ROBINSON'S CONJECTURE FOR CLASSICAL GROUPS

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Abstract. Robinson's conjecture states that the height of any irreducible ordinary character in a block of a finite group is bounded by the size of the central quotient of a defect group. This conjecture had been reduced to quasi-simple groups by Murai. The case of odd primes was settled completely in our predecessor paper. Here, we investigate the 2-blocks of finite quasi-simple classical groups.

1. Introduction

The arithmetic nature of the irreducible character degrees of a finite group has been a fruitful area of research since the very beginnings of the subject. Several deep conjectures in character theory concern the p-parts occurring in character degrees, like for example the McKay conjecture and the Dade conjecture. In 1996 Geoffrey Robinson [13] proposed an extension of Richard Brauer's famous height zero conjecture from 1955, bounding the maximal power of a prime p dividing the degree of an irreducible character of a finite group G in terms of invariants of its p-block:

Conjecture (Robinson). *Let G be a finite group, p a prime and $\chi \in \mathrm{Irr}(G)$ lying in a p-block of G with defect group D. Then*

$$p^{\mathrm{def}(\chi)} \geq |Z(D)| \qquad (RC)$$

with equality if and only if D is abelian.

Here, the *p-defect* of an irreducible character $\chi \in \mathrm{Irr}(G)$ is the integer $\mathrm{def}(\chi)$ such that $|G|_p = p^{\mathrm{def}(\chi)}\chi(1)_p$, where n_p denotes the p-part of an integer n.

We recently succeeded in showing this conjecture for all primes $p \geq 3$, see [8], based on Murai's reduction of (RC) to blocks of quasi-simple groups (see [8, Thm. 2.3]). Here we prove (RC) for the 2-blocks of finite quasi-simple classical groups in odd characteristic:

Theorem 1. *The 2-blocks of covering groups of finite simple linear, unitary, symplectic and orthogonal groups do not provide minimal counterexamples to Robinson's conjecture.*

Thus, by the results in [8], in order to complete the proof of Robinson's conjecture in full generality it only remains to deal with the so-called isolated 2-blocks of quasi-simple groups of exceptional Lie type in odd characteristic.

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After some preparations on 2-blocks of the general linear and unitary groups we treat the principal 2-blocks of the special linear and unitary groups in Section 3, and the remaining finite quasi-simple classical groups in Section 4; the proof of Theorem 1 is then achieved in Section 5.

Throughout the paper, we let ν denote the exponential valuation associated to 2, normalised so that $\nu(2) = 1$. For a finite group H we write $\nu(H)$ for $\nu(|H|)$.

2. The general linear and unitary groups

2.1. **Some notation and background.** Assume $q = p^f$ is a power of a prime p. Let $\overline{\mathbb{F}}$ be an algebraic closure of the finite field $\mathbb{F}_p$. As usual, $\mathbf{G} = \mathrm{GL}_n(\overline{\mathbb{F}})$ denotes the group of all invertible $n \times n$ matrices over $\overline{\mathbb{F}}$. Let $\gamma : \mathbf{G} \to \mathbf{G}$ be the map sending A to $(A^{-1})^t$, where t denotes the transpose of matrices. If F_p is the Frobenius map of $\overline{\mathbb{F}}$ and $F_q = (F_p)^f$, then for $\eta \in \{\pm 1\}$, $F = \gamma^{\frac{1-\eta}{2}} F_q$ induces a Steinberg endomorphism of $\mathbf{G}$ with the finite group of fixed points $\mathbf{G}^F = \mathrm{GL}_n(\eta q)$. Recall that $\mathrm{GL}_n(-q)$ denotes the general unitary group

$$\mathrm{GU}_n(q) = \{ A \in \mathrm{GL}_n(q^2) \mid F_q(A)^t A = I_n \},$$

where I_n is the identity matrix of degree n. We will use the analogous notation $\mathrm{SL}_n(\eta q)$ for $\mathrm{SL}_n(q)$ or $\mathrm{SU}_n(q)$. Denote $\mathbb{F} = \mathbb{F}_{\eta q}$ for $\mathbb{F}_q$ or $\mathbb{F}_{q^2}$, depending on $\eta = \pm 1$.

Let $\mathrm{Irr}(\mathbb{F}[X])$ be the set of monic irreducible polynomials in $\mathbb{F}[X]$ different from X. Denote

$$\mathcal{F}_1 := \{ f \in \mathrm{Irr}(\mathbb{F}_{q^2}[X]) \mid f = \tilde{f} \},$$
$$\mathcal{F}_2 := \{ f\tilde{f} \mid f \in \mathrm{Irr}(\mathbb{F}_{q^2}[X]),\ f \neq \tilde{f} \},$$

where $\tilde{}$ is the permutation of $\mathrm{Irr}(\mathbb{F}_{q^2}[X])$ of order 2 mapping $f(X) = X^m + a_{m-1}X^{m-1} + \cdots + a_0$ to $F_q(a_0^{-1} X^m f(X^{-1}))$. Let

$$\mathcal{F} = \begin{cases} \mathrm{Irr}(\mathbb{F}[X]) & \text{if } \eta = 1; \\ \mathcal{F}_1 \cup \mathcal{F}_2 & \text{if } \eta = -1. \end{cases}$$

As introduced in [9, §1], the polynomials in $\mathcal{F}$ serve as the "elementary divisors" of semisimple elements of $\mathrm{GL}_n(\eta q)$: Given a semisimple element s of $G = \mathrm{GL}_n(\eta q)$, let $\prod_{f \in \mathcal{F}} s_f$ be the primary decomposition of s and $\prod_{f \in \mathcal{F}} L_f$ the corresponding decomposition of the Levi subgroup $L := C_G(s)$. Then $n = \sum_{f \in \mathcal{F}} m_f d_f$, where m_f is the multiplicity of f in the characteristic polynomial of s and d_f denotes the degree of $f \in \mathcal{F}$. Notice that L_f is isomorphic to $\mathrm{GL}_{m_f}((\eta q)^{d_f})$. Correspondingly, the Weyl group W of L can be decomposed as $\prod_f W_f$, where W_f is the Weyl group of L_f.

From now on, we assume p is an odd prime. Let $a := \nu(q^2 - 1) - 1$. We give some elementary lemmas which will be needed in the sequel:

Lemma 2.1. *Let $d = 2^\alpha m$ where m is odd and $\alpha \geq 0$.*

(a) *If $4|(q - \eta)$, then $\nu((\eta q)^d - 1) = a + \alpha$ and $\nu((\eta q)^d + 1) = 1$.*

(b) *Suppose $4|(q + \eta)$. Then, if $\alpha \geq 1$, $\nu((\eta q)^d - 1) = a + \alpha$ and $\nu((\eta q)^d + 1) = 1$; while if $\alpha = 0$, $\nu((\eta q)^m + 1) = a$ and $\nu((\eta q)^m - 1) = 1$.*

Note that, if c is the multiplicative order of some root of $f \in \mathcal{F}$, then $c|((\eta q)^{d_f} - 1)$.

Lemma 2.2. *Let $f \in \mathcal{F}$. Suppose that f has a root of order 2^m, where $m \geq 1$. Then $d_f = 2^{m'}$, where*

$$m' = \begin{cases} m - a & \text{if } m > a, \\ 1 & \text{if } 4|(q+\eta) \text{ and } 1 < m \leq a, \\ 0 & \text{otherwise.} \end{cases}$$

Proof. By definition, we have $\nu((\eta q)^{2^{m'}} - 1) \geq m$, so 2^m divides $(\eta q)^{2^{m'}} - 1$. However, since d_f is the minimal integer such that $2^m|((\eta q)^{d_f} - 1)$, we conclude that $d_f|2^{m'}$.

We now show that $2^{m'} \mid d_f$. If $4|(q-\eta)$, then $\nu((\eta q)^{d_f} - 1) = a + \nu(d_f)$. Thus $2^{m'}|d_f$ and the assertion follows. So, we assume $4|(q+\eta)$. Then we fall into the following three cases: $m > a$; $1 < m \leq a$; $m = 1$. The proof is completed by a case by case checking. □

Lemma 2.3. *Let $a \geq 2$ and $n = 2^{b_1} + \cdots + 2^{b_t}$ the 2-adic expansion of a positive integer n with $b_1 < \cdots < b_t$. Suppose $n = \sum_{i\geq 0} s_i 2^i$, where all s_i are non-negative integers. Then:*

(a) $\sum_i s_i \geq t$;

(b) $a(\sum_i s_i - t) + \sum_{i \geq 0} i s_i \geq \sum_{i=1}^t b_i$;

(c) *if $n \geq 4$ and $s_0 \neq 0$, then $\sum_i s_i - t \geq b_1$.*

In all three cases equality occurs if and only if $n = \sum_i s_i 2^i$ is the 2-adic expansion.

(d) *Let $i_0 = \min\{i \mid s_i > 0\}$ and $k = \min\{a, i_0\}$. Then $a(\sum_i s_i - t) + \sum_i i s_i > k + 1$ unless $n = \sum_i s_i 2^i$ is the 2-adic expansion and either $n = 3$ or $n = 2^h$ with $h \leq a$.*

Proof. By considering $\nu(n)$ we have $i_0 := \min\{i \mid s_i > 0\} \leq b_1$. Then (a) and (b) follow by applying the inductive hypothesis to

$$n - 2^{i_0} = (s_{i_0} - 1)2^{i_0} + \sum_{i > i_0} s_i 2^i = 2^{i_0} + \cdots + 2^{b_1 - 1} + \sum_{i=2}^t 2^{b_i}.$$

We prove (c). If $b_1 = 0$, it is just (a). Thus we assume $b_1 \neq 0$. Then $n - 1 = 1 + \cdots + 2^{b_1-1} + 2^{b_2} + \cdots + 2^{b_t}$. Since $s_0 \neq 0$, $n - 1 = (s_0 - 1) + \sum_{i \neq 0} s_i 2^i$, and thus the assertion follows from (a).

For (d), $a \sum_i (s_i - t) + \sum_i i s_i > k$ follows from (b). Now note that since $a \geq 2$, by a similar argument as in the first paragraph of the proof of [8, Lemma 6.5], if $n = \sum_i s_i 2^i$ is not the 2-adic expansion, then

$$a(\sum_i s_i - t) + \sum_i i s_i \geq a + \sum_i i s_i > k + 1$$

as claimed. Now we assume that $n = \sum_i s_i 2^i$ is the 2-adic expansion. If $t > 1$, then $\sum_i b_i > b_2 + 1 \geq k + 1$ unless $n = 3$. □

2.2. **Sylow 2-subgroups of** $\mathrm{GL}_n(\eta q)$**.** Assume $4|(q-\eta)$. Then by Lemma 2.1, $\nu(q-\eta) = a$. Let R_+^0 be a Sylow 2-subgroup of $\mathrm{GL}_1(\eta q)$. For $b \geq 1$ let R_+^b be the Sylow 2-subgroup $R_+^0 \wr X_b$ of $\mathrm{GL}_{2^b}(\eta q)$ where X_b denotes a Sylow 2-subgroup of the group of permutation matrices of degree 2^b, which is the iterated wreath product $C_2 \wr \cdots \wr C_2$ of b copies of a cyclic group C_2 of order 2. In particular, $\nu(R_+^b) = a2^b + \nu(2^b!)$.

On the other hand, when $4|(q+\eta)$, we denote by R_-^1 the Sylow 2-subgroup of $\mathrm{GL}_2(\eta q)$ as defined in [5], isomorphic to the semi-dihedral group of order 2^{a+2}. For $b \geq 2$ we denote $R_-^b = R_-^1 \wr X_{b-1} \le \mathrm{GL}_{2^b}(\eta q)$, where X_b is as defined above. For convenience we set $R_-^0 := \{\pm 1\}$, the Sylow 2-subgroup of $\mathrm{GL}_1(\eta q)$.

We write the 2-adic expansion of n as $n = 2^{b_1} + \cdots + 2^{b_t}$ with $b_1 < \cdots < b_t$. We recall some facts on 2-blocks and the structure of Sylow 2-subgroups of $\mathrm{GL}_n(\eta q)$; for this, recall that according to Lusztig the set of irreducible characters of a finite reductive group G is partitioned into *Lusztig series* $\mathcal{E}(G,s)$ labelled by semisimple elements s of the Langlands dual group G^*, up to conjugation (see e.g. [7, §13]). The elements of $\mathcal{E}(G,1)$ are the *unipotent characters of* G. Lusztig's *Jordan decomposition* states that $\mathcal{E}(G,s)$ is in bijection with $\mathcal{E}(C_{G^*}(s),1)$ whenever s has connected centraliser in the ambient algebraic group, in such a way that character degrees differ by a factor of $|G^* : C_{G^*}(s)|_{p'}$. For s a semisimple $2'$-element one sets

$$\mathcal{E}_2(G,s) := \bigcup_t \mathcal{E}(G,st)$$

where the union runs over 2-elements $t \in C_{G^*}(s)$. This is known to be a union of 2-blocks of G. For $s = 1$ this contains the trivial character and hence the principal 2-block of G.

Theorem 2.4. *Let $G = \mathrm{GL}_n(\eta q)$ where q is odd, and $\epsilon \in \{\pm\}$ such that $q \equiv \epsilon\eta \pmod 4$.*

(a) *The only unipotent 2-block of G is the principal block $\mathcal{E}_2(G,1)$.*

(b) *A Sylow 2-subgroup of G is given by*

$$R = \prod_{i=1}^{t} R_\epsilon^{b_i}.$$

Thus, R is abelian if and only if $n = 1$.

(c) *The center of R is given by*

$$Z(R) = \prod_{i=1}^{t} R_\epsilon^0 \otimes I_{2^{b_i}}.$$

In particular, $\nu(Z(R)) = t\,\nu(q-\eta)$.

Proof. Part (a) was shown by Broué [2], and (b) follows from [5]. From this (c) can easily be derived. □

2.3. **Robinson's conjecture for** $\mathrm{GL}_n(\eta q)$**.** Now we prove Robinson's conjecture for the principal 2-block of $\mathrm{GL}_n(\eta q)$.

Lemma 2.5. *Let χ be a unipotent character of $\mathrm{GL}_n(\eta q)$. Then $\mathrm{def}(\chi) \ge n\,\nu(q-\eta)$.*

Proof. For $G = \mathrm{GL}_n(q)$ all unipotent characters lie in the principal series. Thus their degree polynomials are not divisible by $q-1$ (see [4, §13.7]) while the order polynomial of G is divisible by $(q-1)^n$, so we obtain the stated bound. The claim for $\mathrm{GU}_n(q)$ follows as the order polynomial as well as the degree polynomials of unipotent characters are obtained by replacing q by $-q$ in those for $\mathrm{GL}_n(q)$. □

Proposition 2.6. *Let $G = \mathrm{GL}_n(\eta q)$, $n \ge 2$, let B be the principal 2-block of G, and R a Sylow 2-subgroup of G. Then $\mathrm{def}(\chi) > \nu(Z(R))$ for any $\chi \in \mathrm{Irr}(B)$.*

Proof. By Theorem 2.4(a) there is a semisimple 2-element $s \in G$ such that $\chi \in \mathcal{E}(G, s)$. For $b \geq 0$ define $\mathcal{F}_b = \{f \in \mathcal{F} \mid d_f = 2^b\}$. Let $\mathcal{F}(s) \subset \mathcal{F}$ be the set of elementary divisors of s and set $\mathcal{F}_b(s) := \mathcal{F}(s) \cap \mathcal{F}_b$. Then by Lemma 2.2 we have $\mathcal{F}(s) = \coprod_b \mathcal{F}_b(s)$ and thus

$$n = \sum_{f \in \mathcal{F}} m_f d_f = \sum_{b \geq 0} \sum_{f \in \mathcal{F}_b} m_f 2^b.$$

Let ψ be the unipotent character of $L := C_G(s)$ in Jordan correspondence with χ. Let $\prod_{f \in \mathcal{F}(s)} L_f$, $\boxtimes_{f \in \mathcal{F}(s)} \psi_f$ be the respective decompositions of L, ψ, corresponding to the primary decomposition of s. By the degree formula for the Jordan decomposition of characters we have

$$\nu(\chi(1)) = \nu(G : L) + \nu(\psi(1))$$

and so, as $\nu(L) = \sum_{f \in \mathcal{F}(s)} \nu(L_f)$,

$$\operatorname{def}(\chi) = \operatorname{def}(\psi) = \sum_{f \in \mathcal{F}(s)} \operatorname{def}(\psi_f).$$

As before, write $n = 2^{b_1} + \cdots + 2^{b_t}$ for the 2-adic expansion of n.

Let first $4|(q - \eta)$, so $\nu(q - \eta) = a$. By Lemmas 2.1 and 2.5,

$$\operatorname{def}(\psi_f) \geq m_f(a + b) \qquad \text{for } f \in \mathcal{F}_b(s).$$

It follows by Lemma 2.3 that

$$\operatorname{def}(\chi) \geq a \sum_{f \in \mathcal{F}} m_f + \sum_b b \sum_{f \in \mathcal{F}_b} m_f \geq at + \sum_{i=1}^{t} b_i > at \tag{1}$$

as $n > 1$. By Theorem 2.4(c) we know that $\nu(Z(R)) = at$, whence the claim in this case.

Now assume that $4|(q + \eta)$. Here by Lemma 2.3(a),

$$\begin{aligned}\operatorname{def}(\chi) &\geq \sum_{f \in \mathcal{F}_0} m_f + \sum_{b \geq 1} \sum_{f \in \mathcal{F}_b} m_f(a + b) \\ &= \sum_{f \in \mathcal{F}} m_f + \sum_{b \geq 1} \sum_{f \in \mathcal{F}_b} m_f(a + b - 1) \geq t + \sum_{b \geq 1} \sum_{f \in \mathcal{F}_b} m_f(a + b - 1),\end{aligned}$$

with equality in the last line if and only if $n = \sum_b \sum_{f \in \mathcal{F}_b} m_f 2^b$ is the 2-adic expansion of n. Furthermore, as $a \geq 2$ the last sum is zero only when $\mathcal{F}_b(s) = \emptyset$ for $b \geq 1$. But in this case $n = 1$, which was excluded, so we conclude by Theorem 2.4(c). □

We note the following for later use.

Remark 2.7. In the notation of Proposition 2.6 assume that $4|(q + \eta)$. If $\operatorname{def}(\chi) - \nu(Z(R)) = 1$ and n is even then the proof of Proposition 2.6 shows that $n = 2$.

3. The groups $\mathrm{SL}_n(\eta q)$

3.1. **Robinson's conjecture for** $\mathrm{SL}_n(\eta q)$**,** n **odd.** The case of $\mathrm{SL}_n(\eta q)$ with n odd is considerably easier than the even degree case.

Proposition 3.1. *Let $H = \mathrm{SL}_n(\eta q)$ with $n \geq 3$ odd and $Z \leq Z(H)$. Then Robinson's conjecture holds for the unipotent 2-blocks of H/Z.*

Proof. Set $G = \mathrm{GL}_n(\eta q)$. Since n is odd, $H \cap \mathbf{O}_2(Z(G)) = 1$, so $G = H_1 \times \mathbf{O}_2(Z(G))$ with $H_1 = \mathbf{O}^2(G)$. Obviously all irreducible characters of G restrict irreducibly to H_1 and the Sylow 2-subgroups of G are the direct products of Sylow 2-subgroups of H_1 with $\mathbf{O}_2(Z(G))$. Thus, if b_1 is a 2-block of H_1 and B is the 2-block of G covering b_1, then Robinson's conjecture holds for b_1 if and only if it holds for B.

Furthermore, $|H_1 : H|$ is odd and so for $\chi \in \mathrm{Irr}(H_1)$ all constituents of $\chi|_H$ have the same defect as χ. Thus, if b is a 2-block of H and b_1 is a 2-block of H_1 covering b then Robinson's conjecture holds for b if and only if it holds for b_1.

Finally, $|Z(H)| = \gcd(n, q-1)$ is odd. Thus if $\bar{b}$ is a 2-block of H/Z, where $Z \leq Z(H)$, and b is the 2-block of H dominating $\bar{b}$, then Robinson's conjecture holds for $\bar{b}$ if and only if it holds for b. The claim thus follows from Proposition 2.6. $\square$

3.2. **Sylow 2-subgroups of $\mathrm{SL}_n(\eta q)$ for n even.** For $n \geq 2$ even, we first determine the centers of Sylow 2-subgroups. Denote $c = \nu(q-\eta)$ and, for $b \geq 0$, set $c(b) := \max\{c-b, 0\}$. Note that $c = a$ if $4|(q-\eta)$ and $c = 1$ if $4|(q+\eta)$.

For a subgroup $H \leq \mathrm{GL}_n(\mathbb{F})$ we set

$$\mathcal{D}(H) := \{\det(A) \mid A \in H\} \leq \mathbb{F}^\times.$$

Lemma 3.2. *Let $b \geq 0$. In the notation of Theorem 2.4 we have*

$$|\mathcal{D}(R_\epsilon^b)| = 2^c \quad \textit{and} \quad |\mathcal{D}(Z(R_\epsilon^b))| = 2^{c(b)}.$$

Proof. If $4|(q-\eta)$ then R_+^b contains $\mathrm{diag}(\zeta, 1, \ldots, 1)$ with $\zeta \in \mathbb{F}^\times$ of order $o(\zeta) = 2^c$, so $|\mathcal{D}(R_+^b)| = 2^c$. Then $|\mathcal{D}(Z(R_+^b))| = 2^{c(b)}$ follows from $\mathcal{D}(Z(R_+^b)) = \{x^{2^b} \mid x \in \mathcal{D}(R_+^0)\}$.

For $\epsilon = -$ this follows from the fact that R_-^1 is conjugate to $\left\langle \begin{pmatrix} \zeta & \\ & \zeta^{\eta q} \end{pmatrix}, \begin{pmatrix} & 1 \\ 1 & \end{pmatrix} \right\rangle$ for $\zeta \in \mathbb{F}^\times$ of order $o(\zeta) = 2^{a+1}$. $\square$

Lemma 3.3. *Let $G = \mathrm{GL}_n(\eta q)$ and $H = \mathrm{SL}_n(\eta q)$ with n even. Suppose that R is a Sylow 2-subgroup of G as in Theorem 2.4(b). Write $Q = R \cap H$. Then $|\mathcal{D}(R)| = 2^c$. Furthermore,*

$$|\mathcal{D}(Z(R))| = 2^{c(b_1)} \quad \textit{and} \quad \nu(R : QZ(R)) = c - c(b_1) = \min\{c, b_1\}.$$

Proof. This follows easily from Lemma 3.2. $\square$

When $4|(q-\eta)$, the Weyl group part of R may not be contained in $\mathrm{SL}_n(\eta q)$. To circumvent this problem, for $b \geq 1$ we will use that $R_+^b = R_+^0 \wr X_b \cong R_+^1 \wr X_{b-1}$, and call X_{b-1} the *Weyl part* and $(R_+^1)^{2^{b-1}}$ the *base group of* R_+^b. Observe that with this convention the Weyl part of R_+^b is contained in $\mathrm{SL}_n(\eta q)$.

Lemma 3.4. *Let $G = \mathrm{GL}_n(\eta q)$ and $H = \mathrm{SL}_n(\eta q)$ with n even. Suppose that R is as in Theorem 2.4(b). Let $R_{0,i}$ be the base group and $R_{W,i} = X_{b_i-1}$ the Weyl part of $R_\epsilon^{b_i}$. Write $R_0 = R_{0,1} \times \cdots \times R_{0,t}$ and $R_W = R_{W,1} \times \cdots \times R_{W,t}$. Denote $Q = R \cap H$. Then*

$$Q = (R_0 \cap H) \rtimes R_W \qquad \textit{and} \qquad Z(Q) = Z(R) \cap H.$$

More specifically, if $R = R_\epsilon^1$, then Q is isomorphic to the generalised quaternion group of order 2^{a+1} and so $Z(Q) = \{\pm I_2\}$.

Proof. The Weyl part $R_{W,i}$ is generated by direct products of tensor products of matrices of the form $\begin{pmatrix} \mathbf{0} & I_2 \\ I_2 & \mathbf{0} \end{pmatrix}$, whence $R_{W,i} \leq H$. Hence $R_W \leq H$, and so $Q = (R_0 \cap H) \rtimes R_W$.

By Lemma 3.2, $\mathcal{D}(R_\epsilon^0) = \mathcal{D}(R_\epsilon^1)$. If $t \geq 2$, then the projections $p_i : Q \to R_\epsilon^{b_i}$ are surjective for all i. Hence $Z(Q) \leq Z(R)$, and so $Z(Q) \leq Z(R) \cap H$. Since the converse inclusion is obvious, we get $Z(Q) = Z(R) \cap H$ in this case.

So now we may assume $R = R_\epsilon^{b+1}$, with $b \geq 0$. If $b = 0$, the assertion is obvious since Q is isomorphic to the generalised quaternion group of order 2^{a+1}. So we may assume $b > 0$. Let $(A_1, \ldots, A_{2^b})\tau$ be an element of $Z(Q)$ with $(A_1, \ldots, A_{2^b}) \in R_0$ and $\tau \in X_b$. Let B be any element of R_ϵ^1 with $\det(B) = \pm 1$, so that $(B, \ldots, B) \in Q$. Now $[(A_1, \ldots, A_{2^b})\tau, (B, \ldots, B)] = 1$. Hence $(A_1, \ldots, A_{2^b}) \in Z(R)$. Moreover, since $[(A_1, \ldots, A_{2^b})\tau, \pi] = 1$ for any $\pi \in X_b$ and X_b transitively permutes the A_i, we have $A_1 = \ldots = A_{2^b}$. It follows that $\tau \in Z(Q)$ and $\tau = 1$, and so $Z(Q) = Z(R) \cap H$, completing the proof. □

3.3. **Robinson's conjecture for** $\mathrm{SL}_n(\eta q)$**,** n **even.** Now we prove Robinson's conjecture for unipotent 2-blocks of $\mathrm{SL}_n(\eta q)$ with even n.

Proposition 3.5. *Let $H = \mathrm{SL}_n(\eta q)$ with $n \geq 2$ even, and let Q be a Sylow 2-subgroup of H. Then for every $\theta \in \mathcal{E}_2(H, 1)$ we have $\mathrm{def}(\theta) > \nu(Z(Q))$.*

Proof. Let $s \in H^*$ be a 2-element such that $\theta \in \mathcal{E}(H, s)$. Let $G = \mathrm{GL}_n(\eta q)$ and let $\chi \in \mathcal{E}_2(G, 1)$ lie above θ. By Theorem 2.4(a), χ lies in the principal 2-block of G, so θ lies in the principal 2-block of H. Let R be a Sylow 2-subgroup of G as described in Theorem 2.4(b) and with $R \cap H = Q$.

Now we have

$$\mathrm{def}(\theta) = \nu(H) - \nu(\theta(1)) = \mathrm{def}(\chi) - \nu(q - \eta) + \nu(\chi(1)/\theta(1))$$

and

$$\nu(Z(Q)) = \nu(Z(R)) - \nu(R : Q) + \nu(R : QZ(R)) = \nu(Z(R)) - \nu(q - \eta) + \min\{c, b_1\}$$

by Lemmas 3.3 and 3.4.

We first let $4|(q - \eta)$. Then $\mathrm{def}(\chi) - \nu(Z(R)) \geq \sum_{i=1}^t b_i$ by (1), so

$$\begin{aligned} \mathrm{def}(\theta) &\geq \nu(Z(R)) - \nu(q - \eta) + \sum_{i=1}^t b_i + \nu(\chi(1)/\theta(1)) \\ &= \nu(Z(Q)) - \min\{c, b_1\} + \sum_{i=1}^t b_i + \nu(\chi(1)/\theta(1)) \geq \nu(Z(Q)) \end{aligned}$$

with equality only when $t = 1$, $c \geq b_1 =: b$, and so $n = 2^b$. In that case there is exactly one element f in $\mathcal{F}(s)$, and $m_f(s) = 1$, $d_f = 2^b = n$. Therefore, in order to complete the proof, it suffices to show that $\nu(\chi(1)/\theta(1)) > 0$. To do this, we compute the number m of irreducible constituents of $\chi|_H$ according to the method of Denoncin [6]. Let ζ be a root of f in $\overline{\mathbb{F}}$. Since G/H is cyclic and isomorphic to $Z(G)$, it follows by [6, Prop. 3.5] that

$$m = |\{z \in Z(G) \mid zs \text{ is } G\text{-conjugate to } s\}|.$$

Now

$$\begin{aligned} m &= |\{\tau \in \mathbb{F}^\times \mid z = \operatorname{diag}(\tau, \ldots, \tau) \in Z(G),\ \tau\zeta \text{ is a root of } f \text{ in } \overline{\mathbb{F}}\}| \\ &= |\{\tau \in \mathbf{O}_2(\mathbb{F}^\times) \mid \tau\zeta \text{ is a root of } f \text{ in } \overline{\mathbb{F}}\}|. \end{aligned}$$

Notice that the roots of f are $\zeta, \zeta^{\eta q}, \ldots, \zeta^{(\eta q)^{n-1}}$. Since $n = 2^b$, we may write $\zeta^{(\eta q)^{\frac{n}{2}}} = \xi\zeta$ with $\xi = \zeta^{(\eta q)^{2^{b-1}}-1} \neq 1$. We claim $\xi^2 = 1$. Indeed, ζ has multiplicative order 2^{a+b}. In addition, by Lemma 2.1 we have $\nu((\eta q)^{2^b} - 1) = \nu((\eta q)^{2^{b-1}} - 1) + 1$. Hence ξ has multiplicative order 2. Thus $\nu(m) > 0$, and we have $\nu(\chi(1)/\theta(1)) > 0$ by Clifford theory.

Now we assume $4|(q+\eta)$. If $Z(R) \nleq H$, then $\nu(R : QZ(R)) = 0$, hence the claim holds by Proposition 2.6. So we may assume $Z(R) \leq H$. Then $\nu(R : QZ(R)) = 1$, and so the claim holds if $\operatorname{def}(\chi) - \nu(Z(R)) \geq 2$. By Remark 2.7, we have $\operatorname{def}(\chi) - \nu(Z(R)) = 1$ only when $n = 2$. But in this case $|Z(Q)| = 2$, and so Robinson's conjecture holds by an old result of Brauer (see [8, Lemma 3.1]). □

3.4. **Robinson's conjecture for central quotients of** $\mathrm{SL}_n(\eta q)$**,** n **even.** In this section, we investigate unipotent 2-blocks of $\mathrm{SL}_n(\eta q)/Z$ for odd q, where $Z \leq Z(\mathrm{SL}_n(\eta q))$. Throughout, $n \geq 2$ is even. As before, we first determine the centers of Sylow 2-subgroups of central quotient groups of $\mathrm{SL}_n(\eta q)$.

Lemma 3.6. *Keep the notation in Lemma 3.4; in particular, n is even. For $1 \neq Z \leq \mathbf{O}_2(Z(H))$ write $\bar{Q} = Q/Z$ and $Z(\bar{Q}) = Z_0/Z$.*

(a) *If $t \geq 2$, then $Z_0 = Z(Q)$.*
(b) *Assume $R = R_\epsilon^b$.*
  (1) *If $b = 1$, then Q/Z is isomorphic to the dihedral group of order 2^a.*
  (2) *If $b > 1$, then $Z_0 = \langle Z(Q), I_{2^b} \otimes \operatorname{diag}(1, -1)\rangle$.*

Thus, if $\bar{Q}$ is not abelian, then $Z(\bar{Q}) = Z(Q)/Z$ unless either $R = R_\epsilon^1$ with $a > 2$, or $R = R_\epsilon^b$ with $b > 1$, in which cases we have $\nu(Z(\bar{Q})) = \nu(Z(Q)) - \nu(Z) + 1$.

Proof. Recall that $|\mathcal{D}(R_-^1)| = 2$ and $\mathcal{D}(R_+^0) = \mathcal{D}(R_+^1)$ by Lemma 3.2.

(a) Suppose $t \geq 2$. We first claim that $Z_0 \subseteq R_0$, where R_0 is the base group of R as in Lemma 3.4. Otherwise, let $(A_1, A_2, \ldots)\tau$ be an element of Z_0, where $1 \neq \tau \in R_W$ and A_i is an element of $R_{0,i}$ for each i; here, $R_{0,i}$ is the base subgroup of $R_\epsilon^{b_i}$ as in Lemma 3.4. Since R has more than one component, arguing as in the last paragraph of the proof of Lemma 3.4, we have that $(A_1, A_2, \ldots) \in Z(R)$. We may take a non-trivial orbit of $\langle\tau\rangle$, say $i_1, i_2 = \tau^{-1}(i_1), \ldots$, and a j outside this orbit (such j exists since $t \geq 2$). In addition, take $(B_1, B_2, \ldots) \in Q$, where $B_i = 1$ for all i except that B_{i_1} and B_j satisfy that the order of B_{i_1} is greater than 2 and $\det(B_{i_1})\det(B_j) = 1$. Then direct calculation shows that $[(A_1, A_2, \ldots)\tau, (B_1, B_2, \ldots)] \notin Z$, which is a contradiction. Thus $Z_0 \subseteq R_0$, as claimed.

Now let $A = (A_1, \ldots, A_t) \in Z_0$, where $A_i \in R_{0,i}$. Then obviously, each component of A is in the center of the corresponding component of Q, and thus $Z_0 = Z(Q)$.

(b) The statement for $b = 1$ is well-known. So assume that $b \geq 2$. Then a slight modification of the argument in (a) shows that $Z_0 \subseteq R_0$, and then the claim follows as there. □

Proposition 3.7. *Let $H = \mathrm{SL}_n(\eta q)$ with $n \geq 2$ even and $1 \neq Z \leq \mathbf{O}_2(Z(H))$. If a Sylow 2-subgroup $\bar{Q}$ of H/Z is non-abelian, then for every ϕ in the principal 2-block of H/Z we have* $\mathrm{def}(\phi) > \nu(Z(\bar{Q}))$.

Proof. Write θ for the inflation of ϕ to H, then θ lies in the principal 2-block of H. It is easy to see that $\mathrm{def}(\phi) = \mathrm{def}(\theta) - \nu(Z)$. Let $G = \mathrm{GL}_n(\eta q)$ and R a Sylow 2-subgroup of G and $Q = R \cap H$, a Sylow 2-subgroup of H. Let $\chi \in \mathcal{E}(G, s)$ with $s \in R$ be such that θ is an irreducible constituent of $\chi|_H$.

We first let $4|(q - \eta)$. If $t \geq 2$, then by Lemma 3.6, $\nu(Z(\bar{Q})) = \nu(Z(Q)) - \nu(Z)$. Hence the result follows from Proposition 3.5. So now we may assume $t = 1$, that is, $R = R_+^b$. Since $\bar{Q}$ is non-abelian, we have $b > 0$. Now Lemma 3.6 yields $\nu(Z(\bar{Q})) = \nu(Z(Q)) - \nu(Z) + 1$.

By the proof of Proposition 3.5, $\mathrm{def}(\theta) > \nu(Z(Q)) + 1$ unless $n = \sum_{f \in \mathcal{F}_b(s)} m_f 2^b$ is the 2-adic expansion and $b \leq a$. Hence $\mathcal{F}_b(s) = \{f\}$, $m_f = 1$ and $o(s) = 2^{a+b}$. Notice that $L = C_G(s)$ is in fact a Coxeter torus of G, and s generates the Sylow 2-subgroup of L, and hence of $H^*/[H^*, H^*]$. But then all characters in $\mathcal{E}(H, s)$ are faithful on $\mathbf{O}_2(Z(H))$. However, this contradicts the obvious fact that $Z \leq \ker(\theta) = \ker(\chi)$. This achieves the proof when $4|(q - \eta)$.

Now assume that $4|(q + \eta)$. As before, if $\nu(Z(\bar{Q})) = \nu(Z(Q)) - \nu(Z)$ then the assertion follows from Proposition 3.5. So by Lemma 3.6 it remains to consider the case $\nu(Z(\bar{Q})) = \nu(Z(Q)) - \nu(Z) + 1$, *i.e.*, either $R = R_-^1$ with $a > 2$, or $R = R_-^b$ with $b \geq 2$. Here, $|Z(\bar{Q})| = 2$ by Lemma 3.6 and so Robinson's conjecture holds by [8, Lemma 3.1]. □

4. Principal 2-blocks of quasi-simple groups of classical type

Let $\mathbf{G}$ be a simple algebraic group of symplectic or orthogonal type over an algebraically closed field of odd characteristic and F a Frobenius endomorphism of $\mathbf{G}$ with respect to an $\mathbb{F}_q$-rational structure, and denote $G = \mathbf{G}^F$. So in particular q is an odd prime power. Throughout we will fix the prime $\ell = 2$, with respect to which defects will be considered. As for the linear and unitary groups, we will need three pieces of information: defects of unipotent characters, centralisers of 2-elements and the centers of Sylow 2-subgroups.

4.1. Unipotent characters of classical groups.

Here we determine lower bounds on defects of unipotent characters of classical groups. Observe that by Lusztig's results the classification and degrees of unipotent characters are insensitive to the isogeny type (see [4, §13.7]), so for our purposes we will not need to specify these here.

Lemma 4.1. *Let χ be a cuspidal unipotent character of a finite group G of classical type. Let 2^b be the precise power of 2 dividing $q + 1$. Then*

$$\mathrm{def}(\chi) = \begin{cases} (b+1)n & \text{for types } B_n(q), C_n(q), \\ (b+1)n - 1 & \text{for types } D_n(q), {}^2D_n(q). \end{cases}$$

Proof. We discuss the various types individually. If $G = B_n(q)$ or $C_n(q)$ then by [4, 13.7] we have that $n = s^2 + s$ for some $s \geq 1$ and according to the formula given in loc. cit.,

$$\mathrm{def}(\chi) = s + b\sum_{i=1}^{s} 2i + \sum_{i=1}^{s}(2i - 1) = s + b(s^2 + s) + s^2 = (b + 1)n.$$

If $G = D_n(q)$ or ${}^2D_n(q)$ then we have that $n = s^2$ for some integer s (which is even in the first case, odd in the second), and

$$\mathrm{def}(\chi) = s - 1 + b\sum_{i=1}^{s}(2i-1) + \sum_{i=1}^{s-1} 2i = s - 1 + bs^2 + s^2 - s = (b+1)n - 1. \quad \square$$

Lemma 4.2. *Let χ be a unipotent character of a finite classical group G.*

(a) *If G is of type $B_n(q), C_n(q)$ with $n \geq 1$, or of type $D_n(q), {}^2D_n(q)$ with $n \geq 2$ then $\mathrm{def}(\chi) > n$.*

(b) *If G is of type $D_n(q^2)$ or ${}^2D_n(q^2)$ with $n \geq 2$ then $\mathrm{def}(\chi) \geq 2n-1$, with equality only possibly when n is a square.*

Proof. Assume that χ lies in the Harish-Chandra series of the cuspidal unipotent character λ of a Levi subgroup $L \leq G$ for an F-stable Levi subgroup $\mathbf{L}$ of an F-stable parabolic subgroup of $\mathbf{G}$. Let 2^a, 2^b be the precise power of 2 dividing $q-1$, $q+1$ respectively. By Lusztig's classification (see [4, §13.7]), then $[\mathbf{L},\mathbf{L}]$ is simple of the same classical type as $\mathbf{G}$, hence λ is as considered in Lemma 4.1. Moreover $\chi(1)$ divides the degree $|G:L|_{p'}\lambda(1)$ of the Harish-Chandra induced character $R_L^G(\lambda)$ (as can be seen for example from [4, Thm. 10.11.5]). Using that $\mathbf{L} = [\mathbf{L},\mathbf{L}]Z^\circ(\mathbf{L})$ we conclude that $2^{\mathrm{def}(\chi)} \geq 2^{\mathrm{def}(\lambda)}|Z^\circ(\mathbf{L})^F|_2$. Now first assume that G is of type $B_n(q)$ or $C_n(q)$ and so L has type $B_u(q)$ or $C_u(q)$ for some $u \geq 1$. Then $|Z^\circ(\mathbf{L})^F| = (q-1)^{n-u}$, and so

$$\mathrm{def}(\chi) \geq a(n-u) + (b+1)u$$

by Lemma 4.1. This is linear in u and hence at least as big as the minimum of its values at $u = 0$ and $u = n$. Hence it is larger than n unless $u = 0$ and $a = 1$. In the latter case χ lies in the principal series. Our claim follows if $\chi(1)$ is not divisible by $(q+1)^n$. If it is divisible by $(q+1)^n$ then χ is 2-cuspidal, and with Ennola duality we obtain $\mathrm{def}(\chi) \geq (a+1)n > n$ from Lemma 4.1. If G has type $D_n(q)$ or ${}^2D_n(q)$, then L has type $D_u(q)$ or ${}^2D_u(q)$, giving

$$\mathrm{def}(\chi) \geq a(n-u) + (b+1)u - 1,$$

and we conclude as before. For types $D_n(q^2)$ or ${}^2D_n(q^2)$ we similarly obtain $\mathrm{def}(\chi) \geq 3n-1$ when $u = 0$ (as $8|(q^2-1)$), and $\mathrm{def}(\chi) \geq (b+1)n-1 = 2n-1$ when $u = n$ is a square. $\square$

4.2. **Centralisers of semisimple elements in classical groups.** We will make use of the primary decomposition of semisimple elements in classical groups. We follow the notation introduced in [10]. As in Section 2 let $\mathrm{Irr}(\mathbb{F}_q[X])$ be the set of non-constant monic irreducible polynomials in $\mathbb{F}_q[X]$ different from X. For $f \in \mathrm{Irr}(\mathbb{F}_q[X])$ let f^* be the polynomial in $\mathrm{Irr}(\mathbb{F}_q[X])$ whose roots (in $\overline{\mathbb{F}}_q$) are the inverses of the roots of f. Denote

$$\begin{aligned}
\mathcal{F}_0 &:= \{\ X-1, X+1\ \}, \\
\mathcal{F}_+ &:= \{\ f \in \mathrm{Irr}(\mathbb{F}_q[X]) \mid f \notin \mathcal{F}_0, f = f^*\ \}, \\
\mathcal{F}_- &:= \{\ ff^* \mid f \in \mathrm{Irr}(\mathbb{F}_q[X]), f \neq f^*\ \}
\end{aligned}$$

and $\mathcal{F} := \mathcal{F}_0 \cup \mathcal{F}_+ \cup \mathcal{F}_-$. For any integer $b \geq 1$ we also set $\mathcal{F}_b := \{f \in \mathcal{F} \mid d_f = 2^b\}$. Here, and later on, d_f denotes the degree of $f \in \mathcal{F}$. Define the *reduced degree* δ_f of $f \in \mathcal{F}$ by

$$\delta_f = \begin{cases} d_f & \text{if } f \in \mathcal{F}_0, \\ \frac{1}{2}d_f & \text{if } f \in \mathcal{F}_+ \cup \mathcal{F}_-. \end{cases}$$

Since the polynomials in $\mathcal{F}_+ \cup \mathcal{F}_-$ have even degree, δ_f is an integer. In addition, we introduce a sign ε_f for $f \in \mathcal{F}_+ \cup \mathcal{F}_-$ defined by

$$\varepsilon_f = \begin{cases} -1 & \text{if } f \in \mathcal{F}_+, \\ \ \ 1 & \text{if } f \in \mathcal{F}_-. \end{cases}$$

Let V be a finite-dimensional symplectic or orthogonal space over $\mathbb{F}_q$ with q odd and $I(V) = \mathrm{Sp}(V)$ or $\mathrm{GO}(V)$ respectively. Given a semisimple element $s \in I(V)$, there exist unique orthogonal decompositions $V = \bigoplus_{f\in\mathcal{F}} V_f$ and $s = \prod_{f\in\mathcal{F}} s_f$, where the V_f are non-degenerate subspaces of V, $s_f \in I(V_f)$, and s_f has minimal polynomial f. The above decomposition of s is often called the primary decomposition of s. Correspondingly, the centraliser of s in $I(V)$ has a decomposition $C_{I(V)}(s) = \prod_f C_f(s)$ with

$$C_f(s) := C_{I(V_f)}(s_f) = \begin{cases} I(V_f) & \text{if } f \in \mathcal{F}_0, \\ \mathrm{GL}_{m_f}(\varepsilon_f q^{\delta_f}) & \text{if } f \in \mathcal{F}_+ \cup \mathcal{F}_-, \end{cases}$$

where m_f is the multiplicity of f in the characteristic polynomial of s_f (and of s).

Now let V be an orthogonal space over $\mathbb{F}_q$ and $I^\circ(V) = \mathrm{SO}(V)$. Let $s \in I(V)$. Then $s \in I^\circ(V)$ if and only if $m_{X+1}(s)$ is even. For more details, see [10, §1]. The following is elementary, see also Lemma 2.2:

Lemma 4.3. *Let $a := \nu(q^2 - 1) - 1$. If $f \in \mathcal{F}_+ \cup \mathcal{F}_-$ has a root (in $\overline{\mathbb{F}}_q$) of order 2^m for some positive integer m, then*

$$\delta_f = \begin{cases} 1 & \text{if } m \leq a, \\ 2^{m-a} & \text{if } m > a. \end{cases}$$

Moreover, $f \in \mathcal{F}_-$ unless $4|(q+1)$ and $m \leq a$.

4.3. **Symplectic groups.**

Lemma 4.4. *Let $G = \mathrm{Sp}_{2n}(q)$ with $n \geq 2$ and q odd, and let $n = 2^{b_1} + \cdots + 2^{b_t}$ be the 2-adic expansion. Then $\mathrm{def}(\chi) \geq t+2$ for every character χ in the principal 2-block of G.*

Proof. As recalled earlier, the principal 2-block lies in $\mathcal{E}_2(G, 1)$. So let $s \in G^*$ be a 2-element such that $\chi \in \mathcal{E}(G, s)$ and let $\tilde{\psi} \in \mathcal{E}(C_{G^*}(s), 1)$ denote the Jordan correspondent of χ (see for example [3, Cor. 15.14]). Then by the degree formula for Jordan decomposition we have

$$\mathrm{def}(\chi) = \nu(|G|) - \nu(\chi(1)) = \nu(|C_{G^*}(s)|) - \nu(\tilde{\psi}(1)).$$

Let ψ be a unipotent character of $C^\circ_{G^*}(s)$ below $\tilde{\psi}$, then by Clifford theory $\tilde{\psi}(1)/\psi(1)$ divides $|C_{G^*}(s) : C^\circ_{G^*}(s)|$, so

$$\mathrm{def}(\chi) = \nu(|C_{G^*}(s)|) - \nu(\tilde{\psi}(1)) \geq \nu(|C^\circ_{G^*}(s)|) - \nu(\psi(1)) = \mathrm{def}(\psi).$$

Thus we need to discuss the defects of unipotent characters of $C^\circ_{G^*}(s)$. By our preliminary observations, $C^\circ_{G^*}(s)$ is isogenous to (and hence has the same unipotent characters as) a product of certain orthogonal, linear and unitary groups, which we will now investigate in detail.

Here, we have that $G^* = \mathrm{SO}_{2n+1}(q)$. Let V be the underlying space of G^*, $s = \prod_f s_f$ the primary decomposition, and $V = \bigoplus_f V_f$ the corresponding orthogonal decomposition of V. Then $C^\circ_{G^*}(s) = \prod_{f\in\mathcal{F}} C_f$ with $C_f = C_{I^\circ(V_f)}(s_f)$.

Observe that $m_2 := m_{X+1}$ is even. Since d_f is even for $f \in \mathcal{F}_+ \cup \mathcal{F}_-$, $m_1 := m_{X-1} \geq 1$ must be odd. Write $\mathcal{F}_b(s) := \{f \in \mathcal{F}_b \mid m_f > 0\}$. By Lemma 4.3, d_f must be a power of 2 if $m_f \neq 0$.

Now $2n + 1 = m_1 + m_2 + \sum_f m_f d_f$ and thus

$$n = \frac{m_1 - 1}{2} + \frac{m_2}{2} + \sum_{b\geq 1}\sum_{f\in\mathcal{F}_b} m_f 2^{b-1}. \tag{2}$$

Then by Lemma 2.3(a),

$$t \leq \frac{m_1 - 1}{2} + \frac{m_2}{2} + \sum_{f\in\mathcal{F}\setminus\mathcal{F}_0} m_f \tag{3}$$

and equality holds if and only if (2) is the 2-adic expansion of n.

Write $\psi = \boxtimes_f \psi_f$, where ψ_f is a unipotent character of C_f. Then by Lemma 4.2(a),

$$\mathrm{def}(\psi_{X-1}) \geq \frac{m_1 - 1}{2} \tag{4}$$

with equality only when $m_1 = 1$, and also by Lemma 4.2(a), if $m_2 > 0$, then

$$\mathrm{def}(\psi_{X+1}) \geq \frac{m_2}{2} \tag{5}$$

with equality possibly only when $m_2 = 2$. If $f \in \bigcup_{b>1} \mathcal{F}_b(s)$, then $\delta_f = 2^{b-1} \geq 2$ is a power of 2 by Lemma 4.3 and $C_f \cong \mathrm{GL}_{m_f}(q^{\delta_f})$. By Lemma 2.5,

$$\mathrm{def}(\psi_f) \geq \nu(q^{\delta_f} - 1)m_f \geq (b + 1)m_f \geq 3m_f \geq m_f + 2. \tag{6}$$

If $f \in \mathcal{F}_1(s)$, then $4|(q + 1)$, $\delta_f = 1$ and $C_f \cong \mathrm{GL}_{m_f}(-q)$, so by Lemma 2.5,

$$\mathrm{def}(\psi_f) \geq m_f\nu(q + 1) \geq 2m_f \geq m_f + 1. \tag{7}$$

By (3), (4), (5), (6) and (7), we can now compare $\mathrm{def}(\chi)$ with $t+2$. If $\bigcup_{b>1} \mathcal{F}_b(s) \neq \emptyset$, then $\mathrm{def}(\chi) \geq t + 2$ by (6). So now assume that $\mathcal{F}_b(s) = \emptyset$ for $b > 1$. Then $n = \frac{m_1-1}{2} + \frac{m_2}{2} + \sum_{f\in\mathcal{F}_1(s)} m_f$ and hence equality does not hold in (3). If $n \geq 4$, then $n-t \geq 3$, and then $\mathrm{def}(\chi) \geq t + 3$ holds. On the other hand when $n = 2, 3$, then $n - t = 1$. If $\mathcal{F}_1(s) \neq \emptyset$, then $\mathrm{def}(\chi) \geq t+2$ holds by (7). So we assume that $\mathcal{F}_1(s) = \emptyset$. If $m_1 > 1$, then the result follows from (4). Hence we may assume further that $m_1 = 1$. Then $n = \frac{m_2}{2}$, and $\mathrm{def}(\chi) \geq t + 2$ holds by (5). This completes the proof. □

Let W be a Sylow 2-subgroup of $\mathrm{Sp}_2(q) = \mathrm{SL}_2(q)$. Clearly, $\nu(Z(W)) = 1$. For any positive integer b, we let $W_b = W \wr X_b$, where X_b is a Sylow 2-subgroup of the symmetric group of degree 2^b (*i.e.*, X_b is isomorphic to $C_2\wr\cdots\wr C_2$ with b factors). Then $\nu(Z(W_b)) = 1$.

Now we let $n = 2^{b_1} + \cdots + 2^{b_t}$, with $0 \le b_1 < \cdots < b_t$, be the 2-adic expansion of n. Then by [5, Thm. 1], a Sylow 2-subgroup R of $G = \mathrm{Sp}_{2n}(q)$ is isomorphic to $W_{b_1} \times \cdots \times W_{b_t}$ and thus $\nu(Z(R)) = t$. The next result follows by a similar proof as for Lemma 3.6

Lemma 4.5. *In the notation above, let $S = G/Z(G)$, $R \in \mathrm{Syl}_2(G)$ and $Q = R/Z(G) \in \mathrm{Syl}_2(S)$. Let $Z_0 \le R$ such that $Z_0/Z(G) = Z(Q)$. Then $Z_0 = Z(R)$ if $t \ge 2$; and $\nu(Z_0) = \nu(Z(R)) + 1$ if $t = 1$.*

From Lemmas 4.4 and 4.5 we immediately deduce the following result:

Proposition 4.6. *Let $G = \mathrm{Sp}_{2n}(q)$ with $n \ge 2$ and q odd, and $Z \le Z(G)$. Then (RC) holds for the principal 2-block of G/Z.*

4.4. **Odd-dimensional orthogonal groups.**

Lemma 4.7. *Let $G = \mathrm{SO}_{2n+1}(q)$ with $n \ge 3$ and q odd, and let $n = 2^{b_1} + \cdots + 2^{b_t}$ be the 2-adic expansion. Then $\mathrm{def}(\chi) \ge t+2$ for every character χ in the principal 2-block of G.*

Proof. Let $s \in G^* \cong \mathrm{Sp}_{2n}(q)$ be a semisimple 2-element such that $\chi \in \mathcal{E}(G, s)$. Let $\psi \in \mathcal{E}(C_{G^*}(s), 1)$ be a unipotent character in Jordan correspondence with χ. As in the proof of Lemma 4.4 we have $\mathrm{def}(\chi) \ge \mathrm{def}(\psi)$.

Let V be the underlying space of G^*, $s = \prod_f s_f$, $V = \bigoplus_f V_f$, and $C_{G^*}(s) = \prod_{f\in\mathcal{F}(s)} C_f$ with $C_f = C_{I(V_f)}(s_f)$. Keep the notation used in the proof of Lemma 4.4. Then m_1 and m_2 are both even. Also, $n = \frac{m_1}{2} + \frac{m_2}{2} + \sum_{b\ge1}\sum_{f\in\mathcal{F}_b} m_f 2^{b-1}$, and so $t \le \frac{m_1}{2} + \frac{m_2}{2} + \sum_{f\in\mathcal{F}\setminus\mathcal{F}_0} m_f$. By Lemma 4.2(a), $\mathrm{def}(\psi_i) \ge \frac{m_i}{2}$ for $i = 1, 2$, with equality only when $m_i \le 2$. For $f \in \mathcal{F}(s) \setminus \mathcal{F}_0$, the formulas for $\mathrm{def}(\psi_f)$ are the same as in (6) and (7).

Now first assume that $n \ge 4$. If $\mathcal{F}_b(s) = \emptyset$ for $b > 1$, then $n = \frac{m_1}{2} + \frac{m_2}{2} + \sum_{f\in\mathcal{F}_1} m_f$. Hence $\mathrm{def}(\chi) \ge n \ge t + 3$ as $n \ge 4$. So now assume $\bigcup_{b>1}\mathcal{F}_b(s) \ne \emptyset$. Also, by (6), it is easy to see that the claim holds unless $|\bigcup_{b>1}\mathcal{F}_b(s)| = 1$ and $m_f = 1$ for the unique polynomial f in $\bigcup_{b>1}\mathcal{F}_b(s)$. If $\mathcal{F}_1(s) \ne \emptyset$, then by (7) we get the result. Hence $\mathcal{F}_1(s) = \emptyset$ and then $\mathcal{F}(s) \setminus \mathcal{F}_0 = \{f\}$. Again by (6), $\mathrm{def}(\psi_f) \ge m_f + 2$, so $n = \frac{m_1}{2} + \frac{m_2}{2} + \delta_f$ must be the 2-adic expansion, and then $m_1 + m_2 \le 2$. Hence $\delta_f \ge n - 1 \ge 3$ and then $\delta_f \ge 4$ since it is a power of 2. Thus by (6), $\mathrm{def}(\psi_f) \ge 4m_f$ and so $\mathrm{def}(\chi) \ge t + 3$, completing the proof in case $n \ge 4$.

Now assume that $n = 3$. By Lemma 4.2(a), for $i \in \{1, 2\}$, if $m_i > 0$ then

$$\mathrm{def}(\psi_i) > \frac{m_i}{2}. \tag{8}$$

Similar to the proof of Lemma 4.4, if $\bigcup\limits_{b>1}\mathcal{F}_b(s) \ne \emptyset$, then by (6) we have $\mathrm{def}(\chi) \ge t + 2$. So we may assume that $\mathcal{F}_b(s) = \emptyset$ for $b > 1$, and then $n = 3 = \frac{m_1}{2} + \frac{m_2}{2} + \sum_{f\in\mathcal{F}_1} m_f$. Note that here $n - t = 1$. By (8), we may assume that $m_1 = m_2 = 0$, and then $\mathcal{F}(s) = \mathcal{F}_1(s)$. Thus $\mathrm{def}(\chi) \ge t + 2$ follows from (7) and we are done. ☐

To discuss the Sylow 2-subgroups, let W be a Sylow 2-subgroup of $\mathrm{GO}_2^\eta(q)$ with $q \equiv \eta 1 \pmod 4$, where $\eta \in \{\pm\}$. Then W is isomorphic to the dihedral group of order 2^{a+1}, where $a = \nu(q^2 - 1) - 1$, and so $\nu(Z(W)) = 1$. For any positive integer b, we let $W_b = W \wr X_b$, where X_b is defined as in Section 2.2. Then $\nu(Z(W_b)) = 1$.

Now let $n = 2^{b_1} + \cdots + 2^{b_t}$ be the 2-adic expansion of n and let $\iota : \mathrm{GO}_{2n}^\eta(q) \to \mathrm{SO}_{2n+1}(q)$, $A \mapsto \mathrm{diag}(A, \det(A))$, be an embedding of $\mathrm{GO}_{2n}^\eta(q)$ into $\mathrm{SO}_{2n+1}(q)$ with $q^n \equiv \eta 1 \pmod 4$.

Then by [5, Thm. 2], the image of $W_{b_1} \times \cdots \times W_{b_t}$ under ι is a Sylow 2-subgroup of $\mathrm{SO}_{2n+1}(q)$.

Now, we consider the simple group $\Omega_{2n+1}(q) = [\mathrm{SO}_{2n+1}(q), \mathrm{SO}_{2n+1}(q)]$. Let $R \cong W_{b_1} \times \cdots \times W_{b_t}$ be a Sylow 2-subgroup of $\mathrm{SO}_{2n+1}(q)$ as above and $Q = R \cap \Omega_{2n+1}(q)$. We give an explicit description of Q. Let $\theta : \mathrm{GO}_2^\eta(q) \to \mathbb{F}^\times/\mathbb{F}^{\times 2}$ be the spinor norm on $\mathrm{GO}_2^\eta(q)$. Let W be the Sylow 2-subgroup of $\mathrm{GO}_2^\eta(q)$ as before, then $W_0 = \ker \theta|_W$ is isomorphic to the dihedral group of order 2^a. In particular, $Z(W) \subseteq W_0$. Denote by $B = W \times \cdots \times W$ the base subgroup of R, then $R = B \rtimes A$ with $A = X_{b_1} \times \cdots \times X_{b_t}$. Then $Q = B_0 \rtimes A$, where B_0 is the subgroup of B consisting of all elements $(w_1, \ldots, w_n)$ satisfying $\theta(w_1) \cdots \theta(w_n) = \bar{1}$; see for example [14, §4].

Proposition 4.8. *Let $S = \Omega_{2n+1}(q)$ with $n \geq 3$ and q odd. Then (RC) holds for the principal 2-blocks of all covering groups of S.*

Proof. For $\Omega_{2n+1}(q)$ we let R, Q be as above. Then $Z(Q) = Z(R)$ by our description of Q. Then by Lemma 4.7, (RC) holds for the principal 2-block of S.

For $2.S = \mathrm{Spin}_{2n+1}(q)$, the assertion follows directly from [8, Lemma 3.1] by the fact that the center of a Sylow 2-subgroup of $\mathrm{Spin}_{2n+1}(q)$ has order 2 by [11, Lemma 4.4]. □

4.5. **Even-dimensional orthogonal groups.** The simply connected group of type D_n is the spin group $\mathrm{Spin}_{2n}^\pm(q)$, and its dual group is the projective conformal special orthogonal group $\mathrm{PCSO}_{2n}^\pm(q)$, the quotient of the conformal special orthogonal group $\mathrm{CSO}_{2n}^\pm(q)$ modulo its central torus. We thus need some control on centralisers of 2-elements in conformal special orthogonal groups. Recall that q is an odd prime power.

Lemma 4.9. *Let $s \in G^* := \mathrm{CSO}_{2n}^\pm(q)$ be a 2-element. Then $C_{G^*}^\circ(s)$ is a product of groups $D_{k_i}(\epsilon_i q)$ (two factors) or $D_{k_i}(\epsilon_i q^2)$ (one factor) with groups of type $\mathrm{GU}_{k_i}(q)$ and $\mathrm{GL}_{k_i}(q^{2^{d_i}})$ for suitable k_i, $d_i \geq 0$ and $\epsilon_i \in \{\pm 1\}$, where $\mathrm{GU}_{k_i}(q)$ only occurs when $4|(q+1)$.*

Proof. Let $s \in \mathbf{G}^* = \mathrm{CSO}_{2n}$. As $\mathrm{CSO}_{2n} = \mathrm{SO}_{2n} \cdot Z(\mathrm{CSO}_{2n})$ we can write $s = s_1 z$ with elements $s_1 \in \mathrm{SO}_{2n}$ and $z \in Z(\mathrm{CSO}_{2n})$ a scalar matrix. Clearly, $C_{\mathbf{G}^*}(s) = C_{\mathbf{G}^*}(s_1)$. Note that s_1 and z need not be F-stable, but as $\mathrm{SO}_{2n} \cap Z(\mathrm{CSO}_{2n}) = \{\pm 1\}$ and both groups are F-stable, we have $F(s_1) = \pm s_1$, and s_1 is F^2-stable. So $C_{\mathbf{G}^*}(s)^{F^2}$ has the structure described above, corresponding to an orthogonal decomposition of $\mathbb{F}_{q^2}^{2n}$ into the s_1-eigenspaces. Now F permutes these eigenspaces according to whether $F(s_1) = s_1$ or $F(s_1) = -s_1$. Thus, the two orthogonal factors of $C_{\mathbf{G}^*}(s)^{F^2}$ are either fixed or permuted and we obtain a collection of type A-factors, as claimed. See also [12, Lemma 2.5] for a more precise statement. □

Lemma 4.10. *Let $G = \mathrm{Spin}_{2n}^\pm(q)$ with $n \geq 4$ and q odd, and let $n = 2^{b_1} + \cdots + 2^{b_t}$ be the 2-adic expansion with $b_1 < \ldots < b_t$. Then $\mathrm{def}(\chi) \geq \max\{t+2, t+b_1+1\}$ for every character χ in the principal 2-block of G.*

Proof. Let $G = \mathrm{Spin}_{2n}^\pm(q)$ with $n \geq 4$. Assume that $\chi \in \mathcal{E}(G, s)$ for some semisimple 2-element $s \in G^* = \mathrm{PCSO}_{2n}^\pm(q)$. Let $\psi \in \mathcal{E}(C_{G^*}^\circ(s), 1)$ be a unipotent character in Jordan correspondence with χ. Then as in the proof of Lemma 4.4 we have $\mathrm{def}(\chi) \geq \mathrm{def}(\psi)$.

Let $\tilde{s} \in \tilde{G}^* := \mathrm{CSO}_{2n}^\pm(q)$ be a preimage of s under the natural map. Then the structure of $C_{\tilde{G}^*}^\circ(\tilde{s})$ is described in Lemma 4.9, and thus also the structure of $C_{G^*}^\circ(s)$. Write

$m_1/2, m_2/2$ for the ranks of the two D-factors, or $m_0/2$ if there is just one over $\mathbb{F}_{q^2}$, and m_f for the ranks of the various GL- and GU-factors with $f \in \mathcal{F} \setminus \mathcal{F}_0$. Then

$$n = m_0 + \frac{m_1}{2} + \frac{m_2}{2} + \sum_{b\geq 1}\sum_{f\in\mathcal{F}_b} m_f 2^{b-1}, \tag{9}$$

$t \leq m_0 + \frac{m_1}{2} + \frac{m_2}{2} + \sum_{f\in\mathcal{F}\setminus\mathcal{F}_0} m_f$, $\mathrm{def}(\psi_f) \geq m_f(b+1)$ for every $f \in \mathcal{F}_b$, and $\mathrm{def}(\psi_0) \geq m_0 - 1$ for the unipotent character ψ_0 from a possible D-factor over $\mathbb{F}_{q^2}$. Thus we have

$$\mathrm{def}(\psi) \geq (m_0 - 1) + \frac{m_1}{2} + \frac{m_2}{2} + \sum_{b\geq 1}\sum_{f\in\mathcal{F}_b}(b+1)m_f \geq t - 1 + \sum_{b\geq 1}\sum_{f\in\mathcal{F}_b} bm_f$$

(where the summand $(m_0 - 1)$ is only present when $m_0 > 0$). By the same argument as in Lemma 4.7, we obtain $\mathrm{def}(\chi) \geq t + 2$ (noting that $n \geq 4$), and even $\mathrm{def}(\chi) \geq t + 3$ when $m_0 = 0$.

For the second bound, by (6) and (7) in the proof of Lemma 4.4 it suffices to show that

$$(m_0 - 1) + \frac{m_1}{2} + \frac{m_2}{2} + \sum_{b\geq 1}\sum_{f\in\mathcal{F}_b} m_f(b+1) \geq t + b_1 + 1.$$

If $m_0 = m_1 = m_2 = 0$, then by Lemma 2.3(b),

$$\begin{aligned}
&\sum_{b\geq 1}\sum_{f\in\mathcal{F}_b} m_f(b+1) - (t + b_1 + 1)\\
&\qquad = 2\Big(\sum_{b\geq 1}\sum_{f\in\mathcal{F}_b} m_f - t\Big) + \sum_{b\geq 1}\sum_{f\in\mathcal{F}_b}(b-1)m_f + t - (b_1 + 1)\\
&\qquad \geq \sum_{i=1}^{t} b_i + t - (b_1 + 1) \geq 0.
\end{aligned}$$

If $m_1 + m_2 > 0$ (and so $m_0 = 0$), then by Lemma 2.3(c),

$$\frac{m_1}{2} + \frac{m_2}{2} + \sum_{b\geq 1}\sum_{f\in\mathcal{F}_b} m_f(b+1) \geq t + b_1 + \sum_{b\geq 1}\sum_{f\in\mathcal{F}_b} m_f b \geq t + b_1.$$

Equality only holds when (9) is the 2-adic expansion of n. Hence $m_1 + m_2 = 2$ and $\mathcal{F}_1(s) = \emptyset$, and then $\bigcup_{b\geq 1}\mathcal{F}_b(s) \neq \emptyset$ since $n \geq 4$. Thus $\sum_{b\geq 1}\sum_{f\in\mathcal{F}_b} m_f b > 0$. So equality does not hold.

Finally, if $m_0 > 0$ (and hence $m_1 = m_2 = 0$) then the 2-adic expansion of $n - 1 = m_0 - 1 + \sum_{b,f} m_f 2^{b-1}$ has $t + b_1 - 1$ terms so by Lemma 2.3(a) we are done unless $\sum_{b,f} bm_f < 2$. But note that $m_0 - 1$ is odd, so unless $m_0 = 2$ its 2-adic expansion has at least 2 terms. Thus either $m_0 = 2$ or $n = m_0$. Both cases are easily dealt with. □

Note that $n \geq 4$ implies that either $t + 2 \geq 4$ or $t + b_1 + 1 \geq 4$, so $\mathrm{def}(\chi) \geq t + 3$ in Lemma 4.10 when $t = 1$.

We next discuss Sylow 2-subgroups. Let W_b be as in the previous subsection. First, assume $q^n \equiv -\eta 1 \pmod 4$. Then we have that $\mathrm{SO}_{2n}^{\eta}(q) = S \times C_2$ with $S = \Omega_{2n}^{\eta}(q)$ simple, and $\mathrm{Spin}_{2n}^{\eta}(q) = 2.S$ is the only proper covering group of S.

Lemma 4.11. *Let R be a Sylow 2-subgroup of $\Omega_{2n}^{\eta}(q)$, with $q^n \equiv -\eta 1 \pmod 4$ and let $n - 1 = 2^{b_1} + \cdots + 2^{b_r}$ be the 2-adic expansion. Then $\nu(Z(R)) = r$.*

Proof. By [14, Thm. 7], a Sylow 2-subgroup of $\Omega_{2n}^{\eta}(q)$ is isomorphic to a Sylow 2-subgroup of $\mathrm{GO}_{2n-2}^{\eta'}(q)$ with $q^{n-1} \equiv \eta' 1 \pmod 4$, i.e., to $W_{b_1} \times \cdots \times W_{b_r}$. The assertion follows. □

Proposition 4.12. *Let $S = \Omega_{2n}^{\eta}(q)$ with $n \geq 4$ and $q^n \equiv -\eta 1 \pmod 4$. Then (RC) holds for the principal 2-blocks of all covering groups of S.*

Proof. If $q^n \equiv -\eta 1 \pmod 4$, then by Lemma 4.11 we have $\nu(Z(R)) = r = t + b_1 - 1$ in the notation of Lemmas 4.10 and 4.11. If χ is a character of $G = 2.S = \mathrm{Spin}_{2n}^{\eta}(q)$ with $Z(G)$ in its kernel, then considering it as a character of S we have $\mathrm{def}(\chi) \geq t + b_1$ which shows the claim. If $\hat{R}$ denotes a Sylow 2-subgroup of G, then clearly $\nu(Z(\hat{R})) \leq r + 1 = t + b_1$. Again, the claim follows with Lemma 4.10. □

We now turn to the more difficult case $q^n \equiv \eta 1 \pmod 4$. Here we have that $\mathrm{SO}_{2n}^{\eta}(q) = 2.S.2$ with $S = \mathrm{P}\Omega_{2n}^{\eta}(q)$ simple, and $\mathrm{Spin}_{2n}^{\eta}(q) = 2^2.S$ if n is even, and $\mathrm{Spin}_{2n}^{\eta}(q) = 4.S$ if n is odd. First note that a Sylow 2-subgroup R of $\mathrm{GO}_{2n}^{\eta}(q)$ is isomorphic to $W_{b_1} \times \cdots \times W_{b_t}$, where $n = 2^{b_1} + \cdots + 2^{b_t}$. Let $Q = R \cap \mathrm{SO}_{2n}^{\eta}(q)$. Then $\nu(Z(Q)) = t$.

Lemma 4.13. *Let $H = \Omega_{2n}^{\eta}(q)$ with $q^n \equiv \eta 1 \pmod 4$ and keep all the notation above.*

(a) *Let $Q_0 = R \cap H$. Then $\nu(Z(Q_0)) = t$.*

(b) *Let $\overline{Q}_0 = Q_0/Z(H)$, a Sylow 2-subgroup of $\mathrm{P}\Omega_{2n}^{\eta}(q)$, and $Z(\overline{Q}_0) = Z_0/Z(H)$. Then $Z_0 = Z(Q_0)$ if $t > 1$; while if $t = 1$, $\nu(Z_0) = \nu(Z(Q_0)) + 1 = 2$ and $|Z(\overline{Q}_0)| = 2$.*

Proof. (a) Let θ be the spinor norm on $\mathrm{GO}_2^{\eta}(q)$. Write $R = B \rtimes A$, where B is the base subgroup, a direct product of copies of W and $A = X_{b_1} \times \cdots \times X_{b_t}$. Then an element $(w_1, \ldots, w_n)a$ with $w_i \in W$ and $a \in A$ is in Q_0 if and only if $\det(w_1) \cdots \det(w_n) = 1$ and $\theta(w_1) \cdots \theta(w_n) = \bar{1}$. Thus $\nu(Z(Q_0)) = t$ follows easily. Part (b) follows by a similar proof as for Lemma 3.6. □

Proposition 4.14. *Let $S = \mathrm{P}\Omega_{2n}^{\eta}(q)$ with $n \geq 4$ and $q^n \equiv \eta 1 \pmod 4$. Then (RC) holds for the principal 2-blocks of all covering groups of S.*

Proof. First assume that n is even. Then by [8, Cor. 2.4] we just need to consider the three groups S, $2.S = \Omega_{2n}^{\eta}(q)$ and $2'.S = \mathrm{HSpin}_{2n}^{\eta}(q)$. (The two half-spin groups are isomorphic under the graph automorphism of order 2.) In the notation of Lemma 4.13 the centers of Sylow 2-subgroups R of these groups satisfy $\nu(Z(R)) \leq t, t, t+1$ respectively, while by Lemma 4.10 the defects of characters belonging to the principal 2-block of G that descend to these groups are at least $t+1, t+2, t+2$ respectively. So (RC) holds in all cases.

Now assume that n is odd. Then the groups to consider are S, $2.S = \Omega_{2n}^{\eta}(q)$ and $4.S$. Here again the centers of Sylow 2-subgroups R satisfy $\nu(Z(R)) \leq t, t, t+1$, while the defects are bounded below by $t+1, t+2, t+3$ respectively. □

5. Proof of the main result

Proof of Theorem 1. Let B be a 2-block of a quasi-simple covering group G of a finite simple classical group S. If S is defined in characteristic 2, our claim is in [8, Thm. 5.1]. So S is a classical group in odd characteristic. By [8, Thm. 3.6] we may assume that G is not an exceptional covering group. Then G is one of the groups considered in the previous sections. According to [3, Thm. 21.14], then G has only one unipotent 2-block, the principal 2-block $\mathcal{E}_2(G, 1)$. This is not a counterexample to (RC) by Propositions 3.1, 3.5, 3.7, 4.6, 4.8, 4.12 and 4.14. If B is not unipotent, then $\mathrm{Irr}(B) \subseteq \mathcal{E}_2(G,s)$ for some semisimple $2'$-element $1 \neq s \in G^*$. If G is of symplectic or orthogonal type, then centralisers of non-trivial $2'$-elements in G^* are proper Levi subgroups. If G is special linear or unitary, then at least the connected components of these centralisers are proper Levi subgroups. In either case, by the reduction theorem of Bonnafé–Rouquier [1] then B is Morita equivalent to a 2-block of a strictly smaller group and thus cannot be a minimal counterexample to (RC) either. □

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