where $t = yxy$ is a quasi-central element (reflection) in the third such coset. So letting $C = x^H$ and $D = yH$, we see that $[C,D]$ is a single H-orbit (namely the graph automorphisms of order 3 with centralizer $G_2(k)$). In particular, G certainly need not have finitely many orbits on $C \times D$. Hence, the analogue of Theorem 11(vii) fails in this situation.

Proposition 8. *Let $G = D_4(k).\mathfrak{S}_3$ and C, D two G°-orbits in distinct cosets of G° of order 2. Then $[C,D]$ is the union of finitely many G°-orbits if and only if (up to order) C consists of reflections.*

Proof. We have seen in Example 13 that if C consists of reflections and D is any class, then $[C,D] = z^{G^\circ}$ where z is a graph automorphism of order 3 with centralizer $G_2(k)$. Thus, by taking closures we may assume that one of the following holds:
(1) C and D are both quasi-semisimple;
(2) C and D are both reflections times long root elements; or
(3) C is quasi-semisimple and D is as in (2).

We can see all these cases in A_1^4 and get a contradiction (assuming that neither C nor D consists of reflections).

Here is another corollary (which could be proved more directly):

Corollary 6. *Let G be an almost simple algebraic group. Suppose that C_1, C_2, C_3 are nontrivial G°-orbits. Then $C_1C_2C_3$ consists of infinitely many G°-orbits unless $G = D_4.\mathfrak{S}_3$ and (up to order) C_1 and C_2 are reflections in different cosets modulo G° and C_3 consists of long root elements.*

Proof. Observe that in the case of $D_4.\mathfrak{S}_3$ for C_1 and C_2 reflections in different cosets and C_3 long root elements, $C_1C_2 = D := z^{G^\circ}$ and so indeed $C_1C_2C_3 = DC_3$ is a finite union of classes by Theorem 8.

In any case other than $D_4(k).\mathfrak{S}_3$ with an outer automorphism, two of the C_i must either be inner or outer and so already C_iC_j consists of infinitely many inner classes again by Theorem 8 (and by working in the normalizer of a torus, infinitely many classes times C_k gives infinitely many classes).

If G is connected, by passing to closures, we may assume that each C_i is either semisimple or unipotent. We know that two semisimple elements do not work and so at least two of the classes must be unipotent. By taking closures, they can be taken to be root elements (and so for roots of distinct lengths). So we reduce to the rank 2 case, i.e. $B_2(k)$ (characteristic 2) or $G_2(k)$ (characteristic 3) by [7]. But then the product of the two unipotent classes contains a regular unipotent element and that is not one of our examples.

8. Infinite Fields

Let's also note the following easy consequence of our results in [7]:

Corollary 7. *If G is a simple compact (real) Lie group and C and D are non-central conjugacy classes of G, then CD is an infinite union of classes.*

Proof. It follows easily from the fact that G is Zariski dense in $\bar{G}$, the complexification of G, that any class is dense in its closure $\bar{C}$ in $\bar{G}$. Now if CD is a finite union of classes, then taking closures shows that $\bar{C}\bar{D}$ is a finite union of $\bar{G}$-classes as well, and as C and D are semisimple classes, by [7, Thm. 1.1] this cannot happen.

For the disconnected case, we would get only the semisimple cases (e.g. in type A). The previous result extends to infinite fields with $G(K)$ an anisotropic simple group (i.e., containing no nontrivial unipotent elements).

The same ideas give the following result.

Corollary 8. *Let K be an infinite field and $G(K)$ some form of a simple algebraic group G over K. Let $C \subset G(K)$ be a non-central conjugacy class. Then CC is an infinite union of classes.*

Proof. Let $\bar{C}$ denote the Zariski closure of C. Then the semisimple part of elements in $\bar{C}$ is unique, since in any rational faithful representation, elements in $\bar{C}$ have the same characteristic polynomial as those in C, as $G(K)$ is Zariski dense in G (see e.g. [16, Cor. 13.3.9]). Thus, if CC consists of finitely many classes, so does $\bar{C}\bar{C}$, which is not possible by [7, Thm. 1.1].

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