# THE PRINCIPAL BLOCK OF A $\mathbb{Z}_\ell$-SPETS AND YOKONUMA TYPE ALGEBRAS

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Abstract. We formulate conjectures concerning the dimension of the principal block of a $\mathbb{Z}_\ell$-spets (as defined in our earlier paper), motivated by analogous statements for finite groups. We show that these conjectures hold in certain situations. For this we introduce and study a Yokonuma type algebra for torus normalisers in $\ell$-compact groups which may be of independent interest.

## 1. Introduction

This paper is a contribution to the following broad question: Do there exist structures associated to finite complex reflection groups that play the same role as finite reductive groups play for finite Weyl groups? In [24, 6, 7], Broué, Michel and the second author discovered that to certain complex reflection groups can be associated data sets, called *spetses*, satisfying properties analogous to those of unipotent characters of finite reductive groups. Their construction is based upon generalisations of Hecke algebras which in turn arise from the braid groups associated to the space of regular orbits of complex reflection groups. On the other hand, $\ell$-adic reflection groups for a prime number $\ell$ arise as the "Weyl" groups of certain topological spaces called *$\ell$-compact groups* which possess much of the structure of compact groups [15, 19]. Thus, spetses and $\ell$-compact groups widen the context of group theory in two different directions — combinatorics/representation theory and algebraic topology.

In [21] we introduced the notion of a $\mathbb{Z}_\ell$-spets, a new object to which both the spetsial theory and $\ell$-compact group theory can be applied. It can be thought of as a finite reductive group which possesses a representation theory in characteristic zero and at the single prime $\ell$. It thus allows one to investigate aspects of the yet not well understood $\ell$-modular representation theory of finite reductive groups in a more general setting. In [21] we used this to exhibit a surprising consistency of spetses data with the famous Alperin weight conjecture. Our results lead us to hope that putting the modular representation theory of finite reductive groups in a broader context will pave the way to a better understanding of some of the deep open problems of representation theory.

In this paper, we develop this theme further. In order to describe our results we recall some features of [21]. Let $\ell$ be a prime number. Formally, a $\mathbb{Z}_\ell$-spets $\mathbb{G} = (W\varphi^{-1}, L)$

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is a spetsial $\ell$-adic reflection group $W$ on a $\mathbb{Z}_\ell$-lattice $L$ together with an element $\varphi \in N_{\mathrm{GL}(L)}(W)$. Via the theory of $\ell$-compact groups if $q$ is a power of a prime different from $\ell$, then under certain conditions, by [4] one can associate a fusion system $\mathcal{F}$ on a finite $\ell$-group $S$ to the pair $\mathbb{G}(q) := (\mathbb{G}, q)$. In this situation, $S$ should be considered as a Sylow $\ell$-subgroup of $\mathbb{G}(q)$. It turns out that the $\ell$-compact theory provides us, for any $s \in S$, with a centraliser which again is a $\mathbb{Z}_\ell$-spets. Using this we attached in [21] a "principal $\ell$-block" $B_0$ to $\mathbb{G}(q)$, with defect group $S$. It consists of a collection $\mathrm{Irr}(B_0)$ of sets ("irreducible characters") in bijection with the unipotent characters attached to the centralisers of elements $s \in S$ (up to $\mathcal{F}$-conjugation), defined in terms of a collection of Hecke algebras. The construction of $B_0$ is modelled on and generalises the case of blocks of finite groups of Lie type in non-describing characteristic. When $W$ is rational and under some natural assumptions on $\ell$, we recover the $\ell$-fusion system $\mathcal{F}_\ell(\mathbb{G}(q))$ and principal $\ell$-block $B_0(\mathbb{G}(q))$ of the associated finite reductive group $\mathbb{G}(q)$. See Section 4 for a description of $B_0$ and comparison with the rational case.

A primary concern in [21] was the translation of certain local-global statements in modular representation theory to purely local statements using the language of $\mathbb{Z}_\ell$-spetses. Here, our considerations are more on the global side, so that actual degrees of characters in $\mathrm{Irr}(B_0)$ (as defined in [21, Def. 6.7]) play a more significant role. Firstly, we investigate the dimension

$$\dim(B_0) := \sum_{\gamma \in \mathrm{Irr}(B_0)} \gamma(1)^2 \in \mathbb{Z}[x]$$

of $B_0$. Motivated by results concerning the divisibility properties of this number for principal blocks of finite groups, we make the following conjecture:

**Conjecture 1.** *Let $\mathbb{G} = (W\varphi^{-1}, L)$ be a simply connected $\mathbb{Z}_\ell$-spets for which $\ell$ is very good, let $q$ be a power of a prime different from $\ell$, and $B_0$ be the principal $\ell$-block of $\mathbb{G}(q)$ with defect group $S$. Then,*

(1) $(\dim(B_0)|_{x=q})_\ell = |S|$; *and*
(2) $(\dim(B_0)|_{x=q})_{\ell'} \equiv |W_{\varphi^{-1}\zeta^{-1}}|_{\ell'} \pmod{\ell}$, *where $\zeta \in \mathbb{Z}_\ell^\times$ is the root of unity with $q \equiv \zeta \pmod{\ell}$, and $W_{\varphi^{-1}\zeta^{-1}}$ is the associated relative Weyl group (see* [21, Thms 2.1 and 3.6] *for the definition and role of the relative Weyl group).*

For a finite group $G$ with principal $\ell$-block $B_0$ the equality $(\dim(B_0))_\ell = |S|$ is closely related to the fact that $|S|$ divides the dimension of each projective indecomposable module $\Phi_\nu$ associated to $\nu \in \mathrm{IBr}(B_0)$. When $B_0$ is as in Conjecture 1, and under the additional hypotheses that $W$ is an $\ell'$-group, $\varphi$ is trivial and $q \equiv 1 \pmod{\ell}$, in Section 4.2 we formulate analogues of $\mathrm{IBr}(B_0)$, decomposition numbers and $\deg \Phi_\nu$ for which we make the following additional conjecture:

**Conjecture 2.** *In the setting of Conjecture* 1 *if $W$ is an $\ell'$-group, $\varphi$ is trivial and $q \equiv 1 \pmod{\ell}$ we have*

(3) $|S|$ *divides* $\deg \Phi_\nu|_{x=q}$ *for all* $\nu \in \mathrm{IBr}(B_0)$.

Conjecture 1 combines the statements of Conjectures 4.2 and 4.3 and Conjecture 2 is restated as Conjecture 4.5 in Section 4. In Proposition 4.6, we prove that Conjectures 1 and 2 hold when $W$ is rational or when $q \equiv 1 \pmod{\ell}$ and $W$ is primitive. The general case for Conjecture 1 appears quite elusive, so we choose to simplify matters by assuming that $W$ is an $\ell'$-group, $q \equiv 1 \pmod{\ell}$ and $\varphi = 1$. Under these conditions we are able to show that our conjectures are closely related to certain previously-studied properties of Hecke algebras which we discuss next.

Recall that the generic Hecke algebra $\mathcal{H}(W, \mathbf{u})$ of $W$ is a certain quotient of the group algebra of the braid group $B(W)$ of $W$ where $\mathbf{u}$ is a set of parameters indexed by conjugacy classes of distinguished reflections in $W$. To each irreducible character $\chi$ of $\mathcal{H}(W, \mathbf{u})$ over a splitting field one associates a Schur element $f_\chi$ which, in turn, is used to define a canonical form

$$t_{W,\mathbf{u}} := \sum_{\chi \in \mathrm{Irr}(\mathcal{H}(W,\mathbf{u}))} \frac{1}{f_\chi} \chi$$

on $\mathcal{H}(W, \mathbf{u})$. This is conjectured to be a symmetrising form [6] (see Conjecture 3.3). It is expected that the form $t_{W,\mathbf{u}}$ behaves well under restriction to parabolic subgroups and with respect to the natural map $B(W) \to W$; if that is the case we say $\mathcal{H}(W, \mathbf{u})$ is *strongly symmetric* (see Definition 3.4). Our main result is the following:

**Theorem 1.** *Let $\mathbb{G} = (W, L)$ be a simply connected $\mathbb{Z}_\ell$-spets. Suppose that $\mathcal{H}(W, \mathbf{u})$ is strongly symmetric, $W$ is an $\ell'$-group and $q \equiv 1 \pmod{\ell}$. Then Conjectures* 1 *and* 2 *hold.*

Since the strongly symmetric condition is known to hold in many cases (see Proposition 3.5), we obtain:

**Corollary 2.** *Suppose $W$ is a spetsial irreducible $\ell'$-group, $\varphi = 1$ and $q \equiv 1 \pmod{\ell}$. Then Conjecture* 1 *holds for $\mathbb{G} = (W, L)$, and Conjecture* 2 *also holds, except possibly when $W$ is primitive of rank at least* 3.

The method behind our proof of Theorem 1 may be even more interesting than the theorem itself. For our approach we introduce and study a Yokonuma type algebra $\mathcal{Y}$ attached to an arbitrary finite $\ell$-adic reflection group. We conjecture that $\mathcal{Y}$ is finitely generated and free over its base ring (Conjecture 5.12) and show that this holds in many cases. It seems likely that if $\ell$ is very good for $(W, L)$, then $\mathcal{Y}$ is a symmetric algebra with respect to a trace form whose Schur elements are derived from those of various parabolic subalgebras of $\mathcal{H}(W, \mathbf{u})$ (Question 5.18).

When $\mathbb{G}$ is a rational spets which satisfies the hypotheses of Theorem 1, the principal block $B_0$ of $\mathbb{G}(q)$ is known to be Morita equivalent to $SW$ over a finite extension of $\mathbb{Z}_\ell$. The endomorphism algebra of the permutation module which captures this equivalence is a Yokonuma algebra isomorphic to a certain $\mathbb{Z}_\ell$-algebra specialisation $\mathcal{Y}_\psi$ of $\mathcal{Y}$ (see Section 5.5). Moreover, the quantities $\dim(B_0)$ and $\deg \Phi_\nu$ ($\nu \in \mathrm{IBr}(B_0)$) can be re-expressed in terms of the corresponding Schur elements of $\mathcal{Y}_\psi$.

This motivates the study of $\mathcal{Y}_\psi$ and associated numerical properties for arbitrary $\ell$-adic reflection groups. In Theorem 5.9 we show that if the parabolic subgroups of the underlying $\ell$-adic reflection group are generated by reflections, then $\mathcal{Y}_\psi$ is finitely generated and free over $\mathbb{Z}_\ell$; a result of Külshammer–Okuyama–Watanabe then allows us to conclude that when the order of $W$ is relatively prime to $\ell$, then $\mathcal{Y}_\psi$ is isomorphic to the group algebra of $SW$ (see Theorem 5.10). We obtain Theorem 1 by applying general results concerning the arithmetic behaviour of Schur elements in symmetric algebras (as discussed in Section 2.3) to $\mathcal{Y}_\psi$, by playing off the form on $\mathcal{Y}_\psi$ inherited from $\mathcal{Y}$ against the standard symmetrising form on the group algebra of $SW$.

We close the introduction by discussing how the hypotheses on $\ell$ and $W$ in Theorem 1 might be relaxed. If $\ell \mid |W|$ we have carried out explicit computations in support of Conjecture 1 when $q \equiv 1 \pmod{\ell}$, such as when $W = G(e, 1, 3)$ for $\ell = 3$ and when $W = G_{29}$ for the bad prime $\ell = 5$ assuming $\mathrm{Irr}(B_0)$ is defined appropriately (see [21, Rem. 6.16]). In these situations, $\mathrm{Irr}(B_0)$ is only partially describable in terms of the Schur elements of the Yokonuma algebra $\mathcal{Y}_\psi$. Even for Weyl groups it is a major open problem to find a description involving the Schur elements of a suitable larger algebra.

**Structure of the paper.** In Section 2 we collect some general material necessary for our proofs. In particular, we discuss certain divisibility properties of Schur elements in symmetric algebras. In Section 3 we recall the construction of Hecke algebras $\mathcal{H}(W, \mathbf{u})$ for complex reflection groups $W$ and the origin of the trace form $t_{W,\mathbf{u}}$, define the property of being strongly symmetric (Definition 3.4) and introduce the relevant specialisations. In Section 4, we recall the description of the principal block of a $\mathbb{Z}_\ell$-spets, introduce the notion of dimension and formulate our main conjectures. Our new Yokonuma type algebra $\mathcal{Y}$ attached to an $\ell$-adic reflection group $(W, L)$ is defined and studied in Section 5. In Theorem 5.7 we describe the structure of $\mathcal{Y}$ over the field of fractions of the base ring and in Theorem 5.10 we obtain, under additional conditions, a similar structural result for $\ell$-adic specialisations of $\mathcal{Y}$. In Sections 5.3 and 5.4 we formulate general freeness and symmetrising form conjectures for $\mathcal{Y}$ and show that the freeness conjecture holds for Coxeter groups (Theorem 5.13) and for most of the imprimitive complex reflection groups (Theorem 5.14). We also discuss the relationship of $\mathcal{Y}$ to the algebra considered by Marin in [32, 33] and in Section 5.5 we show that when $(W, L)$ arises from a Weyl group the principal block of the classical Yokonuma algebra over $\mathbb{Z}_\ell$ is a certain specialisation of $\mathcal{Y}$. Section 5.6 contains the proof of Theorem 1 (Theorem 5.20) and Corollary 2.

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## 2. Background Material

2.1. **Finite generation of modules.** First we record a few general facts on finite generation of modules. The first is a variation on Nakayama's lemma for nilpotent ideals which allows for the dropping of the finite generation hypothesis. Let $R$ be a commutative ring with 1. Recall that the Jacobson radical $J(R)$ of $R$ is the intersection of all maximal left ideals of $R$.

**Lemma 2.1.** *Let $I \subseteq J(R)$ be an ideal of $R$ and let $M, N$ be $R$-modules with $N \subseteq M$ and $M = N + IM$. Suppose that either $M/N$ is finitely generated or that $I$ is nilpotent. Then $N = M$.*

*Proof.* The case that $M/N$ is finitely generated is the usual case of the Nakayama Lemma. Suppose that $I^r = 0$ for some $r \geq 1$. By hypothesis, $M/N = I(M/N)$. Hence $M/N = I^r(M/N) = 0$, showing that $M = N$. □

**Lemma 2.2.** *Let $R$ be a discrete valuation ring with uniformiser $\pi$. Suppose that $N \subseteq M$ are $R$-modules such that $M/\pi M$ is finitely generated, $M/N$ is finitely generated and torsion free, and $\pi^r N = 0$ for some positive integer $r$. Then $M$ is finitely generated.*

*Proof.* Since $M/N$ is finitely generated, it suffices to show that $N$ is finitely generated. Now $\pi^r N = 0$. So, in order to show that $N$ is finitely generated it suffices to show that $\pi^i N/\pi^{i+1}N$ is finitely generated for any $i \geq 1$. Multiplication by $\pi^i$ induces a surjective $R$-module homomorphism from $N/\pi N$ onto $\pi^i N/\pi^{i+1}N$, hence it suffices to show that $N/\pi N$ is finitely generated. By hypothesis, $M/\pi M$ is finitely generated and $N/(\pi M \cap N)$ is isomorphic to a submodule of $M/\pi M$. Thus, as $R$ is Noetherian, $N/(\pi M \cap N)$ is finitely generated. But since $M/N$ is torsion free, $\pi M \cap N = \pi N$. □

2.2. **Clifford theory.** The following is standard Clifford theory adapted to quotients of infinite group algebras with respect to finite normal subgroups. For a ring $R$ and ideal $I$, we will regard without further comment an $R/I$-module as an $R$-module via pullback along the canonical homomorphism $R \to R/I$.

**Lemma 2.3.** *Let $G$ be a group, $T$ a finite normal subgroup of $G$ and $K$ a field of characteristic $0$. For $\theta \in \mathrm{Irr}_K(T)$, denote by $e_\theta$ the corresponding primitive central idempotent of $KT$ and by $G_\theta$ the stabiliser of $\theta$ in $G$ of (finite) index $n_\theta := |G : G_\theta|$. Let $I$ be an ideal of $KG$ and set $I_\theta := e_\theta I e_\theta = I \cap e_\theta KG e_\theta$.*

(a) *There is a $K$-algebra isomorphism*
$$KG/I \cong \prod_\theta \mathrm{Mat}_{n_\theta}(e_\theta KG e_\theta/I_\theta)$$
*where $\theta$ runs over a set of representatives of $G$-orbits on $\mathrm{Irr}_K(T)$. Moreover, $e_\theta KG e_\theta/I_\theta = e_\theta KG_\theta e_\theta/I_\theta = KG_\theta e_\theta/I_\theta$ for all $\theta \in \mathrm{Irr}_K(T)$.*

(b) *The map $(\theta, U) \mapsto \mathrm{Ind}_{G_\theta}^G U$ induces a bijection between the set of pairs $(\theta, U)$, where $\theta$ runs over representatives of $G$-orbits on $\mathrm{Irr}_K(T)$ and $U$ runs over a set of isomorphism classes of simple $KG_\theta e_\theta/I_\theta$-modules, and the set of isomorphism classes of simple $KG/I$-modules.*

*Proof.* Set $Y = KG/I$ and for $a \in KG$ denote by $\bar{a}$ its image in $Y$. Let $X$ be the set of orbits of the conjugation action of $G$ on $\mathrm{Irr}_K(T)$ and for each $x \in X$, let $e_x = \sum_{\theta\in x} e_\theta$. So $\{e_x \mid x \in X\}$ is a set of pairwise orthogonal central idempotents of $KG$ with $1_{KG} = \sum_{x\in X} e_x$. Consequently, $\{\bar{e}_x \mid x \in X\}$ is a set of pairwise orthogonal central idempotents of $Y$ with $1_Y = \sum_{x\in X} \bar{e}_x$. Here, we abuse notation to allow for the possibility that some $\bar{e}_x$ are equal to zero. Thus
$$Y = \bigoplus_{x\in X} Y\bar{e}_x$$
and this is also a decomposition of $Y$ into a direct product of $K$-algebras. Let $x \in X$ and $\theta \in x$. Then,
$$\bar{e}_x = \sum_{g\in G/G_\theta} g\bar{e}_\theta g^{-1}$$

is a decomposition of $\bar{e}_x$ into orthogonal, conjugate idempotents. Hence,

$$Y\bar{e}_x \cong \mathrm{Mat}_{n_\theta}(\bar{e}_\theta Y \bar{e}_\theta)$$

as $K$-algebras. Now the first assertion of (a) follows since the inclusion of $e_\theta KGe_\theta$ in $KG$ induces an isomorphism $e_\theta KGe_\theta/I_\theta \cong \bar{e}_\theta Y\bar{e}_\theta$.

Since $e_\theta g e_\theta = 0$ for any $g \in G \setminus G_\theta$, and since $e_\theta$ is central in $KG_\theta$,

$$e_\theta KGe_\theta/I_\theta = e_\theta KG_\theta e_\theta/I_\theta = KG_\theta e_\theta/I_\theta,$$

proving the second assertion of (a).

Let $V$ be a simple $Y$-module. There exists a unique $x \in X$ such that $\bar{e}_x V \neq 0$ (equivalently $e_x V = V$). By the statement and proof of (a), if $\theta \in x$, then

$$V = \bigoplus_{g \in G/G_\theta} g\bar{e}_\theta V, \tag{$*$}$$

$\bar{e}_\theta V$ is a simple $\bar{e}_\theta Y\bar{e}_\theta$-module, and the map $V \mapsto (\theta, \bar{e}_\theta V)$ induces a bijection between the set of isomorphism classes of simple $Y$-modules and the set of pairs $(\theta, U)$, where $\theta$ runs over representatives of $G$-orbits on $\mathrm{Irr}_K(T)$ and $U$ runs over a set of isomorphism classes of simple $\bar{e}_\theta Y\bar{e}_\theta$-modules. Finally, identifying $\bar{e}_\theta Y\bar{e}_\theta$-modules with $KG_\theta e_\theta/I_\theta$-modules via the isomorphism $KG_\theta e_\theta/I_\theta \cong \bar{e}_\theta Y\bar{e}_\theta$ given in the proof of (a), the equation $(*)$ gives that $V = \mathrm{Ind}_{G_\theta}^G(e_\theta V)$. This proves (b). □

2.3. **Symmetrising forms and divisibility.** Let $R$ be a commutative ring with 1 and let $Y$ be a symmetric $R$-algebra which is free and finitely generated as $R$-module. If $X$ is an $R$-basis of $Y$, then any symmetrising form $t : Y \to R$ determines a dual basis $X^\vee = \{x^\vee \mid x \in X\}$ satisfying

$$t(xy^\vee) = \begin{cases} 1 & \text{for } x = y \in X \\ 0 & \text{for } x, y \in X,\ x \neq y. \end{cases}$$

The *relative projective element of* $Y$ *in* $Z(Y)$ *with respect to* $t$ is defined by $z_t := \sum_{x\in X} xx^\vee$. It depends on $t$ but not on the choice of the basis $X$ (see [22, Sec. 2.11, 2.16] for details).

Now suppose that $t$ is a symmetrizing form on $Y$ and let $s : Y \to R$ be a symmetric form (also known as trace form). Then there exists $u \in Z(Y)$ such that $s = t_u$, where $t_u \in Y^* := \mathrm{Hom}_R(Y, R)$ is defined by $t_u(x) := t(ux)$ for $x \in Y$. The map $u \mapsto t_u$ is an $R$-module isomorphism between $Z(Y)$ and the $R$-module of symmetric forms on $Y$. Moreover, $t_u$ is a symmetrising form if and only if $u \in Z(Y)^\times$. If $X$ is an $R$-basis of $Y$ and $X^\vee$ is the dual basis with respect to $t$ and if $u \in Z(Y)^\times$, then the dual basis of $X$ with respect to $t_u$ is equal to $X^\vee u^{-1}$, and hence the relative projective element in $Z(Y)$ with respect to $t_u$ is equal to $z_t u^{-1}$.

If $Y$ is a split semisimple algebra over a field $K$ then for any symmetric form $s : Y \to K$, there exist elements $s_\chi \in K$, $\chi \in \mathrm{Irr}_K(Y)$, such that $s = \sum_{\chi\in\mathrm{Irr}_K(Y)} s_\chi \chi$. Moreover, we claim that $s$ is a symmetrising form if and only if all $s_\chi$ are non-zero. For this, note that $s$ is a symmetrising form for $Y$ if and only of its restriction to any block of $Y$ is a symmetrising form for the block. Thus we may assume that $Y$ is split simple, that is, a matrix algebra over $K$ and here the claim is immediate from the fact that the trace map on a matrix algebra is a symmetrising form (see for instance [22, Thm 2.11.3]).

**Lemma 2.4.** *Suppose that $K$ is a field and $Y$ is a split semisimple $K$-algebra. Let $s : Y \to K$ be a symmetrising form and let $s_\chi$, $\chi \in \mathrm{Irr}_K(Y)$, be elements of $K$ such that $s = \sum_\chi s_\chi \chi$. Let $z = z_s \in Z(Y)$.*

(a) *If $V$ is a $Y$-module affording the irreducible character $\chi$, then the trace of $z^{-1}$ on $V$ equals $s_\chi$.*

(b) *The trace of $z^{-2}$ in the left regular representation of $Y$ equals $\sum_{\chi\in\mathrm{Irr}_K(Y)} s_\chi^2$.*

*Proof.* A straightforward calculation using the standard bases of matrix algebras shows that

$$z = \sum_{\chi\in\mathrm{Irr}_K(Y)} s_\chi^{-1}\chi(1)e_\chi$$

where $e_\chi \in Z(Y)$ is the central idempotent corresponding to $\chi$. In particular, $z$ is invertible in $Y$ with $z^{-1} = \sum_\chi s_\chi \chi(1)^{-1} e_\chi$. The trace formula is an immediate consequence of this.

For (b) we note that the $K$-linear map

$$s \otimes s : Y \otimes_K Y^{\mathrm{op}} \to K, \qquad (s \otimes s)(y \otimes y') := s(y)s(y'),$$

is a symmetrising form on $Y \otimes_K Y^{\mathrm{op}}$ with corresponding relative projective element $z \otimes z$. Further, $Y \cong \bigoplus_{\chi\in\mathrm{Irr}_K(Y)} V_\chi \otimes V_\chi^*$ as $(Y \otimes_K Y^{\mathrm{op}})$-modules, where $V_\chi$ is an irreducible $Y$-module with character $\chi$ and $V_\chi^*$ is an irreducible $Y^{\mathrm{op}}$-module with character $\chi$. So, by part (a) applied with $Y \otimes_K Y^{\mathrm{op}}$ in place of $Y$, $s \otimes s$ in place of $s$ and $V_\chi \otimes V_\chi^*$ in place of $V$, $\sum_\chi s_\chi^2$ equals the trace of $z^{-1} \otimes z^{-1}$ on $Y$. Since $z$ is central in $Y$, this trace is just the trace of left multiplication by $z^{-2}$ on $Y$. □

**Lemma 2.5.** *Let $G$ be a finite group and $K$ a field such that $KG$ is split semisimple. Let $t$ be the canonical symmetrising form on $KG$. Let $u \in Z(KG)^\times$, let $\alpha \in K$ be the coefficient of 1 when $u^2$ is written as a $K$-linear combination of elements of $G$ and suppose that $t_u = \sum_\chi s_\chi \chi$, $s_\chi \in K$. Then*

$$\sum_{\chi\in\mathrm{Irr}_K(Y)} s_\chi^2 = \frac{\alpha}{|G|}.$$

*Proof.* By a straightforward calculation using the set of group elements as basis, the relative projective element of $KG$ with respect to $t$ is $|G|1_{KG}$. Hence the relative projective element of $KG$ with respect to $t_u$ is $|G|u^{-1}$. Now the result follows from Lemma 2.4 since for any $y \in KG$, the trace of the action of $y$ on $KG$ via the left regular representation equals $|G|\beta$, where $\beta$ is the coefficient of 1 in the standard basis presentation of $y$. □

**Lemma 2.6.** *Let $\mathcal{O}$ be a complete discrete valuation ring with field of fractions $K$ of characteristic zero and residue field $\mathcal{O}/J(\mathcal{O})$ of characteristic $\ell$. Let $G = T \rtimes W$ be the semidirect product of a finite abelian $\ell$-group $T$ with an $\ell'$-group $W$ acting faithfully on $T$. Let $t$ be the canonical symmetrising form on $KG$ and $s$ a symmetrising form on $KG$ with $s = t_u$, $u \in Z(KG)^\times$. Let $\alpha$ be the coefficient of 1 when $u^2$ is expressed as a $K$-linear combination of elements of $G$.*

*Suppose that the restriction of $s$ to $\mathcal{O}G$ takes values in $\mathcal{O}$ and for any $x \in T$, $s(x) = \delta_{x,1}$. Then $u \in Z(\mathcal{O}G)$, $\alpha \in \mathcal{O}$ and $\alpha \equiv 1 \pmod{\ell}$.*

*Proof.* The restriction $t' := t|_{\mathcal{O}G}$ is a symmetrising form $\mathcal{O}G \to \mathcal{O}$ and by hypothesis $s' := s|_{\mathcal{O}G} : \mathcal{O}G \to \mathcal{O}$ is a symmetric form. Thus there exists $u' \in Z(\mathcal{O}G)$ such that $s' = t'_{u'}$. Extending scalars, by linearity we have that $s = t_{u'}$. Since by hypothesis $s = t_u$, we conclude $u = u' \in Z(\mathcal{O}G)$. This proves the first and second assertions.

For a conjugacy class $C$ of $G$, denote by $\hat{C}$ the corresponding class sum. Let

$$u = 1 + \sum_C \alpha_C \hat{C}$$

where $C$ runs over the non-identity conjugacy classes of $G$. Then for $1 \neq x \in T$ we have $0 = s(x) = t(ux) = \alpha_D$ where $D$ is the conjugacy class of $G$ containing $x^{-1}$. It follows that

$$u = 1 + \sum_{C \in \mathcal{C}} \alpha_C \hat{C}$$

where $\mathcal{C}$ is the set of conjugacy classes of $G \setminus T$ and

$$\alpha = 1 + \sum_{C \in \mathcal{C}} \alpha_C \alpha_{C^{-1}} |C|,$$

where $C^{-1}$ denotes the class containing the inverses of the elements of $C$. Since $W$ acts faithfully on $T$, $\ell$ divides $|C|$ for all $C \in \mathcal{C}$. Since $u \in Z(\mathcal{O}G)$, all $\alpha_C$ are elements of $\mathcal{O}$ and we obtain the last assertion. □

The following is a consequence of Tate duality for symmetric algebras over complete discrete valuation rings exhibited in [16]. For an integral domain $\mathcal{O}$ with field of fractions $K$ and $Y$ an $\mathcal{O}$-algebra which is finitely generated free as $\mathcal{O}$-module, we set $KY := K \otimes_{\mathcal{O}} Y$.

**Proposition 2.7.** *Suppose $\mathcal{O}$ is a complete discrete valuation ring with field of fractions $K$ of characteristic zero, $Y$ is a symmetric $\mathcal{O}$-algebra such that $KY$ is split semisimple. Let $s : KY \to K$ be a symmetrising form with $s = \sum_\chi s_\chi \chi$. Suppose that the restriction of $s$ to $Y$ takes values in $\mathcal{O}$.*

*Let $U$ be a projective $Y$-module and suppose that there is an isomorphism of $KY$-modules*

$$K \otimes_{\mathcal{O}} U \cong \bigoplus_{\chi \in \mathrm{Irr}(KY)} V_\chi^{d_\chi},$$

*where $V_\chi$ is a simple $KY$-module with character $\chi$ and $d_\chi \in \mathbb{N}_0$. Then*

$$\sum_{\chi \in \mathrm{Irr}(KY)} s_\chi d_\chi \in \mathcal{O}.$$

*Proof.* Let $t : Y \to \mathcal{O}$ be a symmetrising form and let $z$ be the relative projective element of $Y$ with respect to $t$. By [16, Prop. 2.2], for any $\gamma \in \mathrm{End}_Y(U)$, the trace of $z^{-1}\gamma$ on $K \otimes_{\mathcal{O}} U$ lies in $\mathcal{O}$. Here by $z^{-1}\gamma \in \mathrm{End}_{KY}(K \otimes_{\mathcal{O}} U)$ we denote the composition of (the extension to $K$ of) $\gamma$ with multiplication by $z^{-1}$.

Denote by $\tilde{t} : KY \to K$ the $K$-linear extension of $t$ to $KY$. Then $\tilde{t}$ is a symmetrising form for $KY$ with relative projective element $z$. Hence $s = \tilde{t}_u$ for some $u \in Z(KY)$. Since the restriction of $s$ to $Y$ takes values in $\mathcal{O}$, we have as in the proof of Lemma 2.6 that $u \in Z(Y)$. Let $\gamma : U \to U$ be multiplication by $u$. Since $u \in Z(Y)$, $\gamma \in \mathrm{End}_Y(U)$. Thus the trace of $z^{-1}\gamma$ on $K \otimes_{\mathcal{O}} U$ is an element of $\mathcal{O}$. Further, $zu^{-1}$ is the relative projective element of $KY$ with respect to $s$ and $z^{-1}\gamma$ is multiplication by $(zu^{-1})^{-1}$. Thus the result follows from Lemma 2.4. $\square$

The first statement of the next proposition is a theorem of Brauer. The second is also well known to experts; we provide a proof of it for the convenience of the reader.

**Proposition 2.8.** *Let $G$ be a finite group, $S$ a Sylow $\ell$-subgroup of $G$ and $H$ a subgroup of $G$ containing $N_G(S)$. Denote by $B_0(G)$ (respectively $B_0(H)$) the principal $\ell$-block of $G$ (respectively $H$). Then,*

$$(\dim B_0(G))_\ell = (\dim B_0(H))_\ell = |S|$$

*and*

$$(\dim B_0(G))_{\ell'} \equiv (\dim B_0(H))_{\ell'} \pmod{\ell}.$$

*Proof.* Let $\mathcal{O}$ be a complete discrete valuation ring in characteristic zero with algebraically closed residue field of characteristic $\ell$. Let $b \in \mathcal{O}G$ be the block idempotent corresponding to $B_0(G)$ and $c \in \mathcal{O}H$ the one corresponding to $B_0(H)$. The group $S$ is a defect group of the principal block $B_0(G)$. Hence as $\mathcal{O}[G \times G]$-module, $B_0(G)$ has vertex $\Delta S = \{(x,x) \mid x \in S\}$ (see [22, Rem. 6.7.14]). Further, the $\mathcal{O}[H \times H]$-module $B_0(H)$ is the Green correspondent of $B_0(G)$ in $H \times H$ (see [22, Thm 6.7.2], and note that by [22, Thm 6.13.14], $B_0(G)$ and $B_0(H)$ are Brauer correspondents). Thus, by properties of the Green correspondence (see [22, Thm 5.2.1])

$$\mathrm{Ind}_{H\times H}^{G\times G}(\mathcal{O}Hc) = \mathcal{O}Gb \oplus Y$$

where every indecomposable $\mathcal{O}[G \times G]$-module summand of $Y$ has vertex of order strictly smaller than $|\Delta S| = |S|$. Comparing $\mathcal{O}$-ranks,

$$\frac{|G|^2}{|H|^2}\mathrm{rk}(\mathcal{O}Hc) = \mathrm{rk}(\mathcal{O}Gb) + \mathrm{rk}(Y).$$

By a standard application of Green's indecomposability theorem and the properties of vertices and sources (see [22, Sec. 5.1, Thm 5.12.3]), the $\ell$-part of the rank of any indecomposable $\mathcal{O}X$-lattice, for $X$ a finite group, is greater than or equal to $\frac{|X|_\ell}{|Q|}$ where $Q$ is a vertex of the lattice. Thus, $(\mathrm{rk}(Y))_\ell$ is strictly greater than $|S|$ and we obtain

$$\frac{|G|^2}{|H|^2}\mathrm{rk}(\mathcal{O}Hc) \equiv \mathrm{rk}(\mathcal{O}Gb) \pmod{\ell|S|}.$$

Now by a theorem of Brauer (see [22, Thm 6.7.13]), the $\ell$-part of $\dim(B_0(G)) = \mathrm{rk}(\mathcal{O}Gb)$ equals $|S|$ and similarly for $B_0(H)$. The result follows since $|G : H| \equiv 1 \pmod{\ell}$ by Sylow's theorem. $\square$

2.4. **Tits deformation theorem.** We recall some features of Tits' deformation theorem. Let $R$ and $R'$ be integral domains with field of fractions $K$ and $K'$ respectively and let $\psi : R \to R'$ be a ring homomorphism. Let $Y$ be an $R$-algebra which is finitely generated and free as $R$-module and let $Y'$ denote the $R'$-algebra $R' \otimes Y$ obtained via extension of scalars through $\psi$. Let $t : Y \to R$ be an $R$-linear map and let $t' : Y' \to R'$ be its $R'$-linear extension through $\psi$.

**Theorem 2.9.** *Suppose that $KY$ and $K'Y'$ are both split semisimple.*

(a) *The map $\psi$ induces a bijection* $\mathrm{Irr}(KY) \to \mathrm{Irr}(K'Y')$, $\chi \mapsto \chi'$, *such that*

$$\chi'(1 \otimes y) = \psi^*(\chi(y)) \qquad \text{for all } y \in Y.$$

*Here $\psi^* : R^* \to K^*$ is an extension of $\psi$ to the integral closure $R^*$ of $R$ in $K$ and $K^*$ is some extension field of $K'$. The bijection preserves dimensions of underlying simple modules.*

(b) *If $t$ is the restriction to $Y$ of a linear combination $\sum_{\chi \in \mathrm{Irr}(KY)} s_\chi \chi$ with $s_\chi \in R_{\mathfrak{p}}$, then the $R'$-linear map $t' : Y' \to R'$ is the restriction to $Y'$ of $\sum_{\chi \in \mathrm{Irr}(KY)} \psi(s_\chi)\chi'$. Here $\mathfrak{p}$ is the kernel of $\psi$, $R_{\mathfrak{p}}$ is the corresponding localisation and $\psi(s_\chi)$ is the image of $s_\chi$ under the unique extension of $\psi$ to a ring homomorphism $R_{\mathfrak{p}} \to K'$.*

*Proof.* For part (a) see [14, Thm 68.17, Cor. 68.20]. Note that the bijection given in [14] is between the sets $\mathrm{Irr}(\bar{K}Y)$ and $\mathrm{Irr}(\bar{K}'Y')$ where $\bar{K}$ and $\bar{K}'$ are algebraic closures of $K$ and $K'$ respectively. Since $KY$ and $K'Y'$ are split, this descends via restriction on both sides to a bijection $\mathrm{Irr}(KY) \to \mathrm{Irr}(K'Y')$. Now (b) is an immediate consequence of (a). $\square$

## 3. Hecke algebras and their Schur elements

Let $W$ be a finite complex reflection group, that is, a finite subgroup of $\mathrm{GL}_n(\mathbb{C})$ for some $n \geq 1$ generated by complex reflections. We denote by $\mathbb{Q}_W$ the field generated by the traces of elements of $W$, a finite extension of $\mathbb{Q}$. Attached to any reflection $r \in W$ is its reflecting hyperplane $H = \ker(1-r)$ in $\mathbb{C}^n$; its point-wise stabiliser $W_H$ in $W$ is cyclic, generated by reflections. We say that $r \in W_H$ is the *distinguished reflection* associated to $H$ if $r$ generates $W_H$ and has non-trivial eigenvalue $\exp(2\pi\mathbf{i}/o(r))$ of smallest possible argument.

3.1. **From braid groups to trace forms.** Let $B(W)$ be the topological braid group of $W$ (see Broué–Malle–Rouquier [8]), that is, the fundamental group of the space of regular orbits of $W$ on $\mathbb{C}^n$. So there is an associated exact sequence

$$1 \to P(W) \to B(W) \to W \to 1,$$

with kernel the pure braid group $P(W)$. Let $A := \mathbb{Z}_W[\mathbf{u}^{\pm 1}]$, where $\mathbb{Z}_W$ denotes the ring of integers of $\mathbb{Q}_W$ and where $\mathbf{u} = (u_{rj})$ are algebraically independent elements indexed by conjugacy classes of distinguished reflections $r \in W$ and $1 \leq j \leq o(r)$. By [8, Def. 4.21] the *(generic) Hecke algebra* $\mathcal{H}(W, \mathbf{u})$ of $W$ is defined to be the quotient of the group algebra $A[B(W)]$ of $B(W)$ over $A$ by the ideal generated by the elements (called *deformed order relations*)

$$\prod_{j=1}^{o(r)} (\mathbf{r} - u_{rj}) \tag{$\mathcal{H}$}$$

for $\mathbf{r}$ running over the braid reflections of $B(W)$ (introduced as *generators of the monodromy around a hyperplane* in [8, 2B]), with $r$ denoting the image of $\mathbf{r}$ in $W$, a distinguished reflection. We will write $A[B(W)] \to \mathcal{H}(W, \mathbf{u})$, $\mathbf{x} \mapsto h_{\mathbf{x}}$, for the associated quotient map.

We have the following result, first conjectured in [5], with the final cases having been established by Chavli, Marin and Tsuchioka, respectively, see [10, 34, 37] and the references therein:

**Theorem 3.1** ('Freeness Conjecture'). *The algebra $\mathcal{H}(W,\mathbf{u})$ is $A$-free of rank $|W|$.*

By results of the second author [27, Cor. 4.8] there is a positive integer $z$ such that the field of fractions $K_W$ of $\tilde{A} := \mathbb{Z}_W[\tilde{\mathbf{u}}^{\pm 1}] \supseteq A$ is a splitting field of $\mathcal{H}(W,\mathbf{u})$, where $\tilde{\mathbf{u}} = (\tilde{u}_{rj})$ are such that $\tilde{u}_{rj}^z = \exp(-2\pi\mathbf{i}j/o(r))u_{rj}$ for all $r, j$.

Furthermore, by [28, Prop. 7.1] to each irreducible character $\chi$ of $K_W \otimes_A \mathcal{H}(W,\mathbf{u})$ is associated an element $f_\chi \in \tilde{A}$ called the *Schur element* of $\chi$. The Schur element $p_W := p_W(\mathbf{u}) := f_{1_W}$ of the trivial character $1_W$ is called the *Poincaré polynomial* of $\mathcal{H}(W,\mathbf{u})$ (a Laurent polynomial in the $\tilde{u}_{rj}$). For simply-laced Coxeter groups, this is in fact the homogenisation of the usual Poincaré polynomial of $W$.

The collection $\{(\chi, f_\chi) \mid \chi \in \mathrm{Irr}(\mathcal{H}(W,\mathbf{u}))\}$ is Galois-invariant, so in particular

$$t_{W,\mathbf{u}}(h) := \sum_{\chi \in \mathrm{Irr}(\mathcal{H}(W,\mathbf{u}))} \frac{1}{f_\chi}\chi(h)$$

lies in $\mathrm{Frac}(A)$ for all $h \in \mathcal{H}(W,\mathbf{u})$. Thus, this defines a symmetric $A$-linear map

$$t_{W,\mathbf{u}} : \mathcal{H}(W,\mathbf{u}) \to \mathrm{Frac}(A), \quad h \mapsto t_{W,\mathbf{u}}(h),$$

called the *canonical trace form on* $\mathcal{H}(W,\mathbf{u})$. We denote also by $t_{W,\mathbf{u}}$ its $\mathrm{Frac}(A)$-linear extension to $\mathcal{H}_{\mathrm{Frac}(A)}(W,\mathbf{u}) := \mathrm{Frac}(A) \otimes_A \mathcal{H}(W,\mathbf{u})$. This form satisfies the following property:

**Proposition 3.2.** *There exists a set $\mathcal{B} \subset \mathcal{H}(W,\mathbf{u})$ consisting of monomials in images of braid reflections with $1 \in \mathcal{B}$, such that $t_{W,\mathbf{u}}(b) = \delta_{1,b}$ for $b \in \mathcal{B}$, and the images in $W$ of the $b \in \mathcal{B}$ under the canonical map form a system of representatives of the conjugacy classes of $W$.*

*Proof.* First note that the claim easily reduces to the case of irreducible reflection groups. For those, it holds by the construction of the Schur elements $f_\chi$, see [25] and [28] for the exceptional types, and [17, Thm 1.3, Lemma 4.3 and §4.5] for the infinite series. □

Furthermore, $t_{W,\mathbf{u}}$ satisfies a duality with respect to a certain central element, but this will not be of importance here. In analogy with the case of finite Coxeter groups, the theory of spetses predicts the following, see [6, Thm-Ass. 2.1]:

**Conjecture 3.3.** *The form $t_{W,\mathbf{u}}$ takes values in $A$ and is a symmetrising form on $\mathcal{H}(W,\mathbf{u})$.*

The above conjecture has been established for all imprimitive complex reflection groups $G(e,1,m)$, for example, by Malle–Mathas [31, Thm].

Let $W_0 \leq W$ be a reflection subgroup. We denote by $\mathcal{H}(W_0,\mathbf{u}_0)$ the Hecke algebra of $W_0$ whose parameters $\mathbf{u}_0$ consist of those parameters for $W$ whose corresponding reflections are, up to conjugacy, contained in $W_0$. Since non-conjugate reflections in $W_0$ might be conjugate in $W$, $\mathcal{H}(W_0,\mathbf{u}_0)$ is a specialisation of the generic Hecke algebra of $W_0$ corresponding to an identification of certain of its parameters. It follows from the Freeness Conjecture (Theorem 3.1) that $\mathcal{H}(W_0,\mathbf{u}_0)$ is naturally a subalgebra of $\mathcal{H}(W,\mathbf{u})$. Moreover, by the explicit results in [27, Thm 5.2] the field $K_W$ is also a splitting field for $\mathcal{H}(W_0,\mathbf{u}_0)$. By $t_{W_0,\mathbf{u}_0}$ we will mean the corresponding specialisation of the canonical form of the generic Hecke algebra of $W_0$. Recall that, by a theorem of Steinberg, all parabolic subgroups of $W$, that is, stabilisers in $W \leq \mathrm{GL}_n(\mathbb{C})$ of subspaces of $\mathbb{C}^n$, are reflection subgroups of $W$.

**Definition 3.4.** We will say that $\mathcal{H}(W,\mathbf{u})$ is *strongly symmetric* if the following holds:

(1) $t_{W,\mathbf{u}}$ is a symmetrising form on $\mathcal{H}(W,\mathbf{u})$ and there is a section $W \to \mathbf{W} \subset B(W)$ of the natural map $B(W) \to W$ containing 1 whose image in $\mathcal{H}(W,\mathbf{u})$ is an $A$-basis of $\mathcal{H}(W,\mathbf{u})$ with $t_{W,\mathbf{u}}(h_\mathbf{w}) = \delta_{\mathbf{w},1}$ for all $\mathbf{w} \in \mathbf{W}$; and

(2) for any parabolic subgroup $W_0 \leq W$, $t_{W_0,\mathbf{u}_0}$ is a symmetrising form on $\mathcal{H}(W_0,\mathbf{u}_0)$ and $t_{W,\mathbf{u}}|_{\mathcal{H}(W_0,\mathbf{u}_0)} = t_{W_0,\mathbf{u}_0}$.

Note that the assertion that (2) holds is referred to as the parabolic trace conjecture in [11]. Strong symmetry is known to be satisfied in many cases; here we use the Shephard–Todd notation for irreducible complex reflection groups:

**Proposition 3.5.** *For the following irreducible groups, $\mathcal{H}(W,\mathbf{u})$ is strongly symmetric:*

(a) *for $W$ a Coxeter group;*

(b) *for $W = G(e,p,n)$ with $n \neq 2$ or $p$ odd; and*

(c) *for $W = G_i$, $i \in \{4,5,6,7,8\}$.*

*Proof.* First note that property (2) of being strongly symmetric follows for a parabolic subgroup $W_0$ of $W$ if there exists $\mathbf{W} \subset B(W)$ as in (1) such that

(2') $\{h_\mathbf{w} \mid w \in W_0\}$ is an $A$-basis of $\mathcal{H}(W_0,\mathbf{u}_0)$ and $t_{W_0,\mathbf{u}_0}(h_\mathbf{w}) = \delta_{\mathbf{w},1}$ for all $w \in W_0$.

For Coxeter groups, (1) and (2') hold for any section $\mathbf{W}$ consisting of reduced expressions in the standard generators, see [18, Prop. 8.1.1]. For $W = G(e,p,n)$ the existence of $\mathbf{W}$ satisfying (1) is shown in [31, Thm 5.1]. Since any parabolic subgroup of $W$ is a Young subgroup, that is, a product of symmetric groups with a group $G(e,p,n')$ for $n' \leq n$, the Ariki–Koike basis of $\mathcal{H}(W,\mathbf{u})$ considered in [31] also satisfies (2'). (See also [6, p. 177]). The claim for $G_i$, $i \in \{4,5,6,7,8\}$, follows from the explicit results in [2]. □

3.2. **Specialisations.** By a *specialisation* we will mean a ring homomorphism $\psi : A' \to R$ where $A'$ and $R$ are commutative rings with $A' \supseteq A$. We then set $\mathcal{H}_\psi(W,\mathbf{u}) := R \otimes (A' \otimes_A \mathcal{H}(W,\mathbf{u}))$. If $\psi$ is inclusion we will sometimes write $\mathcal{H}_R(W,\mathbf{u})$ instead of $\mathcal{H}_\psi(W,\mathbf{u})$. The restriction of any specialisation $\psi$ to a subring of $A'$ will again be denoted by $\psi$ as will the composition of $\psi$ with any inclusion $R \hookrightarrow R'$.

We will consider certain types of specialisations. For the remainder of this section, let $R$ be an integral domain containing $\mathbb{Z}_W$ and let $K$ be its field of fractions. Let $\psi_1 : R[\tilde{\mathbf{u}}] \to R$ be the $R$-linear homomorphism defined by $\psi_1(\tilde{u}_{rj}) := 1$ for all $r$ and $j$. So, $\psi_1$ restricts to the specialisation

$$A \to \mathbb{Z}_W, \quad u_{rj} \mapsto \exp(2\pi \mathbf{i} j/o(r)).$$

By Bessis [1, Thm 0.1], the Hecke algebra maps to the group algebra $RW$ of $W$ under $\psi_1$. Combining this with the Freeness Conjecture, the Tits deformation theorem gives that $\mathcal{H}_{\tilde{K}}(W,\mathbf{u}) := \tilde{K} \otimes_A \mathcal{H}(W,\mathbf{u})$ is isomorphic to the group algebra $\tilde{K}W$, where $\tilde{K} = \mathrm{Frac}(R[\tilde{\mathbf{u}}])$ (see Theorem 2.9 and note that $K_W \subseteq \tilde{K}$). Thus, we may and will identify the irreducible characters of $\mathcal{H}_{\tilde{K}}(W,\mathbf{u})$ with $\mathrm{Irr}(W)$ via the bijection $\chi_\phi \leftrightarrow \phi$ induced by $\psi_1$. This also induces a labelling of Schur elements of $\mathcal{H}(W,\mathbf{u})$ by $\mathrm{Irr}(W)$ and we will henceforth denote them as $f_\phi$, $\phi \in \mathrm{Irr}(W)$. Since $K_W$ is also a splitting field for the Hecke algebra $\mathcal{H}(W_0,\mathbf{u}_0)$ of any reflection subgroup $W_0$ of $W$ we also have $\mathcal{H}_{\tilde{K}}(W_0,\mathbf{u}) \cong \tilde{K}W_0$. We will similarly identify the irreducible characters of $\mathcal{H}(W_0,\mathbf{u}_0)$ over $\tilde{K}$ with $\mathrm{Irr}(W_0)$.

Now let $q$ be a prime power and suppose that $R$ contains $q^{\pm 1/z}$. We also consider $R$-linear specialisations of the form

$$\psi_q : R[\tilde{\mathbf{u}}^{\pm 1}] \to R, \quad \tilde{u}_{rj} \mapsto q^{a_{rj}/z},$$

for integers $a_{rj}$. Any such $\psi_q$ restricts to a specialisation

$$A \to \mathbb{Z}_W, \quad u_{rj} \mapsto \zeta_{o(r)}^j q^{a_{rj}},$$

with $\zeta_{o(r)} \in \mathbb{Z}_W$ an $o(r)$th primitive root of unity.

**Lemma 3.6.** *For all $\phi \in \mathrm{Irr}(W)$ we have $\psi_1(f_\phi) = \phi(1)/|W|$ and $\psi_q(f_\phi) \neq 0$.*

*Proof.* The assertion on $\psi_1$ follows since by construction the set $\mathcal{B}$ from Proposition 3.2 specialises under $\psi_1$ to a system of representatives of the conjugacy classes of $W$. Next, the following can be observed from the explicit form of the Schur elements (and was first stated explicitly in [12, Thm 4.2.5]): any $f_\phi$ is a product of a scalar, a monomial in the $\tilde{u}_{rj}$ and a product of cyclotomic polynomials $\Psi_i$ over $K_W$ evaluated at monomials $M_i$ in the $\tilde{u}_{rj}^{\pm 1}$ of total degree 0. Thus we need to see that $\psi_q(\Psi_i(M_i)) \neq 0$. This is clear if $\psi_q(M_i)$ is not a root of unity. Now $\psi_q(M_i)$ can only be a root of unity, if the powers of $q$ cancel completely in $\psi_q(M_i)$, which means that $\psi_q(M_i) = \psi_1(M_i)$ and so $\psi_q(\Psi_i(M_i)) = \psi_1(\Psi_i(M_i))$. But the latter is a factor of $\psi_1(f_\phi) = \phi(1)/|W|$ and hence non-zero. $\square$

For the next result we note that symmetrising forms remain symmetrising after specialisation, that is if $\theta : \mathcal{O} \to \mathcal{O}'$ is a ring homomorphism, $Y$ is an $\mathcal{O}$-algebra which is finitely generated and free as $\mathcal{O}$-module and $\tau : Y \to \mathcal{O}$ is a symmetrising form, then the induced $\mathcal{O}'$-linear form $\tau'$ on $Y' := \mathcal{O}' \otimes_{\mathcal{O}} Y$ satisfying $\tau'(1 \otimes y) = \tau(y)$ for $y \in Y$ is a symmetrising form on $Y'$.

**Lemma 3.7.** *Assume that $\mathcal{H}(W, \mathbf{u})$ is symmetric over $A$ with respect to the form $t_{W,\mathbf{u}}$ with Schur elements $f_\phi \in \tilde{A}$, $\phi \in \mathrm{Irr}(\mathcal{H}(W, \mathbf{u}))$, and let $K = \mathrm{Frac}(R)$. The algebra $K\mathcal{H}_{\psi_q}(W, \mathbf{u})$ is split semisimple. Let $t'$ be the induced form on $\mathcal{H}_{\psi_q}(W, \mathbf{u})$. For each $\phi \in \mathrm{Irr}(W)$, $\psi_q(f_\phi)$ is the corresponding Schur element of $t'$.*

*Proof.* By Lemma 3.6 we have $\psi_q(f_\phi) \neq 0$ for all $\phi \in \mathrm{Irr}(W)$. Now $\psi_q$ is a concatenation of specialisation maps whose kernel is a prime ideal of height 1. So by [12, Thm 2.4.12], $K\mathcal{H}_{\psi_q}(W, \mathbf{u})$ is also split semisimple. The result thus follows by Tits' deformation theorem (Theorem 2.9). $\square$

Finally, for an indeterminate $x$ we will also consider the *spetsial specialisation*

$$\psi_{\mathrm{s}} : R[\tilde{\mathbf{u}}^{\pm 1}] \to R[x^{\pm \frac{1}{z}}], \qquad \tilde{u}_{rj} \mapsto \begin{cases} x^{\frac{1}{z}} & \text{if } j = o(r), \\ 1 & \text{if } 1 \leq j < o(r) \end{cases}$$

(so $\psi_{\mathrm{s}}(u_{r,o(r)}) = x$, $\psi_{\mathrm{s}}(u_{rj}) = \exp(2\pi \mathbf{i} j/o(r))$ for $j < o(r)$). Clearly $\psi_1$ factorises through $\psi_{\mathrm{s}}$ by composition with $x^{1/z} \mapsto 1$ as does any specialisation $\psi_q$ for the case $a_{rj} = 1$ if $j = o(r)$ and $a_{rj} = 0$ if $j < o(r)$ by composition with $x^{1/z} \mapsto q^{1/z}$. We write $\psi_{\mathrm{s},q}$ for this latter specialisation, which will become important in Section 5.6. The spetsial specialisation links Schur elements of Hecke algebras to unipotent character degrees of spetses (see Section 4.4).

## 4. Conjectures for the principal block of a $\mathbb{Z}_\ell$-spets

In this section we define the dimension of the principal block of a $\mathbb{Z}_\ell$-spets, introduced in [21, §6.2], and propose some conjectures around this notion.

4.1. **The principal block of a $\mathbb{Z}_\ell$-spets.** Let $\ell$ be a prime and let $q$ be a prime power not divisible by $\ell$. Recall that (under some conditions) the set of characters of the principal $\ell$-block of a finite reductive group $G$ over $\mathbb{F}_q$ can be described as a union of sets of characters in bijection with the principal $e$-Harish Chandra series of unipotent characters of dual-centraliser subgroups $C_{G^*}(s)^*$ where $G^*$ is the "Langlands dual" of $G$ and $s$ runs over conjugacy classes of $\ell$-elements of $G^*$. The principal block of a $\mathbb{Z}_\ell$-spets as constructed in [21] is modelled on this description: unipotent characters and the appropriate Harish-Chandra series are provided by the theory of spetses whereas the indexing set of $\ell$-elements and corresponding centralisers comes from the theory of $\ell$-compact groups and fusion systems. In the next paragraph, we briefly recall this construction. Before doing so, we point out that the description of the principal block of a $\mathbb{Z}_\ell$-spets does not involve going over to the dual as one would expect from analogy with the group case. The reason for this is that the fusion system construction that we rely on is for the moment only available for simply connected $\ell$-compact groups. However, this departure does not lead to an inconsistency in the group case under the conditions on $\ell$ with which we are concerned (see [21, Prop. 6.8]).

We assume from now till the end of the section that $\ell > 2$. Let $W \leq \mathrm{GL}(L)$ be a finite *spetsial* (see [26, §3]) $\ell$-adic reflection group on a $\mathbb{Z}_\ell$-lattice $L$, and let $X$ be the associated connected $\ell$-compact group (see [4, Thm 1.1]). We assume moreover that $X$ is simply connected and that $\ell$ is *very good* for $(W, L)$, in the sense of [21, Def. 2.4]. Let $\varphi \in N_{\mathrm{GL}(L)}(W)$ be of $\ell'$-order and $\mathbb{G} = (W\varphi^{-1}, L)$ be the associated simply connected $\mathbb{Z}_\ell$-spets. For example any Weyl group $W$ for which $\ell$ is very good (in the classical sense) determines a $\mathbb{Z}_\ell$-spets satisfying the above conditions.

Set $\mathbb{G}(q) := (W\varphi^{-1}, L, q)$. In [4], Broto and Møller showed how to attach to these data a fusion system $\mathcal{F}$ on a finite $\ell$-group $S$, via the associated $\ell$-compact group $X$ (see [21, Thm 3.2]). Here, $S$ is an extension of a homocyclic $\ell$-group $T$ (the *toral part*) by a Sylow $\ell$-subgroup of the associated relative Weyl group $W_{\varphi^{-1}\zeta^{-1}}$ (see [21, Thm 3.6]). Note that $S$ and $\mathcal{F}$ only depend on the $\ell$-part $\ell^a$ of $q - \zeta$, where $\zeta \in \mathbb{Z}_\ell^\times$ is the root of unity with $q \equiv \zeta \pmod{\ell}$, not on $q$ itself.

If $\mathbb{G}(q)$ arises from a connected reductive algebraic group $\mathbf{G}$ over $\overline{\mathbb{F}_q}$ with Weyl group $W$ and a Frobenius morphism $F : \mathbf{G} \to \mathbf{G}$ with respect to an $\mathbb{F}_q$-structure acting as $\varphi$ on $W$, then $S$ is a Sylow $\ell$-subgroup of $\mathbf{G}^F$ and $\mathcal{F}$ is the $\ell$-fusion system of $\mathbf{G}^F$ on $S$ (see [21, Rem. 3.3 and Sec. 5.3]). In particular, $T = (\mathbf{T}^F)_\ell$ where $\mathbf{T}$ is a maximal $e$-split torus of $\mathbf{G}$.

Under our assumptions, for any $s \in S$ the centraliser $W(s) := C_W(s)$ of $s$ is again an $\ell$-adic reflection group, a reflection subgroup of $W$ (see the proof of [21, Thm 5.2] or [21, Prop. 2.3])), and by [21, Prop. 6.2], it is again spetsial. We let $C_\mathbb{G}(s) := (W(s)\varphi_s^{-1}, L)$ be the associated $\mathbb{Z}_\ell$-spets, where $\varphi_s \in N_{W\varphi}(W(s))$ is defined as in [21, §5.2].

Now recall from [24] (for the infinite series of irreducible complex reflection groups) and [7] (for the primitive ones) that associated to $\mathbb{G}$ as well as to the various $C_\mathbb{G}(s)$ there are sets of unipotent characters $\mathrm{Uch}(\mathbb{G})$ and $\mathrm{Uch}(C_\mathbb{G}(s))$, respectively. If $W$ is a Weyl group, these are just the unipotent characters of an associated finite reductive group. For $s \in S$ we let

$$\mathcal{E}(\mathbb{G}, s) = \{\gamma_{s,\lambda} \mid \lambda \in \mathrm{Uch}(C_{\mathbb{G}}(s))\}$$

denote a set in bijection with $\mathrm{Uch}(C_{\mathbb{G}}(s))$ and call it the *characters of* $\mathbb{G}$ *in the series* $s$. The sets $\mathrm{Uch}(C_{\mathbb{G}}(s))$ are in canonical bijection for conjugate elements $s$. Moreover, these sets only depend on $\ell^a$, not on $q$ itself. Any unipotent character $\lambda$ comes with a degree (polynomial) $\lambda(1) \in \mathbb{Z}_\ell[x]$. The *degree of* $\gamma_{s,\lambda} \in \mathcal{E}(\mathbb{G}, s)$ is defined as

$$\gamma_{s,\lambda}(1) := |\mathbb{G} : C_{\mathbb{G}}(s)|_{x'}\, \lambda(1) \in \mathbb{Z}_\ell[x].$$

Here, $|\mathbb{G} : C_{\mathbb{G}}(s)|_{x'}$ means the prime-to-$x$ part of the polynomial $|\mathbb{G}|/|C_{\mathbb{G}}(s)| \in \mathbb{Z}_\ell[x]$, where $|\mathbb{G}|, |C_{\mathbb{G}}(s)|$ are the respective order polynomials; note that the latter divides the former by [21, Lemma 6.6].

Now by [24, Folg. 3.16 and 6.11] and [7, 4.31] for any $\mathbb{Z}_\ell$-spets $\mathbb{H}$, for any root of unity $\eta$ the set of unipotent characters $\mathrm{Uch}(\mathbb{H})$ is naturally partitioned into so-called $\eta$-Harish-Chandra series, and one among them, the *principal* $\eta$*-Harish-Chandra series* $\mathcal{E}(\mathbb{H}, 1, \eta)$ of $\mathrm{Uch}(\mathbb{H})$ containing $1_{\mathbb{G}}$, is in bijection with the irreducible characters of the corresponding Springer–Lehrer relative Weyl group. In particular, $\mathcal{E}(\mathbb{H}, 1, 1)$ is in bijection with $\mathrm{Irr}(W^\varphi)$.

With this, for $\zeta$ as above denote by $\mathcal{E}(\mathbb{G}, s)_\zeta$ the subset of $\mathcal{E}(\mathbb{G}, s)$ in bijection with the principal $\zeta$-Harish-Chandra series of $\mathrm{Uch}(C_{\mathbb{G}}(s))$, and hence also in bijection with the irreducible characters of the relative Weyl group $W(s)_{\varphi_s^{-1}\zeta^{-1}}$.

The following definition from [21, §6.2] is inspired by the results of Cabanes–Enguehard on unipotent $\ell$-blocks of finite reductive groups; indeed, if $\mathbb{G}$ is a rational spets for which $\ell$ is very good, then what we define are exactly the characters in the principal $\ell$-block of the corresponding finite group of Lie type $\mathbb{G}(q)$ (see [21, Prop. 6.8]):

**Definition 4.1.** Let $\mathbb{G} = (W\varphi^{-1}, L)$ be a simply connected $\mathbb{Z}_\ell$-spets with $\varphi$ of $\ell'$-order such that $\ell$ is very good for $\mathbb{G}$ (in the sense of [21, Def. 2.4]) and $q$ a prime power with $q \equiv \zeta \pmod{\ell}$. The *characters in the principal block* $B_0$ *of* $\mathbb{G}(q)$ are

$$\mathrm{Irr}(B_0) := \coprod_{s \in S/\mathcal{F}} \mathcal{E}(\mathbb{G}, s)_\zeta,$$

where the union runs over a set $S/\mathcal{F}$ of representatives $s$ of $\mathcal{F}$-conjugacy classes in $S$. The *dimension of* $B_0$ is defined as

$$\dim(B_0) = \sum_{\gamma \in \mathrm{Irr}(B_0)} \gamma(1)^2 = \sum_{s \in S/\mathcal{F}} \sum_{\lambda \in \mathcal{E}(C_{\mathbb{G}}(s), 1, \zeta)} \gamma_{s,\lambda}(1)^2 \in \mathbb{Z}_\ell[x].$$

Again, these do not depend on $q$ but only on $\ell^a = (q - \zeta)_\ell$.

The dimension of the principal $\ell$-block $B_0$ of a finite group $G$ with Sylow $\ell$-subgroup $S$ satisfies $(\dim B_0)_\ell = |S|$ (see Proposition 2.8). We conjecture that our dimension of the principal block $\dim(B_0)$ of $\mathbb{G}(q)$ behaves similarly:

**Conjecture 4.2.** *Let $\mathbb{G} = (W\varphi^{-1}, L)$ be a simply connected $\mathbb{Z}_\ell$-spets such that $\ell$ is very good for $\mathbb{G}$. Let $q$ be a prime power not divisible by $\ell$ and $S$ the associated $\ell$-group. Then*

$$(\dim(B_0)|_{x=q})_\ell = |S|.$$

By further analogy with the group case (see Proposition 2.8) we also conjecture the following global-local statement:

**Conjecture 4.3.** *In the setting of Conjecture* 4.2, *if* $\zeta \in \mathbb{Z}_\ell^\times$ *with* $q \equiv \zeta \pmod{\ell}$ *we have*

$$\big(\dim(B_0)|_{x=q}\big)_{\ell'} \equiv |W_{\varphi^{-1}\zeta^{-1}}|_{\ell'} \pmod{\ell}.$$

Note that Conjectures 4.2 and 4.3 combine to form Conjecture 1.

**Example 4.4.** We describe $S$, $\mathcal{F}$, and $\mathrm{Irr}(B_0)$ in a special case relevant to Theorem 1. Suppose that $q \equiv 1 \pmod{\ell}$ and $\varphi = 1$. Then $T$ may be identified with $L/\ell^a L$ where $\ell^a || (q-1)$, $W = W_{\varphi^{-1}\zeta^{-1}}$ and the action of $W$ on $T$ in $S$ is the one inherited from the action of $W$ on $L$. Assume in addition that $W$ is an $\ell'$-group. Then the condition that $\ell$ is very good for $(W, L)$ always holds (see [21, Prop. 2.6]). Further, $S = T$ and $\mathcal{F}$ is the $\ell$-fusion system $\mathcal{F}_{SW}(S)$ of the group $SW$ on $S$ (see [21, Thm 3.4] and [4, Thm 9.8]). Moreover, for any $s \in S = T$ the subgroup $W(s)_{\varphi_s^{-1}\zeta^{-1}}$ equals $W(s)$ and hence $\mathcal{E}(\mathbb{G}, s)_1$ is in bijection with $\mathrm{Irr}(W(s))$. So $\mathrm{Irr}(B_0)$ is in bijection with $W$-classes of pairs $(s, \phi)$ where $s \in S$ and $\phi \in \mathrm{Irr}(W(s))$.

4.2. **Decomposition numbers.** Suppose in this subsection that $|W|$ is prime to $\ell$, that $\ell \mid (q-1)$ and that $\varphi = 1$. Recall the description of the principal block $B_0$ under these assumptions given in Example 4.4.

Since $W$ is an $\ell'$-group and $S$ an $\ell$-group, we may identify $\mathrm{IBr}(SW)$ with the subset $\mathrm{Irr}(W)$ of $\mathrm{Irr}(SW)$. Similarly, we think of the unipotent characters in $B_0$ as the irreducible Brauer characters of $B_0$ and set

$$\mathrm{IBr}(B_0) := \mathcal{E}(\mathbb{G}, 1)_1 \subseteq \mathrm{Irr}(B_0).$$

We associate decomposition numbers, and formal degrees of projective indecomposable characters to $B_0$ as follows.

The $\mathcal{F}$-classes of elements of $S$ are the $W$-conjugacy classes of $S$. Further, since $(|W|, \ell) = 1$, the Glauberman–Isaacs correspondence gives that the actions of $W$ on $S$ and on $\mathrm{Irr}(S)$ are permutation isomorphic. Thus there is a bijection between the set of $W$-classes of $\mathrm{Irr}(S)$ and the set of $W$-classes of $S$ such that if the class of $s \in S$ corresponds to the class of $\hat{s} \in \mathrm{Irr}(S)$, then $W_s = W_{\hat{s}}$ where $W_s, W_{\hat{s}}$ denotes the stabiliser in $W$ of $s$, $\hat{s}$ respectively. Note that $W_s$ was denoted $W(s)$ in Example 4.4. Such a bijection between the set of $W$-classes of $\mathrm{Irr}(S)$ and of $S$ will be called *$W$-equivariant* if in addition the (class of) $1 \in S$ is sent to the (class of the) trivial character of $S$. Note that a $W$-equivariant bijection always exists.

By Clifford theory,

$$\mathrm{Irr}(SW) = \coprod_{\theta \in \mathrm{Irr}(S)/W} \mathrm{Irr}(SW|\theta),$$

where the union runs over a set $\mathrm{Irr}(S)/W$ of representatives $\theta$ of $W$-conjugacy classes of $\mathrm{Irr}(S)$ and where $\mathrm{Irr}(SW|\theta)$ denotes the set of irreducible characters of $SW$ covering $\theta$. Moreover, since $|W|$ is prime to $\ell$, $\mathrm{Irr}(SW|\theta)$ is in bijection with $\mathrm{Irr}(W_\theta)$.

By the description of $\mathrm{Irr}(B_0)$ given in Example 4.4, $|\mathrm{Irr}(B_0)| = |\mathrm{Irr}(SW)|$. A bijection $\Theta : \mathrm{Irr}(SW) \to \mathrm{Irr}(B_0)$, $\gamma \mapsto \hat{\gamma}$, will be said to be $W$-equivariant if there exists a $W$-equivariant bijection $\mathrm{Irr}(S) \to S$ such that for corresponding elements $s \in S$ and $\hat{s} \in \mathrm{Irr}(S)$, $\Theta$ restricts to a bijection $\mathrm{Irr}(SW|\hat{s}) \to \mathcal{E}(\mathbb{G}, s)_1$. Since $\mathrm{IBr}(SW)$ is in bijection with $\mathrm{Irr}(W|1)$, any $W$-equivariant bijection $\mathrm{Irr}(SW) \to \mathrm{Irr}(B_0)$ restricts to a bijection $\mathrm{IBr}(SW) \to \mathrm{IBr}(B_0)$.

Let $\mathrm{Irr}(SW) \to \mathrm{Irr}(B_0)$, $\gamma \mapsto \hat{\gamma}$, be a $W$-equivariant bijection. We declare the decomposition matrix of $B_0$ to be the $\ell$-decomposition matrix of $SW$ via this bijection, that is if $d_{\gamma\nu}$ is the decomposition number in $SW$ corresponding to $\gamma \in \mathrm{Irr}(SW)$ and $\nu \in \mathrm{IBr}(SW)$, then we regard $d_{\gamma\nu}$ also as the decomposition number for $\hat{\gamma} \in \mathrm{Irr}(B_0)$ and $\hat{\nu} \in \mathrm{IBr}(B_0)$. Recall that for any $\gamma \in \mathrm{Irr}(SW)$, we have $\gamma(1) = \sum_{\nu\in\mathrm{IBr}(SW)} d_{\gamma\nu}\nu(1)$. In Proposition 4.11 we show that the analogous equations hold in $B_0$. We define

$$\deg \Phi_{\hat{\nu}} := \sum_{\gamma\in\mathrm{Irr}(SW)} d_{\gamma\nu} \deg(\hat{\gamma}) \in \mathbb{Z}_\ell[x] \qquad \text{for } \hat{\nu} \in \mathrm{IBr}(B_0),$$

to be the *formal degrees of projective indecomposable characters of* $B_0$. The following is a restatement of Conjecture 2.

**Conjecture 4.5.** *In the setting of Conjecture* 4.2*, if* $W$ *has order coprime to* $\ell$*,* $\varphi$ *is trivial and* $q \equiv 1 \pmod{\ell}$*, then for some* $W$*-equivariant bijection* $\mathrm{Irr}(SW) \xrightarrow{\sim} \mathrm{Irr}(B_0)$ *we have that* $|S|$ *divides* $(\deg \Phi_{\hat{\nu}})|_{x=q}$ *for all* $\hat{\nu} \in \mathrm{IBr}(B_0)$.

4.3. **The rational and the primitive cases.** We'll prove the above conjectures for most $W$ of order coprime to $\ell$ in Theorem 5.20. For the moment, let us see why they hold in the rational case:

**Proposition 4.6.** *Conjectures* 4.2*,* 4.3*, and* 4.5 *hold if* $\mathbb{G}$ *is a rational spets underlying a finite reductive group.*

*Proof.* Let $\mathbf{G}$ be a connected reductive group over an algebraically closed field of characteristic $p$ and $F : \mathbf{G} \to \mathbf{G}$ a Frobenius endomorphism with respect to an $\mathbb{F}_q$-structure, such that $\mathbb{G}$ is the underlying spets. That is, $\varphi$ is the automorphism of $W$ induced by $F$. Recall that $S$ may be identified with a Sylow $\ell$-subgroup of $\mathbf{G}^F$ and $\mathcal{F}$ with the fusion system $\mathcal{F}_{\mathbf{G}^F}(S)$ (see [21, Rem. 3.3(a) and Sec. 5.3]). Let $d$ be the order of $\zeta$, hence the order of $q$ modulo $\ell$. By [21, Prop. 6.8], there is a degree preserving bijection between $\mathrm{Irr}(B_0)$ and the set of irreducible characters of the principal $\ell$-block $B_0(\mathbf{G}^F)$. Note that the bijection given in [21, Prop. 6.8] is stated to preserve defects but it is easy to check from the setup that for any irreducible character $\gamma_{s,\lambda}$ in $B_0$, $\gamma_{s,\lambda}(1)|_{x=q}$ is the degree of the corresponding character of $B_0(\mathbf{G}^F)$. Now the assertion regarding Conjecture 4.2 follows from Proposition 2.8.

As described in Section 4.1, $S = T.(W_1)_\ell$ with $W_1 := W_{\varphi^{-1}\zeta^{-1}}$. Under our assumptions on $\ell$, $\mathbf{L} := C_{\mathbf{G}}(T)$ is a Levi subgroup, and moreover $N_{\mathbf{G}}(S)^F \le N_{\mathbf{G}}(T)^F = N_{\mathbf{G}}(\mathbf{L})^F$ (see [30, Thm 5.9 and 5.14]). Let $H := N_{\mathbf{G}}(\mathbf{L})^F$. Then Proposition 2.8 shows that in order to prove Conjecture 4.3 for $\mathbb{G}$ it suffices to see that $(\dim B_0(H))_{\ell'} \equiv |W_1|_{\ell'} \pmod{\ell}$. Now $B_0(H)$ is isomorphic to the principal block of $H/O_{\ell'}(\mathbf{L}^F)$. Since $S/T$ acts faithfully on $T$, $T$ is a Sylow $\ell$-subgroup of $\mathbf{L}^F$, so $H/O_{\ell'}(\mathbf{L}^F) \cong T(N_{\mathbf{G}}(\mathbf{L})^F/\mathbf{L}^F) \cong TW_1$, the latter by the definition of relative Weyl groups. The result follows as $TW_1$ has a unique $\ell$-block.

Finally, we prove Conjecture 4.5 in this situation. So assume $\varphi = 1$ and $q \equiv 1 \pmod{\ell}$. Then $W_1 = W$. As recalled above, there is a degree preserving bijection $\mathrm{Irr}(B_0) \to \mathrm{Irr}(B_0(\mathbf{G}^F))$. On the other hand, by a result of Puig [36, Thm 5.5, Cor. 5.10], as explained in [9, Prop. 8.11], the principal block of $\mathbf{G}^F$ over a suitably large complete discrete valuation ring $\mathcal{O}$ of characteristic 0, is Morita equivalent to the group algebra $\mathcal{O}[SW]$ (note here $S = T$ as $|W|$ is prime to $\ell$ and that $W = N_{\mathbf{G}}(\mathbf{T})^F/\mathbf{T}^F$, where $\mathbf{T}$ is a

Sylow 1-torus of $\mathbf{G}$ with $T$ the Sylow $\ell$-subgroup of $\mathbf{T}^F$). In particular the decomposition numbers of $\mathcal{O}[SW]$ are decomposition numbers of $B_0(\mathbf{G}^F)$. Thus $(\deg \Phi_{\hat{\nu}})|_{x=q}$ is the dimension of a projective indecomposable module of $B_0(\mathbf{G}^F)$. Now Conjecture 4.5 follows since the dimension of any projective indecomposable module of a finite group algebra $\mathcal{O}G$ is divisible by $|G|_\ell$ (for instance, apply Proposition 2.7 with respect to the standard symmetrising form on $\mathcal{O}G$.) □

To deal with the primitive cases, we use the following:

**Lemma 4.7.** *Let $(W, L)$ be a finite $\ell$-adic reflection group with $|W|$ prime to $\ell$ and let $W_0 \le W$ be a parabolic subgroup. For $1 \le k \le \mathrm{rk}(W_0)$ there exist $b_k^{W_0} \in \mathbb{Z}$ such that for any prime power $q \equiv 1 \pmod{\ell}$ and $\mathcal{F}$ the fusion system attached to $(W, L, q)$, on a homocyclic $\ell$-group $T$ of exponent $a$, the number of $W$-orbits ($\mathcal{F}$-classes) of elements of $T$ with stabiliser conjugate to $W_0$ is given by*

$$\frac{1}{|N_W(W_0) : W_0|} \prod_{k=1}^{\mathrm{rk}(W_0)} (\ell^a - b_k^{W_0}).$$

*Proof.* Let $\mathcal{A}$ denote the set of 1-eigenspaces of reflections in $T$ and denote by $\mathcal{L} = \mathcal{L}(\mathcal{A})$ the lattice of all intersections of elements of $\mathcal{A}$ with minimal element $T$. By Steinberg's theorem (see [21, Prop. 2.3]), each $Y \in \mathcal{L}$ is the centraliser in $T$ of some parabolic subgroup. Thus by inclusion/exclusion, the number of $W$-orbits of elements of $T$ with stabiliser conjugate to $W_0$ is given by the Euler characteristic of the sublattice $\{Y \in \mathcal{L} \mid C_T(W_0) \le Y\}$ divided by $|N_W(W_0) : W_0|$. This Euler characteristic has the stated form by [35, Thm 1.2]. □

The tables in [35] explicitly list the integers $b_k^{W_0}$ (and the quantities $|N_W(W_0) : W_0|$) for all parabolic subgroups $W_0$ of all exceptional complex reflection groups $W$. An immediate consequence is the following result:

**Proposition 4.8.** *Conjectures 4.2 and 4.3 hold for all primitive spetsial $\ell$-adic reflection groups with $q \equiv 1 \pmod{\ell}$.*

*Proof.* If $W$ is a Weyl group, the claim follows from Proposition 4.6. For the remaining Coxeter groups, it follows from Theorem 5.20. Otherwise since $\ell$ is very good, we must have $\ell \nmid |W|$ and

$$W \in \{G_4, G_6, G_8, G_{14}, G_{24}, G_{25}, G_{26}, G_{27}, G_{29}, G_{32}, G_{33}, G_{34}\}.$$

For all of these only $\varphi = 1$ is possible, by [6, Prop. 3.13]. We explicitly calculate $\dim(B_0)$ as a polynomial in $x$ using Lemma 4.7 and the tables in [7, App.]. The required congruences for $\dim(B_0)|_{x=q}$ are readily checked via the substitution $q \mapsto 1 + r\ell^a$ for $r \in \mathbb{Z}$. □

4.4. **Character degrees and Schur elements.** The proof of Theorem 1 goes through the connection between unipotent character degrees of spetses and Schur elements of corresponding Hecke algebras. We describe this connection in the relevant special case. Let $\mathbb{G} = (W, L)$ be a simply connected $\mathbb{Z}_\ell$-spets such that $\ell$ is very good for $\mathbb{G}$ and $q$ a prime power and let $\psi_{\mathrm{s}} : \mathbb{Z}_\ell[\tilde{\mathbf{u}}^{\pm 1}] \to \mathbb{Z}_\ell[x^{1/z}]$ be the spetsial specialisation described in

Section 3.2 with $R = \mathbb{Z}_\ell$. The degrees of the unipotent characters $\mathrm{Uch}(\mathbb{G})$ of $\mathbb{G}$ in the principal 1-Harish-Chandra series $\mathcal{E}(\mathbb{G},1)_1$ are given by

$$\psi_{\mathrm{s}}(f_{1_W})/\psi_{\mathrm{s}}(f_\phi) \qquad \text{for } \phi \in \mathrm{Irr}(W)$$

(see [24, Sätze 3.14, 6.10] for the infinite series and [7, Ax. 4.16] for the exceptional types). This leads to the following formula for character degrees in the principal block $B_0$ of $\mathbb{G}(q)$.

**Lemma 4.9.** *Assume $q \equiv 1 \pmod{\ell}$. Then the degrees of the characters in $\mathrm{Irr}(B_0)$ are given by*

$$\coprod_{s\in S/\mathcal{F}} \Big\{ \frac{|C_{\mathbb{G}}(\mathbb{T})|_{x'}}{|C_{C_{\mathbb{G}}(s)}(\mathbb{T}_s)|_{x'}} \frac{\psi_{\mathrm{s}}(p_W)}{\psi_{\mathrm{s}}(f_{s,\phi})} \mid \phi \in \mathrm{Irr}(W(s)) \Big\}$$

*where $\mathbb{T}$, $\mathbb{T}_s$ is a Sylow 1-torus of $\mathbb{G}$, $C_{\mathbb{G}}(s)$, respectively, and the $f_{s,\phi}$ denote the Schur elements of the Hecke algebra $\mathcal{H}(W(s),\mathbf{u}_s)$ of $W(s)$, with the parameters $\mathbf{u}_s$ inherited from $\mathcal{H}(W,\mathbf{u})$.*

*Proof.* As mentioned above for any $s \in S$ the degrees of the unipotent characters in the principal 1-series of $\mathcal{E}(C_{\mathbb{G}}(s),1)$ are given by

$$\big\{\psi_{\mathrm{s}}(p_{W(s)})/\psi_{\mathrm{s}}(f_{s,\phi}) \mid \phi \in \mathrm{Irr}(W(s))\big\} \subset \mathbb{Z}_\ell[x].$$

Now by [28, Prop. 8.1] we have $\psi_{\mathrm{s}}(p_W) = \psi_{\mathrm{s}}(f_1) = |\mathbb{G} : C_{\mathbb{G}}(\mathbb{T})|_{x'}$ and accordingly $\psi_{\mathrm{s}}(p_{W(s)}) = |C_{\mathbb{G}}(s) : C_{C_{\mathbb{G}}(s)}(\mathbb{T}_s)|_{x'}$. Since by definition the degree of $\gamma_{s,\phi} \in \mathrm{Irr}(B_0)$ is

$$|\mathbb{G} : C_{\mathbb{G}}(s)|_{x'}\ \psi_{\mathrm{s}}(p_{W(s)})/\psi_{\mathrm{s}}(f_{s,\phi}),$$

our claim follows. □

**Lemma 4.10.** *In the situation of Lemma 4.9, assume moreover that $|W|$ is coprime to $\ell$. Then the degrees of the characters in $\mathrm{Irr}(B_0)$ are given by*

$$\coprod_{s\in S/\mathcal{F}} \big\{\psi_{\mathrm{s}}(p_W)/\psi_{\mathrm{s}}(f_{s,\phi}) \mid \phi \in \mathrm{Irr}(W(s))\big\},$$

*where $f_{s,\phi}$ denotes the Schur elements of the Hecke algebra $\mathcal{H}(W(s),\mathbf{u}_s)$.*

*Proof.* If $\ell$ does not divide $|W|$ then we have $S = T$, that is, the centraliser of any $\ell$-element $s \in S$ contains the Sylow 1-torus $\mathbb{T}$, whence $\mathbb{T}_s = \mathbb{T}$ for all $s$. Now the centraliser of a Sylow 1-torus is a maximal torus since the coset $W\phi$ always contains a 1-regular element by [29, Prop. 3.3]; for this note that none of the exceptions in loc. cit. is spetsial. So in fact $C_{\mathbb{G}}(\mathbb{T}) = C_{C_{\mathbb{G}}(s)}(\mathbb{T})$ and the stated formula follows from Lemma 4.9. □

By analogy with the group case, we now establish a Brauer reciprocity formula for the Brauer characters and decomposition numbers defined in Section 4.2.

**Proposition 4.11.** *Suppose that $W$ is $\ell$-adic spetsial of order coprime to $\ell$, $\varphi = 1$ and $q \equiv 1 \pmod{\ell}$. Then for any $W$-equivariant bijection $\hat{}\ : \mathrm{Irr}(SW) \to \mathrm{Irr}(B_0)$ we have*

$$\deg(\hat\gamma) = \sum_{\chi\in\mathrm{IBr}(SW)} d_{\gamma,\chi} \deg(\hat\chi).$$

*Proof.* By Clifford theory the ordinary irreducible characters of $SW = TW$ are obtained as

$$\mathrm{Irr}(TW) = \{\mathrm{Ind}_{TW_\theta}^{TW}(\theta \otimes \nu) \mid \theta \in \mathrm{Irr}(T),\ \nu \in \mathrm{Irr}(W_\theta)\}.$$

Since $T = O_\ell(TW)$ and $|W|$ is prime to $\ell$, $\mathrm{IBr}(TW)$ consists of the restrictions to $\ell'$-classes of the irreducible characters $1 \otimes \nu$, $\nu \in \mathrm{Irr}(W)$, and we may (and will) thus identify $\mathrm{IBr}(TW)$ with $\mathrm{Irr}(W)$. The $\ell$-decomposition numbers of $TW$ are then described as follows: If $\eta \in \mathrm{Irr}(TW)$ then its restriction to $\ell'$-classes $\eta^0$ can be considered as character of $W$, and the multiplicity of $\chi \in \mathrm{IBr}(TW) = \mathrm{Irr}(W)$ in $\eta^0$ is just $\langle \eta, \chi \rangle$. That is, if $\eta = \mathrm{Ind}_{TW_\theta}^{TW}(\theta \otimes \nu)$ as above, then this multiplicity is $\langle \nu, \chi|_{W_\theta} \rangle$.

Now assume that $\gamma \in \mathrm{Irr}(B_0)$. Then there is $s \in T$ and $\lambda \in \mathrm{Uch}(C_{\mathbb{G}}(s))_1$ such that $\gamma = \gamma_{s,\lambda}$. Let $\phi \in \mathrm{Irr}(W(s))$ be the irreducible character indexing $\lambda \in \mathrm{Uch}(C_{\mathbb{G}}(s))_1$. Now by Lemma 4.10 we have $\gamma_{s,\lambda}(1) = \psi_{\mathrm{s}}(p_W)/\psi_{\mathrm{s}}(f_{s,\phi})$. On the other hand, for $\gamma' \in \mathcal{E}(\mathbb{G}, 1)_1$ labelled by $\chi \in \mathrm{Irr}(W)$ we have $\gamma'(1) = \psi_{\mathrm{s}}(p_W)/\psi_{\mathrm{s}}(f_\chi)$. Thus the required equality reads

$$\psi_{\mathrm{s}}(f_{s,\phi})^{-1} = \sum_{\chi \in \mathrm{Irr}(W)} \langle \phi, \chi|_{W_s} \rangle \psi_{\mathrm{s}}(f_\chi)^{-1}.$$

But this holds for spetsial $W$ by the validity of 1-Howlett–Lehrer theory, see [24, Sätze 3.14, 6.10] for the infinite series and [7, Ax. 4.16] for the exceptional types. □

## 5. Yokonuma type algebras for torus normalisers for $\ell$-adic reflection groups

When the order of $W$ is prime to $\ell$ our definition of principal block and Conjectures 4.2, 4.3 and 4.5 are related to a generalisation of Yokonuma algebras to $\ell$-adic reflection groups. The classical Yokonuma algebra was defined as the endomorphism algebra of the permutation representation of a finite Chevalley group on a maximal unipotent subgroup [38]. It is a deformation of the group algebra of the normaliser of a maximally split torus, to which it becomes isomorphic over a splitting field (see [23, §34]). We propose to extend this construction over the $\ell$-adic integers to "torus normalisers" arising from $\ell$-compact groups attached to arbitrary $\ell$-adic reflection groups. This will allow us in Section 5.6 to prove Conjectures 4.2, 4.3 and 4.5 in the case $q \equiv 1 \pmod{\ell}$ and $\varphi = 1$.

5.1. **Definition and first properties.** Let $\ell$ be a prime and $W$ be a finite $\ell$-adic reflection group, that is, $W \leq \mathrm{GL}(L)$ with $L = \mathbb{Z}_\ell^n$. Let $q$ be a prime power with $q \equiv 1 \pmod{\ell}$ and $a$ the positive integer such that $\ell^a || (q-1)$. Let $T = L/\ell^a L$. Then $T$ is homocyclic of exponent $\ell^a$ and is equipped with a natural action of $W$. For any reflection $r \in W$ we set $T_r := [T, r] := \langle [t, r] \mid t \in T \rangle \leq T$.

The topological braid group $B := B(W)$ of $W$ (see Section 3.1) acts naturally on $T$ through its quotient $W$. We let $\widehat{B}$ be the semidirect product of $T$ with $B$. Observe that $P(W)$ acts trivially on $T$, so $\widehat{B}$ is a (non-split) extension of $T \times P(W)$ by $W$.

Recall from Section 3.1 the indeterminates $\mathbf{u} = (u_{rj})$ attached to $W$. We define a new set $\mathbf{v} = (v_{rj})$ of indeterminates by the linear relations

$$u_{rj} = \zeta_{o(r)}^j (1 + |T_r| v_{rj}) \qquad \text{for } r \in W,\ 1 \leq j \leq o(r),$$

where, for any $k|(\ell-1)$, $\zeta_k \in \mathbb{Z}_\ell$ denotes a primitive $k$th root of unity. Let $\hat{A} = \mathbb{Z}_\ell[\mathbf{v}, \mathbf{u}^{-1}]$.

**Definition 5.1.** Define $\mathcal{Y}(W,a,\mathbf{v})$ to be the quotient of the group algebra $\hat{A}[\widehat{B}]$ of $\widehat{B} = T \rtimes B$ over $\hat{A}$ by the ideal generated by the deformed order relations

$$\prod_{j=1}^{o(r)} \big(\mathbf{r} - \zeta_{o(r)}^j(1+v_{rj}E_r)\big) \qquad \text{with } E_r := \sum_{t\in T_r} t \in \hat{A}[T], \tag{†}$$

where $\mathbf{r}$ runs over the braid reflections of $B \leq \widehat{B}$ and $r$ denotes the image of $\mathbf{r}$ in $W$. We will write $x \mapsto y_x$ for the canonical map $\hat{A}[\widehat{B}] \to \mathcal{Y}(W,a,\mathbf{v})$.

When $W$ is a Weyl group, the deformed order relation (†) generalises the quadratic one from the classical Yokonuma algebra [20, Thm 2(2.1)], see also [32, 2.2(3)]. In Section 5.5, we show that in this case a suitable specialisation of $\mathcal{Y}$ is isomorphic to a truncation of the classical Yokonuma algebra.

As far as we can tell, there is no direct relation between the algebra $\mathcal{Y}(W,a,\mathbf{v})$ defined above and the "cyclotomic Yokonuma–Hecke algebra" considered by Chlouveraki–d'Andecy [13, §2] for the reflection group $W = G(d,1,n)$; in their algebra, the underlying reflection group $W$ only acts via its quotient $G(1,1,n) \cong \mathfrak{S}_n$ on the torus. On the other hand, our construction is related to an algebra defined by Marin [32], see Remark 5.16 below.

Henceforth, for simplicity we set $\mathcal{Y} := \mathcal{Y}(W,a,\mathbf{v})$. Note that the specialisation $\psi_1 : \mathbb{Z}_\ell[\mathbf{u}^{\pm 1}] \to \mathbb{Z}_\ell$, $u_{rj} \mapsto \zeta_{o(r)}^j$, extends to a homomorphism $\hat{A} \to \mathbb{Z}_\ell$, $v_{rj} \mapsto 0$.

**Lemma 5.2.** *The following hold:*

(a) *Under the specialisation $\psi_1 : \hat{A} \to \mathbb{Z}_\ell$, $v_{rj} \mapsto 0$ (so $u_{rj} \mapsto \zeta_{o(r)}^j$), the algebra $\mathcal{Y}$ specialises to the group algebra of $TW$.*

(b) *The quotient of $\mathcal{Y}$ by the ideal $I$ generated by the $\{y_t - 1 \mid t \in T\}$ is isomorphic to the extension $\hat{A} \otimes_A \mathcal{H}(W,\mathbf{u})$ of the generic Hecke algebra of $W$.*

(c) *The natural $\hat{A}$-module homomorphism $\mathcal{Y} \to \mathcal{H}(W,\mathbf{u})$, $y_\mathbf{r} \mapsto h_\mathbf{r}$, in* (b) *has a splitting $\mathcal{H}(W,\mathbf{u}) \to \hat{A}[\ell^{-1}] \otimes_{\hat{A}} \mathcal{Y}$ given by $h_\mathbf{r} \mapsto |T|^{-1}\sum_{t\in T} y_t y_\mathbf{r}$.*

*Proof.* The first parts follows directly from the deformed order relation (†) and the corresponding result of Bessis [1] for $B$. For (b), let $J_1$ be the ideal of $\hat{A}[\widehat{B}]$ generated by the elements $\{t-1 \mid t \in T\}$. Then $I$ is the ideal of $\hat{A}[\widehat{B}]$ generated by $J_1$ and the elements (†) as $\mathbf{r}$ runs over the braid reflections. Let $L$ be the ideal of $\hat{A}[\widehat{B}]$ generated by $J_1$ and the $\prod_{j=1}^{o(r)}(\mathbf{r} - u_{rj})$ as $\mathbf{r}$ runs over braid reflections. Then $\mathcal{H}(W,\mathbf{u}) = \hat{A}[B]/L$. For an element $x = \prod_{j=1}^{o(r)}(\mathbf{r} - \zeta_{o(r)}^j(1+E_r v_{rj})) \in J_1$ of the form (†) set $x' = \prod_{j=1}^{o(r)}(\mathbf{r} - u_{rj})$. Then $x + J_1 = x' + J_1$, whence $I = L$.

For (c), note that $|T|^{-1}\sum_{t\in T} y_t$ is a central idempotent of $\hat{A}[\ell^{-1}] \otimes_{\hat{A}} \mathcal{Y}$. $\square$

We make some further straightforward observations. First, since for $\ell > 2$ all reflections $r$ in an $\ell$-adic reflection group have order prime to $\ell$, in that case $T_r \cong \mathbb{Z}/\ell^a\mathbb{Z}$.

For all braid reflections $\mathbf{r}$, the element $E_r$ commutes with $\mathbf{r}$ and $E_r^2 = |T_r|E_r$. Thus, over $\hat{A}[\ell^{-1}]$, the element $E_r' = |T_r|^{-1}E_r$ is idempotent. Multiplying (†) with $E_r'$ respectively

$1 - E'_r$ we obtain the elements

$$(1 - E'_r)(\mathbf{r}^{o(r)} - 1) \quad \text{and} \quad E'_r \prod_{j=1}^{o(r)} (\mathbf{r} - u_{rj}), \qquad (\dagger')$$

which generate the same ideal as (†) over $\hat{A}[\ell^{-1}]$. Thus, (†) "interpolates" between the group relation for $r$ and the deformed Hecke algebra relation ($\mathcal{H}$) for $\mathbf{r}$. Let us also note the following:

**Lemma 5.3.** *The constant coefficient in the deformed order relation* (†) *is invertible in* $\hat{A}[T]$.

*Proof.* The constant term in the polynomial relation (†) for a braid reflection $\mathbf{r}$ is (up to a root of unity) a product of factors $1+v_{rj}E_r$, which has inverse $1-\zeta_{o(r)}^j v_{rj}/u_{rj}E_r \in \hat{A}[T]$. □

We now give a more tangible description of $\mathcal{Y}(W, a, \mathbf{v})$. Recall that the braid group $B$ has a presentation in terms of certain sets of braid reflections together with so-called braid relations, encoded in *braid diagrams*, such that adding the order relations for the chosen braid reflections, we obtain a presentation of $W$ (see [8, Thm 2.27] and Bessis [1, Thm 0.1]). Choose reflections $r_1, \ldots, r_m$ in $W$ corresponding to a braid diagram for $B$. It is known that any distinguished reflection of $W$ is then conjugate to one of the $r_i$, and by their construction all braid reflections projecting onto a fixed reflection of $W$ are conjugate in $B$. Then using Lemma 5.3 we see that $\mathcal{Y}$ is the associative unital $\hat{A}$-algebra generated by elements $\{y_t, y_{\mathbf{r}_i} \mid t \in T,\ 1 \le i \le m\}$ subject to

- the $y_t$ satisfy the same relations as the corresponding group elements $t$ (i.e., they generate a subalgebra isomorphic to (possibly a quotient of) the group algebra $\hat{A}[T]$);
- the action relations between the $t, r_i$, with $t$ replaced by $y_t$ and $r_i$ by $y_{\mathbf{r}_i}$;
- the braid relations between the $y_{\mathbf{r}_i}$; and
- the deformed order relations (†) for the $y_{\mathbf{r}_i}$.

5.2. **On the structure of specialised Yokonuma type algebras.** We show that under some additional hypothesis certain specialisations of $\mathcal{Y}$ are isomorphic to the group algebra of $TW$. The main results are Theorem 5.7 and Theorem 5.10.

For a specialisation $\psi : \hat{A} \to R$ to a commutative ring $R$, let $\mathcal{Y}_\psi := R \otimes_{\hat{A}} \mathcal{Y}$ denote the extension of scalars by $\psi$. Then $\mathcal{Y}_\psi$ is the quotient of the group algebra $R\widehat{B}$ by the ideal

$$\Big\langle \prod_{j=1}^{o(r)} \big(\mathbf{r} - \zeta_{o(r)}^j (1 + \psi(v_{rj})E_r)\big) \mid \mathbf{r} \in B \text{ braid reflection} \Big\rangle.$$

Let $W_0$ be a parabolic subgroup of $W$ and $B_0 = B(W_0)$ be its braid group. In [8, §2D] is constructed an embedding $B_0 \hookrightarrow B$, well-defined up to $P$-conjugation, where $B = B(W)$, $P = P(W)$. By [8, Prop. 2.29 and 2.18] this satisfies:

**Lemma 5.4.** *Let $W_0$ be a parabolic subgroup of $W$. Let $\widetilde{B}_0$ be the inverse image of $W_0$ in $B$ and let $\tilde{P}_0$ be the subgroup of $P$ generated by the elements $\mathbf{r}^{o(r)}$, as $r$ runs over the distinguished reflections in $W \setminus W_0$. Let $B_0$ be the braid group of $W_0$. The above inclusion $B_0 \hookrightarrow B$ has image contained in $\widetilde{B}_0$ and induces an isomorphism $B_0 \xrightarrow{\sim} \widetilde{B}_0/\tilde{P}_0$.*

So we have the following diagram with exact columns:

$$\begin{array}{ccccc}
1 & & 1 & & 1 \\
\downarrow & & \downarrow & & \downarrow \\
P_0 & \hookrightarrow & P = \tilde{P}_0 \rtimes P_0 & = & P \\
\downarrow & & \downarrow & & \downarrow \\
B_0 & \hookrightarrow & \widetilde{B}_0 = \tilde{P}_0 \rtimes B_0 & \hookrightarrow & B \\
\downarrow & & \downarrow & & \downarrow \\
W_0 & = & W_0 & \hookrightarrow & W \\
\downarrow & & \downarrow & & \downarrow \\
1 & & 1 & & 1
\end{array}$$

**Remark 5.5.** Examples show that the isomorphism $B_0 \xrightarrow{\sim} \widetilde{B}_0/\tilde{P}_0$ might more generally hold for all reflection subgroups $W_0$ of $W$ generated by distinguished reflections (see e.g. [8, Prop. 3.24]), so that the assumptions of the subsequent Theorem 5.7 might be relaxed accordingly. We will not need this here.

If $W_\theta$ is a parabolic subgroup of $W$, we will denote by $\mathbf{u}_\theta$ the set $\mathbf{u}_0$ in the notation of Section 3.1 if $W_\theta$ is the reflection subgroup $W_0$.

**Lemma 5.6.** *Let $R$ be an integral domain containing the* $\exp(T)$*th roots of unity with field of fractions $K$ of characteristic* $0$ *and let $\psi : \hat{A} \to R$ be a specialisation. Let $I$ be the ideal of $R\widehat{B}$ generated by*

$$\Big\{ \prod_{j=1}^{o(r)} \big(\mathbf{r} - \zeta_{o(r)}^j (1 + \psi(v_{rj}) E_r)\big) \mid \mathbf{r} \in B \textit{ braid reflection} \Big\}.$$

*Let $\theta \in \mathrm{Irr}_K(T)$ and $e_\theta \in KT$ the corresponding central idempotent. Suppose that the stabiliser $W_\theta$ of $\theta$ is a parabolic subgroup of $W$. Then $e_\theta R\widehat{B} e_\theta / e_\theta I e_\theta \cong \mathcal{H}_\psi(W_\theta, \mathbf{u}_\theta)$ as $R$-algebras. Here we regard $R\widehat{B}$ as a subset of $K\widehat{B}$.*

Note that the assumption that $R$ contains the $\exp(T)$th roots of unity is needed in order to ensure that any ordinary irreducible character of $T$ is $R$-valued.

*Proof.* Let $\widetilde{B}_\theta$ be the full inverse image of $W_\theta$ in $B$. For a braid reflection $\mathbf{r} \in B$ set

$$i_{\mathbf{r}} := \prod_{j=1}^{o(r)} \big(\mathbf{r} - \zeta_{o(r)}^j (1 + \psi(v_{rj}) E_r)\big) = (1 - E_r')(\mathbf{r}^{o(r)} - 1) + E_r' \prod_{j=1}^{o(r)} (\mathbf{r} - \psi(u_{rj}))$$

where $E_r' = |T_r|^{-1} E_r$. Then

$$\{ e_\theta x i_{\mathbf{r}} y e_\theta \mid x, y \in \widehat{B},\ \mathbf{r} \in B \text{ braid reflection} \}$$

generates $e_\theta I e_\theta$ as $R$-module. The set of braid reflections is invariant under conjugation by $B$, and $v_{rj} = v_{r'j}$ whenever $r$ and $r'$ are conjugate. Thus, if $x = tg$, $y = hs$ with $t, s \in T$ and $g, h \in B$ and $\mathbf{r}$ is a braid reflection, then

$$e_\theta x i_{\mathbf{r}} y e_\theta = \theta(t)\theta(s) e_\theta i_{{}^g\mathbf{r}} g h e_\theta, \quad \text{where } \theta(t), \theta(s) \in R.$$

Thus, $e_\theta I e_\theta$ is the $R$-span of $\{ e_\theta i_{\mathbf{r}} x e_\theta \mid x \in B,\ \mathbf{r} \in B \text{ braid reflection} \}$.

Now $r \in W_\theta$ if and only if $\theta(t^{-1}t^r) = 1$ for all $t \in T$. Thus, $e_\theta E'_r = e_\theta$ if $r \in W_\theta$ and zero otherwise, and so

$$e_\theta i_{\mathbf{r}} = \begin{cases} e_\theta(\mathbf{r}^{o(r)} - 1) & \text{if } \mathbf{r} \notin W_\theta, \\ e_\theta \prod_j (\mathbf{r} - \psi(u_{rj})) & \text{if } \mathbf{r} \in W_\theta. \end{cases}$$

Further, since $\mathbf{r}^{o(r)} \in P$ commutes with $T$, we have $e_\theta(\mathbf{r}^{o(r)} - 1)xe_\theta = (\mathbf{r}^{o(r)} - 1)e_\theta x e_\theta$ and if $r \in W_\theta$, then

$$e_\theta \prod_j (\mathbf{r} - \psi(u_{rj}))xe_\theta = \prod_j (\mathbf{r} - \psi(u_{rj}))e_\theta x e_\theta.$$

For any $x \in B$, $e_\theta x e_\theta = xe_\theta$ if $x \in \widetilde{B}_\theta$ and zero otherwise. Hence $e_\theta I e_\theta$ is the $R$-span of

$$\big\{(\mathbf{r}^{o(r)} - 1)xe_\theta \mid x \in \widetilde{B}_\theta,\ r \notin W_\theta\big\} \cup \Big\{\prod_j (\mathbf{r} - \psi(u_{rj}))xe_\theta \mid x \in \widetilde{B}_\theta,\ r \in W_\theta\Big\}.$$

By the same argument we have that $e_\theta R\widehat{B}e_\theta = e_\theta R[T\widetilde{B}_\theta]e_\theta$ and since $e_\theta$ is $T\widetilde{B}_\theta$-stable and $e_\theta$ is idempotent we also have $e_\theta R[T\widetilde{B}_\theta]e_\theta = R[T\widetilde{B}_\theta]e_\theta$. Since $\theta$ is linear and $\widetilde{B}_\theta$-stable, there is an $R$-algebra isomorphism $R\widetilde{B}_\theta \cong R[T\widetilde{B}_\theta]e_\theta$ given by $x \mapsto xe_\theta$. This induces an isomorphism

$$R\widetilde{B}_\theta/J \cong R[T\widetilde{B}_\theta]e_\theta/e_\theta I e_\theta$$

where $J \trianglelefteq R\widetilde{B}_\theta$ is the ideal generated by $\{\mathbf{r}^{o(r)} - 1 \mid r \notin W_\theta\} \cup \{\prod_j (\mathbf{r} - \psi(u_{rj})) \mid r \in W_\theta\}$. By Lemmas 5.4 and 5.2(b), $R\widetilde{B}_\theta/J \cong \mathcal{H}_\psi(W_\theta, \mathbf{u}_\theta)$. □

**Theorem 5.7.** *Assume all stabilisers $W_\theta$ of elements $\theta \in \mathrm{Irr}(T)$ are parabolic subgroups of $W$. Let $K$ be a field of characteristic $0$ containing the $|T|$th roots of unity and let $\psi : \hat{A} \to K$ be a ring homomorphism. Then*

(a)
$$\mathcal{Y}_\psi \cong \prod_\theta \mathrm{Mat}_{|W:W_\theta|}(K) \otimes_K \mathcal{H}_\psi(W_\theta, \mathbf{u}_\theta)$$
*as $\theta$ runs over a set of representatives of $W$-orbits on $\mathrm{Irr}_K(T)$;*

(b) $\dim_K \mathcal{Y}_\psi = |TW|$.

(c) *Suppose that $K_W \subseteq K$ and $\psi$ is the inclusion homomorphisms. Then $\mathcal{Y}_\psi \cong K[TW]$.*

*Proof.* Part (a) is immediate from Lemmas 2.3 and 5.6 applied with $R = K$. Part (b) follows from part (a) by Theorem 3.1 and Lemma 2.3(b) applied with $G = TW$.

Now assume $K$ and $\psi$ are as in (c). As explained in Section 3.2, for all $\theta \in \mathrm{Irr}(T)$, $\mathcal{H}_\psi(W_\theta, \mathbf{u}_\theta) = K \otimes_{\hat{A}} \mathcal{H}(W_\theta, \mathbf{u}_\theta) \cong KW_\theta$. Now (c) follows from (a) and Lemma 2.3(a) applied with $G = TW$ and $I = 0$, noting that $K(TW)_\theta e_\theta \cong KW_\theta$. □

We now turn to specialisations of $\hat{A}$ to finite extensions of $\mathbb{Z}_\ell$. For the rest of this subsection, the following notation will be in effect. Let $\mathbb{Z}_\ell \subseteq \mathcal{O}$ be a complete discrete valuation ring with uniformiser $\pi$, residue field $k$ and field of fractions $K$. Let $\psi : \hat{A} \to \mathcal{O}$ be a $\mathbb{Z}_\ell$-algebra homomorphism and denote by $\bar{\psi}$ the composition of $\psi$ with the canonical map $\mathcal{O} \to k$. Recall that for $x \in \hat{A}[\widehat{B}]$ we denote by $y_x$ its image in $\mathcal{Y}$. For $y \in \mathcal{Y}$ let $\tilde{y} := 1_\mathcal{O} \otimes y \in \mathcal{Y}_\psi$, and $\bar{y} := 1_k \otimes y \in \mathcal{Y}_{\bar{\psi}}$.

**Lemma 5.8.** *Let* $\mathbf{W}$ *be a set of coset representatives of* $P$ *in* $B$ *and let* $J$ *be the ideal of* $\mathcal{Y}_\psi$ *generated by* $\{\tilde{y}_t - 1 \mid t \in T\}$ *and* $\pi$. *Then,* $kW \cong \mathcal{Y}_\psi/J$ *via the map which sends* $w \in W$ *to* $\tilde{y}_{\mathbf{w}} + J$ *for* $\mathbf{w} \in \mathbf{W}$ *lifting* $w$.

*Proof.* For a distinguished reflection $r \in W$, $\ell$ divides $|T_r|$, hence $u_{rj} \in I$. Now the result follows from Lemma 5.2(b) (suitably adapted to the coefficient ring $\mathcal{O}$). $\square$

**Theorem 5.9.** *Let* $\mathbf{W}$ *be a set of coset representatives of* $P$ *in* $B$. *If all stabilisers* $W_\theta$ *of elements* $\theta \in \mathrm{Irr}(T)$ *are parabolic subgroups of* $W$, *then* $X := \{\tilde{y}_t\tilde{y}_{\mathbf{w}} \mid t \in T,\ \mathbf{w} \in \mathbf{W}\}$ *is an* $\mathcal{O}$*-basis of* $\mathcal{Y}_\psi$.

*Proof.* We first show that $\{\bar{y}_t\bar{y}_{\mathbf{w}} \mid t \in T,\ \mathbf{w} \in \mathbf{W}\}$ generates $\mathcal{Y}_{\bar{\psi}}$ as $k$-vector space. Let $R \subseteq \mathcal{Y}_{\bar{\psi}}$ be the $k$-span of $\{\bar{y}_t \mid t \in T\}$. Then $R$ is a commutative $k$-subalgebra of $\mathcal{Y}_{\bar{\psi}}$. Let $Q$ be the ideal of $R$ generated by $\{\bar{y}_t - 1 \mid t \in T\}$. Since $T$ is a finite abelian $\ell$-group, $Q$ is a nilpotent ideal of $R$. We consider $\mathcal{Y}_{\bar{\psi}}$ as a left $R$-module. Since $T$ is normal in $\widehat{B}$, the $R$-submodule $Q\mathcal{Y}_{\bar{\psi}}$ of $\mathcal{Y}_{\bar{\psi}}$ is an ideal of $\mathcal{Y}_{\bar{\psi}}$. Further, $Q\mathcal{Y}_{\bar{\psi}}$ is the image of the ideal $J$ of Lemma 5.8 in $\mathcal{Y}_\psi/\pi\mathcal{Y}_\psi \cong \mathcal{Y}_{\bar{\psi}}$ and for any $\mathbf{w} \in \mathbf{W}$, $\bar{y}_{\mathbf{w}} + Q\mathcal{Y}_{\bar{\psi}} = \tilde{y}_{\mathbf{w}} + J$. So, by Lemma 5.8, $\{\bar{y}_{\mathbf{w}} + Q\mathcal{Y}_{\bar{\psi}} \mid \mathbf{w} \in \mathbf{W}\}$ generates $\mathcal{Y}_{\bar{\psi}}/Q\mathcal{Y}_{\bar{\psi}}$ as $k$-vector space and hence as $R$-module. Applying Lemma 2.1 with $M = \mathcal{Y}_{\bar{\psi}}$ and the nilpotent ideal $Q$ we obtain that $\{\bar{y}_{\mathbf{w}} \mid \mathbf{w} \in \mathbf{W}\}$ generates $\mathcal{Y}_{\bar{\psi}}$ as $R$-module. Since $R$ is generated by $\{\bar{y}_t \mid t \in T\}$ as $k$-vector space we have the required result.

Now we claim that in order to prove the theorem it suffices to prove that $\mathcal{Y}_\psi$ is finitely generated as $\mathcal{O}$-module. Indeed, suppose that $\mathcal{Y}_\psi$ is finitely generated as $\mathcal{O}$-module. Then by the previous paragraph and the standard Nakayama lemma (Lemma 2.1) applied to the ring $\mathcal{O}$ and ideal $\pi\mathcal{O}$, the set $X = \{\tilde{y}_t\tilde{y}_{\mathbf{w}} \mid t \in T, \mathbf{w} \in \mathbf{W}\}$ generates $\mathcal{Y}_\psi$ as $\mathcal{O}$-module. Then $1 \otimes X$ generates $K \otimes_{\mathcal{O}} \mathcal{Y}_\psi$ as $K$-vector space and since the latter has dimension $|TW|$ by Theorem 5.7(b), $X$ is an $\mathcal{O}$ basis of $\mathcal{Y}_\psi$.

It remains only to show that $\mathcal{Y}_\psi$ is finitely generated as $\mathcal{O}$-module. For this, suppose first that $\mathcal{O}$-contains the $|T|$th roots of unity. Let $I$ be the ideal of $\mathcal{O}\widehat{B}$ generated by the (†)-relations, let

$$I' = \bigoplus_{\theta,\mu \in \mathrm{Irr}(T)} e_\theta I e_\mu \subseteq \bigoplus_{\theta,\mu \in \mathrm{Irr}(T)} e_\theta \mathcal{O}\widehat{B} e_\mu$$

with $e_\theta \in KT$ as in Lemma 5.6 and let $\tilde{I} = \mathcal{O}\widehat{B} \cap I'$. Since $I'$ is an ideal of $\bigoplus e_\theta\mathcal{O}\widehat{B}e_\mu$, $\tilde{I}$ is an ideal of $\mathcal{O}\widehat{B}$ containing $I$. On the other hand, $|T|e_\theta \in \mathcal{O}\widehat{B}$ for all $\theta$, hence $|T|^2 e_\theta I e_\mu \subseteq I$ and $|T|^2\tilde{I} \subseteq I$. The kernel of the composition of the inclusion $\mathcal{O}\widehat{B} \hookrightarrow \bigoplus e_\theta\mathcal{O}\widehat{B}e_\mu$ with the surjection $\bigoplus e_\theta\mathcal{O}\widehat{B}e_\mu \twoheadrightarrow \bigoplus e_\theta\mathcal{O}\widehat{B}e_\mu/I'$ is $\tilde{I}$. Thus, $\mathcal{O}\widehat{B}/\tilde{I}$ is isomorphic to a submodule of $\bigoplus e_\theta\mathcal{O}\widehat{B}e_\mu/I'$. On the other hand,

$$\bigoplus_{\theta,\mu \in \mathrm{Irr}(T)} e_\theta\mathcal{O}\widehat{B}e_\mu/I' \cong \bigoplus_{\theta,\mu \in \mathrm{Irr}(T)} e_\theta\mathcal{O}\widehat{B}e_\mu/e_\theta I e_\mu.$$

If $\mu = {}^x\theta$ for $x \in B$, then $e_\theta\mathcal{O}\widehat{B}e_\mu/e_\theta I e_\mu \cong e_\theta\mathcal{O}\widehat{B}e_\theta/e_\theta I e_\theta$ via right multiplication by $x$ and it follows from Lemma 5.6 and Theorem 3.1 that $e_\theta\mathcal{O}\widehat{B}e_\theta/e_\theta I e_\theta$ is finitely generated free as $\mathcal{O}$-module. If $\theta, \mu$ are in different $W$-orbits, then $e_\theta\mathcal{O}\widehat{B}e_\mu = 0$. By the above displayed equation, $\bigoplus e_\theta\mathcal{O}\widehat{B}e_\mu/I'$ is finitely generated free as $\mathcal{O}$-module. Since $\mathcal{O}$ is a principal ideal domain, and since $\mathcal{O}\widehat{B}/\tilde{I}$ is isomorphic to a submodule of $\bigoplus e_\theta\mathcal{O}\widehat{B}e_\mu/I'$, it follows that $\mathcal{O}\widehat{B}/\tilde{I}$ is finitely generated free as $\mathcal{O}$-module. Also, $\pi^r(\tilde{I}/I) = 0$ for $r$ equal to twice the $\pi$-adic valuation of $|T|$. We saw above that $\mathcal{Y}_\psi/\pi\mathcal{Y}_\psi \cong \mathcal{Y}_{\bar{\psi}}$ is finitely generated as $k$-vector space and hence as $\mathcal{O}$-module. Thus by Lemma 2.2 applied with $M = \mathcal{Y}_\psi = \mathcal{O}\widehat{B}/I$ and $N = \tilde{I}/I$ we have $\mathcal{Y}_\psi$ is finitely generated as $\mathcal{O}$-module.

Now consider the general case. Let $\mathcal{O}'$ be a finite extension of $\mathcal{O}$ containing the $|T|$th roots of unity and let $\psi'$ be the composition of $\psi$ with inclusion of $\mathcal{O}$ in $\mathcal{O}'$. By the previous part, applied with $\psi'$ in place of $\psi$, $\mathcal{Y}_{\psi'}$ is finitely generated as $\mathcal{O}'$-module. Since $\mathcal{O}'$ is a finite extension of $\mathcal{O}$, $\mathcal{Y}_{\psi'}$ is also finitely generated as $\mathcal{O}$-module. Since $\mathcal{O}$ is a direct summand of $\mathcal{O}'$ as $\mathcal{O}$-module, the inclusion $\mathcal{O} \hookrightarrow \mathcal{O}'$ is pure. Thus the map $\mathcal{Y}_\psi \to \mathcal{O}' \otimes_{\mathcal{O}} \mathcal{Y}_\psi \cong \mathcal{Y}_{\psi'}$, $y \mapsto 1 \otimes y$, is injective and consequently $\mathcal{Y}_\psi$ is isomorphic to an $\mathcal{O}$-submodule of $\mathcal{Y}_{\psi'}$. Since $\mathcal{O}$ is Noetherian and since as shown above $\mathcal{Y}_{\psi'}$ is finitely generated as $\mathcal{O}$-module, $\mathcal{Y}_\psi$ is finitely generated as $\mathcal{O}$-module. □

The following is an application of a theorem of Külshammer, Okuyama and Watanabe (see [22, Thm 4.8.2]). Recall that if $R$ is a commutative ring and $C$ is a subalgebra of an $R$-algebra $B$, then $B$ is *relatively $C$-separable* if $B$ is a direct summand of $B \otimes_C B$ as a $(B, B)$-bimodule.

**Theorem 5.10.** *Suppose that $W$ is an $\ell'$-group. Then there exists an $\mathcal{O}$-algebra isomorphism $\mathcal{Y}_\psi \cong \mathcal{O}[TW]$ sending $\tilde{y}_t$ to $t$ for any $t \in T$.*

*Proof.* Let $\mathbf{W}$ be a set of coset representatives of $P$ in $B$. Since $W$ is an $\ell'$-group, $W_\theta$ is a parabolic subgroup of $W$ for all $\theta \in \mathrm{Irr}(T)$. Thus, by Theorem 5.9, $\{\tilde{y}_t\tilde{y}_{\mathbf{w}} \mid t \in T,\ \mathbf{w} \in \mathbf{W}\}$ is an $\mathcal{O}$-basis of $\mathcal{Y}_\psi$. Further, we may regard $\mathcal{O}T$ as an $\mathcal{O}$-subalgebra of $\mathcal{Y}_\psi$ via the identification of $\mathcal{O}T$ with the subalgebra generated by $\{\tilde{y}_t \mid t \in T\}$. Under this identification, again via Theorem 5.9, there is a homomorphism of $(\mathcal{O}T, \mathcal{O}T)$-bimodules $\gamma : \mathcal{O}[TW] \to \mathcal{Y}_\psi$ defined by $\gamma(tw) = \tilde{y}_t\tilde{y}_{\mathbf{w}}$.

Let $J$ be the ideal of $\mathcal{Y}_\psi$ generated by $\{\tilde{y}_t - 1 \mid t \in T\}$ and $\pi$. It follows from Lemma 5.8 that the composition of $\gamma$ with the natural surjection $\mathcal{Y}_\psi \to \mathcal{Y}_\psi/J$ is an $\mathcal{O}$-algebra homomorphism. Now we may apply [22, Thm 4.8.2] to obtain an $\mathcal{O}$-algebra homomorphism $\sigma : \mathcal{O}[TW] \to \mathcal{Y}_\psi$ extending the $\mathcal{O}$-algebra homomorphism $\mathcal{O}[TW] \to \mathcal{Y}_\psi/J$ obtained above from $\gamma$ and satisfying $\sigma(t) = \tilde{y}_t$ for all $t \in T$. For this one needs to have that $J$ is contained in the radical of $\mathcal{Y}_\psi$ and that $\mathcal{O}[TW]$ is relatively $\mathcal{O}T$-separable. The second condition holds since $W$ is an $\ell'$-group (see [22, Prop. 2.6.9]) whereas the first condition holds since $T$ is a finite normal $\ell$-subgroup of $\widehat{B}$ and, by Theorem 5.9, $\mathcal{Y}_\psi$ is finitely generated as $\mathcal{O}$-module.

The surjectivity of $\sigma$ follows by Nakayama's lemma since the composition of $\sigma$ with $\mathcal{Y}_\psi \to \mathcal{Y}_\psi/I$ is surjective and then the injectivity follows since both algebras are free of the same rank. □

**Remark 5.11.** The assumption of Theorem 5.7 on stabilisers is satisfied whenever $\ell$ is very good for $W$, e.g., when $|W|$ is coprime to $\ell$, or if $W = G(e, 1, n)$ with $e \geq 2$, see [21, Prop. 2.3].

5.3. **Freeness.** We propose the following, analogous to the (now proven) Freeness Conjecture (Theorem 3.1) for cyclotomic Hecke algebras:

**Conjecture 5.12.** *The algebra $\mathcal{Y}$ is free over $\hat{A}$ of rank $|TW|$. More precisely, there is a section $W \to \mathbf{W} \subset B$ of the natural map $B \to W$ containing* 1 *such that $\{y_t y_{\mathbf{w}} \mid t \in T, \mathbf{w} \in \mathbf{W}\}$ is an $\hat{A}$-basis of $\mathcal{Y}$.*

For Weyl groups and parameters occurring in finite reductive groups, the freeness follows from the construction as an endomorphism algebra, and the dimension from the number of double cosets of a maximal unipotent subgroup, that is, the Bruhat decomposition; see Lusztig [23, 34.2–34.10] for a detailed investigation. We propose a proof in the case of finite Coxeter groups and for most infinite series of complex reflection groups.

**Theorem 5.13.** *Conjecture* 5.12 *holds for any finite Coxeter group.*

*Proof.* Assume that $W$ is a Coxeter group and choose a presentation of $B$ on braid reflections $\mathbf{r}_1, \ldots, \mathbf{r}_m \in B$ mapping to the Coxeter generators of $W$. Clearly, the set of all monomials in the $y_t, y_{\mathbf{r}_i}$ forms a generating system for $\mathcal{Y}$ as an $\hat{A}$-module. By the 'action relations' any such monomial can be rewritten into an $\hat{A}$-linear combination of elements $y_t y_{\mathbf{w}}$ with $t \in T$ and $\mathbf{w}$ a monomial in the generators $\mathbf{r}_i$, $1 \le i \le m$. Now by Matsumoto's lemma, by using the braid relations plus the quadratic relations (†) expressing $y_{\mathbf{r}_i}^2$ as a linear combination of smaller powers of $y_{\mathbf{r}_i}$, $\mathbf{w}$ can be rewritten into an $\hat{A}[T]$-linear combination of elements from a fixed set $\mathbf{W} \subset B$ of reduced expressions of elements of $W$.

Thus any monomial in the generators is an $\hat{A}$-linear combination of elements $y_t y_{\mathbf{w}}$ with $t \in T$ and $\mathbf{w} \in \mathbf{W}$. Since the $y_t$ satisfy the same relations as the corresponding $t \in T$, there are at most $|T|$ distinct elements $y_t$, so we have identified a generating system for $\mathcal{Y}$ of cardinality $|TW|$. By Theorem 5.7 this must be free over $K$, hence an $\hat{A}$-basis of $\mathcal{Y}$. □

**Theorem 5.14.** *Conjecture* 5.12 *holds for $W = G(e, p, n)$ with $e|(\ell - 1)$ for any divisor $p$ of $e$, except possibly when $n = 2$, $e, p$ are both even and $p \neq e$.*

*Proof.* The group $W = G(e, p, n)$ is a normal reflection subgroup of $W_1 := G(e, 1, n)$ of index $p$. First assume that $p < e$. Then the braid group $B$ of $W$ is normal in the braid group $B_1$ of $W_1$ of index $p$ by [8, §3.B1]. Also, the corresponding tori $T$ can be identified, such that $T.B$ is normal in $T.B_1$ of index $p$. A system of coset representatives is given by $\{\mathbf{r}_1^i \mid 0 \le i \le p - 1\}$, where $\mathbf{r}_1 \in B_1$ lifts a distinguished reflection $r \in W_1$ of order $e$. Let $\zeta_e$ be a primitive $e$th root of unity and set $p' := e/p$. Recall the parameters $u_{rj}$, $1 \le j \le p'$, for $\mathcal{Y}$ at the reflection $r$. Let $K$ be a sufficiently large extension of $\mathrm{Frac}(\hat{A})$. Consider parameters $u'_{rj} := u_{rj}^{1/p}$ for $1 \le j \le p'$, and $u'_{r,j+p'} := \zeta_e^{p'} u'_{rj}$ for $1 \le j \le e - p'$. Now over $K$, the relation (†) for $y_1 := y_{\mathbf{r}_1}$ can be rewritten as

$$(1 - E')(y_1^e - 1) + E' \prod_{j=1}^{e} (y_1 - u'_{rj}) = 0,$$

with $E' := |T_r|^{-1} E_r$ (see (†′) above). Note that $\prod_{i=0}^{p-1} (y_1 - u'_{r,j+p'i}) = y_1^p - u_{rj}$ for any $j$. Similarly, over $K$ the relation (†) for the generator $y_1^p$ of $\mathcal{Y}$ can be written as

$$(1 - E')(y_1^{pp'} - 1) + E' \prod_{j=1}^{p'} (y_1^p - u_{rj}) = 0.$$

Thus, we obtain the same relation for $y_1^p$ in $\mathcal{Y}$ as before.

Let $\mathcal{Y}_1$ be the quotient of $\hat{A}[T.B_1]$ by the deformed order relations, which agree with those for $\mathcal{Y}$ as we just saw and hence can be written over $\hat{A}$, except in the excluded case $n = 2$, $e, p$ both even, when $G(e, p, 2)$ contains an additional class of reflections.

We claim that the conjecture holds for $W_1$. Our proof for this closely follows some arguments in Bremke–Malle [3]. Let $\mathbf{r}_1, \ldots, \mathbf{r}_n \in B$ be braid reflections corresponding to the standard presentation, so that $\mathbf{r}_2, \ldots, \mathbf{r}_n$ generate the braid group on $n$ strands (of type $A_{n-1}$) and $(\mathbf{r}_1\mathbf{r}_2)^2 = (\mathbf{r}_2\mathbf{r}_1)^2$. Set $y_2 := y_{\mathbf{r}_2}$. Now by Lemma 5.15 below for any $a, b \geq 1$ we have

$$y_2y_1^ay_2y_1^b = \alpha y_1^by_2y_1^ay_2 + \sum_{i=1}^{b}(\alpha_iy_1^{a+b-i}y_2y_1^i + \alpha_i'y_1^iy_2y_1^{a+b-i})$$

for suitable $\alpha \in \hat{A}[T]^\times$ and $\alpha_i, \alpha_i' \in \hat{A}[T]$. With this, one deduces as in the proof of [3, Prop. 2.4] that there is a set $\mathcal{B}_1 \subset B_1$ of cardinality $|W_1|$ consisting of monomials in the $\mathbf{r}_i$, as in [3, Lemma 1.5], such that any monomial in the $y_t, y_i$ can be rewritten in $\mathcal{Y}_1$ into an $\hat{A}$-linear combination of the $|TW_1|$ products $\mathcal{B} := \{y_ty_{\mathbf{w}} \mid t \in T,\ \mathbf{w} \in \mathcal{B}_1\}$. Thus, $\mathcal{B}$ is linearly independent over $K$ by Theorem 5.7 and so an $\hat{A}$-basis of $\mathcal{Y}_1$. This proves our claim for $W_1$.

Now $\hat{A}[T.B_1] = \bigoplus_{i=0}^{p-1} \hat{A}[T.B]\mathbf{r}_1^i$ is $\mathbb{Z}/p\mathbb{Z}$-graded and multiplication by $\mathbf{r}_1$ defines $\hat{A}$-module isomorphisms between the summands. Furthermore, the defining ideal $I$ for $\mathcal{Y}$ in $\hat{A}[T.B]$ is contained in the defining ideal $I_1$ of $\mathcal{Y}_1$ in $\hat{A}[T.B_1]$, and $I_1 = \bigoplus_{i=0}^{p-1} I\mathbf{r}_1^i$ is graded. So $\mathcal{Y}_1 = \hat{A}[T.B_1]/I_1 = \bigoplus_{i=0}^{p-1} \mathcal{Y}y_1^i$ and multiplication with $y_1$ induces $\hat{A}$-module isomorphisms between the summands on the right. By construction the $\hat{A}$-basis $\mathcal{B}$ of $\mathcal{Y}_1$ has the property that $\mathcal{B} = \bigcup_{i=0}^{p-1}(\mathcal{B} \cap \mathcal{Y}y_1^i)$, hence $\mathcal{B} \cap \mathcal{Y}$ is an $\hat{A}$-free generating system of $\mathcal{Y}$.

Finally assume that $p = e$. Since $G(e, e, 2)$ is a Coxeter group (the dihedral group of order $2e$), by Theorem 5.13 we may assume $n \geq 3$. In this case, the braid group $B$ of $W = G(e, e, n)$ is a normal subgroup of index $e$ of the quotient $\bar{B}_1 = B_1/\langle \mathbf{r}_1^e \rangle$ of the braid group of $W_1$ (see [8, Prop. 3.24]). Thus, $\mathcal{Y}$ is an $\hat{A}$-subalgebra of $\hat{A}[T.B_1]/I$ where $I$ is generated by $\mathbf{r}_1^e - 1$ and the relations (†) for $\mathbf{r}_2, \ldots, \mathbf{r}_n$. We can now argue precisely as in the previous case. □

The following was used in the preceding proof:

**Lemma 5.15.** *Let $\mathcal{Y} = \mathcal{Y}(G(e, 1, n))$ and $y_1, y_2 \in \mathcal{Y}$ images of braid reflections satisfying $y_2y_1y_2y_1 = y_1y_2y_1y_2$ and such that the corresponding reflections $r_1, r_2 \in W$ have order $e, 2$ respectively. Then for all integers $a, b \geq 1$ there exist $\alpha \in \hat{A}[T]^\times$ and $\alpha_i, \alpha_i' \in \hat{A}[T]$ such that*

$$y_2y_1^ay_2y_1^b = \alpha y_1^by_2y_1^ay_2 + \sum_{i=1}^{b}(\alpha_iy_1^{a+b-i}y_2y_1^i + \alpha_i'y_1^iy_2y_1^{a+b-i}).$$

*Proof.* Write the relation (†) for $y_2$ as $y_2^2 = \lambda y_2 + \mu$ with $\mu \in \hat{A}[T]^\times$ and $\lambda \in \hat{A}[T]$, so $y_2^{-1} = \mu^{-1}y_2 - \lambda\mu^{-1}$. The relation between $y_1, y_2$ implies $y_1^ay_2y_1y_2 = y_2y_1y_2y_1^a$ for all $a \geq 1$.

Thus we find

$$\begin{aligned} y_2y_1^ay_2y_1 &= y_2y_1^ay_2y_1\cdot y_2y_2^{-1} = y_2^2y_1y_2y_1^ay_2^{-1} = (\lambda y_2+\mu)y_1y_2y_1^a(\mu^{-1}y_2-\lambda\mu^{-1})\\ &= \mu y_1y_2y_1^ay_2\mu^{-1} + \lambda y_2y_1y_2y_1^ay_2^{-1} - \mu y_1y_2y_1^a\lambda\mu^{-1}\\ &= \mu y_1y_2y_1^ay_2\mu^{-1} + \lambda y_1^ay_2y_1 - \mu y_1y_2y_1^a\lambda\mu^{-1}\\ &= \mu' y_1y_2y_1^ay_2 + \lambda y_1^ay_2y_1 - \lambda' y_1y_2y_1^a \end{aligned}$$

for suitable $\mu'\in\hat{A}[T]^\times$ and $\lambda'\in\hat{A}[T]$, giving the claim for $b=1$. For $b=2$, using the previous result twice we find

$$\begin{aligned} y_2y_1^ay_2y_1^2 &= (\mu' y_1y_2y_1^ay_2 + \lambda y_1^ay_2y_1 - \lambda' y_1y_2y_1^a)y_1\\ &= \mu' y_1y_2y_1^ay_2y_1 + \lambda y_1^ay_2y_1^2 - \lambda' y_1y_2y_1^{a+1}\\ &= \mu' y_1(\mu' y_1y_2y_1^ay_2 + \lambda y_1^ay_2y_1 - \lambda' y_1y_2y_1^a) + \lambda y_1^ay_2y_1^2 - \lambda' y_1y_2y_1^{a+1}\\ &= \mu'' y_1^2y_2y_1^ay_2 + \sum_{i=1}^{2}(\alpha_i y_1^{a+2-i}y_2y_1^i + \alpha_i' y_1^iy_2y_1^{a+2-i}) \end{aligned}$$

for suitable $\mu'',\alpha_i,\alpha_i'$. A straightforward induction yields the claim for arbitrary $b$. □

**Remark 5.16.** Marin [32, Def. 5.4] defines for arbitrary complex reflection groups $W$ an $\hat{A}$-algebra $M$ attached to $W$ as follows: let $\mathcal{L}$ be the lattice of intersections of the hyperplane arrangement of $W$. Then $M$ is the quotient of the group algebra over $\hat{A}$ of the semidirect product $\mathcal{L}\rtimes B(W)$ by the deformed order relations (†) for the braid reflections of $B(W)$. This algebra is generated by images of braid reflections $\mathbf{r}'$ and idempotents $e_r$, $r\in W$ a reflection (by [32, 5.1]). Marin shows [33, Thm 1.3] that $M$ is a free $\hat{A}$-module of rank $|W||\mathcal{L}|$. If $W$ is $\ell$-adic, there is a natural morphism

$$i_W: M\to \hat{A}[\ell^{-1}]\otimes_{\hat{A}}\mathcal{Y},\quad \mathbf{r}'\mapsto y_{\mathbf{r}},\ e_r\mapsto \ell^{-a}E_r,$$

from Marin's algebra to ours. We expect this to be injective, but in general far from surjective, since his algebra is free of rank independent of $\ell$ (compare to Theorem 5.7).

5.4. **A trace form.** Assume for the rest of the section that $W_\theta$ is a parabolic subgroup of $W$ for all $\theta\in\mathrm{Irr}(T)$; this holds whenever $\ell$ is very good for $(W,L)$, see Remark 5.11. Let $K$ be an extension of $\mathbb{Q}_\ell$ by the $\ell^a$th roots of unity. Let $\tilde{\mathbf{u}}=(\tilde{u}_{rj})$ be as in Section 3.1 and let $\tilde{K}=\mathrm{Frac}(K[\tilde{\mathbf{u}}])$. Recall from Section 3.2 that for any $\theta\in\mathrm{Irr}(T)$, $\tilde{K}\otimes\mathcal{H}(W_\theta,\mathbf{u}_\theta)\cong\tilde{K}W_\theta$ is split semisimple and the irreducible characters of $\mathcal{H}(W_\theta,\mathbf{u}_\theta)$ over $\tilde{K}$ are identified with $\mathrm{Irr}(W_\theta)$. Here, as before we denote by $\mathbf{u}_\theta$ the set $\mathbf{u}_0$ in the notation of Section 3.1 if $W_\theta$ is the reflection subgroup $W_0$. Then, with $U$ a $\tilde{K}\otimes\mathcal{H}(W_\theta,\mathbf{u}_\theta)$-module affording $\phi$ and $U_{\theta,\phi}:=\mathrm{Ind}_{\hat{B}_\theta}^{\hat{B}}(U)$, Theorem 5.7(a) shows

$$\mathrm{Irr}(\mathcal{Y}_{\tilde{K}})=\{U_{\theta,\phi}\mid \theta\in\mathrm{Irr}(T)/W,\ \phi\in\mathrm{Irr}(W_\theta)\}.$$

We let $\chi_{\theta,\phi}$ denote the character of $U_{\theta,\phi}$.

We consider the following non-degenerate trace form $\mathcal{Y}\to\tilde{K}$:

$$\tau:=\tau_{\mathcal{Y}}:=\sum_{\theta/W}\sum_{\phi\in\mathrm{Irr}(W_\theta)}\frac{1}{f_{\theta,\phi}}\chi_{\theta,\phi}. \tag{1}$$

Here, for any $\theta \in \mathrm{Irr}(T)$, $f_{\theta,\phi} \in \tilde{A}$ is the Schur element of the Hecke algebra $\mathcal{H}(W_\theta, \mathbf{u}_\theta)$ indexed by $\phi$ as in Section 3.1.

**Proposition 5.17.** *Assume that $W_\theta$ is a parabolic subgroup of $W$ for all $\theta \in \mathrm{Irr}(T)$. Assume also that $\mathcal{H}(W, \mathbf{u})$ is strongly symmetric with respect to $\mathbf{W} \subset B$ as in Definition 3.4. Then we have*

$$\tau(y_t y_{\mathbf{w}}) = \delta_{t,1}\delta_{\mathbf{w},1}|T| \qquad \text{for any } t \in T,\ \mathbf{w} \in \mathbf{W}.$$

*Proof.* For $\theta \in \mathrm{Irr}(T)$ let $C_\theta$ be a system of coset representatives of $W_\theta$ in $W$, and $\mathbf{C}_\theta \subseteq \mathbf{W}$ the corresponding system of coset representatives of $T.\widetilde{B}_\theta$ in $T.B$. Let $\theta \in \mathrm{Irr}(T)$ and $\phi \in \mathrm{Irr}(W_\theta)$, and let $U$ be a corresponding representation of $\mathcal{H}(W_\theta, \mathbf{u}_\theta)$ (which we consider as a representation of $\widehat{B}_\theta = T\widetilde{B}_\theta$ as above). Let us set $U^0(x) := U(x)$ if $x \in T.\widetilde{B}_\theta$ and 0 otherwise. Then

$$U_{\theta,\phi}(y_t y_{\mathbf{w}}) = \sum_{\mathbf{x}\in\mathbf{C}_\theta} U^0((y_t y_{\mathbf{w}})^{\mathbf{x}}) = \sum_{\mathbf{x}\in\mathbf{C}_\theta, y_{\mathbf{w}}^{\mathbf{x}}\in\widetilde{B}_\theta} \theta(y_t^{\mathbf{x}}) U(y_{\mathbf{w}}^{\mathbf{x}}),$$

so

$$\sum_{\phi\in\mathrm{Irr}(W_\theta)} \frac{1}{f_{\theta,\phi}} \chi_{\theta,\phi}(y_t y_{\mathbf{w}}) = \sum_{\mathbf{x}\in\mathbf{C}_\theta, y_{\mathbf{w}}^{\mathbf{x}}\in\widetilde{B}_\theta} \theta(y_t^{\mathbf{x}}) \sum_{\phi\in\mathrm{Irr}(W_\theta)} \frac{1}{f_{\theta,\phi}} \chi_{\theta,\phi}(y_{\mathbf{w}}^{\mathbf{x}}) = \sum_{\mathbf{x}\in\mathbf{C}_\theta, y_{\mathbf{w}}^{\mathbf{x}}\in\widetilde{B}_\theta} \theta(y_t^{\mathbf{x}}) t_{W_\theta,\mathbf{u}_\theta}(y_{\mathbf{w}}^{\mathbf{x}})$$

with $t_{W_\theta,\mathbf{u}_\theta}$ as in Definition 3.4. By the choice of $\mathbf{W}$ we have

$$t_{W_\theta,\mathbf{u}_\theta}(y_{\mathbf{w}}^{\mathbf{x}}) = t_{W,\mathbf{u}}(y_{\mathbf{w}}^{\mathbf{x}}) = t_{W,\mathbf{u}}(y_{\mathbf{w}}) = \delta_{\mathbf{w},1}.$$

Thus the form $\tau$ evaluates to

$$\tau(y_t y_{\mathbf{w}}) = \sum_{\theta/\mathcal{F}} \sum_{\mathbf{x}\in\mathbf{C}_\theta} \theta(y_t^{\mathbf{x}})\delta_{\mathbf{w},1} = \sum_{\theta\in\mathrm{Irr}(\mathbf{T})} \theta(y_t)\delta_{\mathbf{w},1} = \delta_{t,1}\delta_{\mathbf{w},1}|T|,$$

as desired. □

It seems natural to ask the following:

**Question 5.18.** *Let $(W, L)$ be a simply connected $\ell$-adic reflection group for which $\ell$ is very good. Does the form $|T|^{-1}\tau$ take values in $\hat{A}$ and is it then a symmetrising form on $\mathcal{Y}$ over $\hat{A}$?*

Note that an affirmative answer to the first part of Question 5.18 follows under the assumptions of Proposition 5.17.

5.5. **Relation to classical Yokonuma algebras.** Suppose that $W$ is the Weyl group with respect to a maximally split torus $\mathbf{T}_0$ of a connected reductive group $\mathbf{G}$ with an $\mathbb{F}_q$-structure defined by a split Frobenius map $F : \mathbf{G} \to \mathbf{G}$. Set $G = \mathbf{G}^F$ and set $T_0 = \mathbf{T}_0^F$. Let $\mathbf{U}$ be the unipotent radical of an $F$-stable Borel subgroup of $\mathbf{G}$ containing $\mathbf{T}_0$, and let $U = \mathbf{U}^F$. Let $\ell$ be a prime dividing $q - 1$ and set $e_U = |U|^{-1} \sum_{u\in U} u \in \mathbb{Z}_\ell G$. Then

$$\mathcal{Y}' := \mathrm{End}_{\mathbb{Z}_\ell G}(\mathbb{Z}_\ell[G/U]) = e_U \mathbb{Z}_\ell G e_U$$

is the associated classical Yokonuma Hecke algebra [38].

Let $r_1, \ldots, r_m$ be Coxeter generators of the Weyl group $W$. For $t \in T_0$, let $t' = e_U t e_U$ and for each $i$, let $E_i := E_{[T_0,r_i]} e_U$, where for any subgroup $A \leq G$ we denote by $E_A$ the sum of elements of $A$. By [20, Thm 2], $\mathcal{Y}'$ has a generating set $\{t', s_i \mid t \in T_0,\ 1 \leq i \leq m\}$ such that

- the $t'$ satisfy the same relations as the corresponding group elements $t$;
- the action relations between the $t, r_i$, with $t$ replaced by $t'$ and $r_i$ by $s_i$ hold in $\mathcal{Y}'$;
- the braid relations between the $r_i$, with $r_i$ replaced by $s_i$ hold in $\mathcal{Y}'$; and
- $s_i^2 = 1 - q^{-1}(E_i - s_iE_i)$.

Note that Juyumaya–Kannan work over the complex numbers but it can be checked from the explicit description of the $s_i$s in terms of the standard generators coming from the Bruhat decomposition, that the above holds over any ring in which $q$ is invertible.

**Proposition 5.19.** *Suppose that there is a $W$-equivariant isomorphism between $T$ and the Sylow $\ell$-subgroup of $T_0$. Let $H$ be the $\ell'$-Hall subgroup of $T_0$, let $e_H = |H|^{-1}E_H$ be the principal block idempotent of $\mathbb{Z}_\ell T_0$ and set $f = e_He_U$. Let $\psi_q : \hat{A} \to \mathbb{Z}_\ell$ be the specialisation corresponding to $u_{\mathbf{r}1} \mapsto -1$, $u_{\mathbf{r}2} \mapsto q$ for all $\mathbf{r}$. Then there is an isomorphism of $\mathbb{Z}_\ell$-algebras $\mathcal{Y}_{\psi_q} \cong f\mathcal{Y}'f$.*

*Proof.* Note that $f$ is an idempotent of $\mathbb{Z}_\ell G$. By considering the generating set of $\mathcal{Y}'$ described above, one sees that $f$ is central in $\mathcal{Y}'$ and for any $x \in H$, $xf = f$, hence $\{t'f, s_if \mid t \in T_0,\ 1 \le i \le m\}$ is a generating set for $\mathcal{Y}'f = f\mathcal{Y}'f$.

It follows from the description of $\mathcal{Y}$ via generators and relations given after Lemma 5.3 that there is a surjective $\mathbb{Z}_\ell$-algebra homomorphism $\mathcal{Y}_{\psi_q} \to \mathcal{Y}'f$ which sends the image $\tilde{y}_t \in \mathcal{Y}_{\psi_q}$ of $y_t$ to $\frac{q^{-1}\ell^a}{q-1}t'f$ and $\tilde{y}_{\mathbf{r}_i}$ to $-s_if$. By Theorem 5.13, $\mathcal{Y}_{\psi_q}$ is $\mathbb{Z}_\ell$-free of rank $|TW|$ and the same is true for $\mathcal{Y}'f$. The last assertion can be seen by considering the standard basis $\{e_Une_U \mid n \in N_G(T_0)\}$ of $\mathcal{Y}'$ given by the Bruhat decomposition. Thus $\mathcal{Y}_{\psi_q}$ is isomorphic to $f\mathcal{Y}'f$ as claimed. □

5.6. **Proofs of Theorem 1 and Corollary 2.** Throughout this subsection $\ell > 2$ is a prime, $q$ is a prime power with $q \equiv 1 \pmod{\ell}$ and $a > 0$ such that $\ell^a || (q-1)$. Let $\mathbb{G} = (W, L)$ be a simply connected $\mathbb{Z}_\ell$-spets with $W$ an $\ell'$-group. Let $\mathcal{F}$ be the fusion system associated to $(W, L)$ as described in Section 4.1, with underlying $\ell$-group $S$. Recall that with the stated assumptions we have $S = T \cong (\mathbb{Z}/\ell^a)^n$ where $T$ is the homocyclic group $L/\ell^aL$ of exponent $\ell^a$. We let $\mathcal{Y}$ be the Yokonuma algebra associated to $(W, L, q)$ as in Section 5.

Recall the indeterminates $\tilde{u}_{rj}$ with $\tilde{u}_{rj}^z = \zeta_{o(r)}^{-j}u_{rj}$. By [27, Cor. 4.8], $z$ may be chosen to divide the order of the group of roots of unity in $\mathbb{Q}_\ell$, that is, $\ell - 1$. As $\ell^a || (q-1)$, by Hensel's lemma there is a unique root of $X^z - q \in \mathbb{Z}_\ell[X]$ in $\mathbb{Z}_\ell$, say $q^{1/z}$, with $\ell^a || (q^{1/z} - 1)$. Let $\mathbb{Z}_\ell \subseteq \mathcal{O}$ be a complete discrete valuation ring containing the $|T|$th roots of unity. Let

$$\psi_{s,q} : \mathcal{O}[\tilde{\mathbf{u}}^{\pm 1}] \to \mathcal{O}, \qquad \tilde{u}_{rj} \mapsto \begin{cases} q^{\frac{1}{z}} & \text{if } j = o(r), \\ 1 & \text{if } 1 \le j < o(r), \end{cases}$$

be the specialisation $\psi_{s,q}$ from Section 3.2 with $R = \mathcal{O}$. Since $|T_r|$ divides $\ell^a$, $\psi_{s,q}$ extends to an $\mathcal{O}$-linear homomorphism $\mathcal{O}[\tilde{\mathbf{u}}^{\pm 1}, \mathbf{v}] \to \mathcal{O}$ which we still denote $\psi_{s,q}$. Let $\tilde{K} = \mathrm{Frac}(\mathcal{O}[\tilde{\mathbf{u}}^{\pm 1}])$ and $K = \mathrm{Frac}(\mathcal{O})$.

We restate Theorem 1.

**Theorem 5.20.** *Let $\mathbb{G}$ be as above. Suppose that $\mathcal{H}(W, \mathbf{u})$ is strongly symmetric as in Definition 3.4. Then Conjectures 4.2, 4.3 and 4.5 hold for $\mathbb{G}$.*

*Proof.* Let $W_0$ be a reflection subgroup of $W$. By Lemma 3.7, $K \otimes_{\mathcal{O}} \mathcal{H}_{\psi_{s,q}}(W_0, \mathbf{u}_0)$ is split semisimple and $\psi_{s,q}$ induces a bijection $\operatorname{Irr}(\tilde{K} \otimes_A \mathcal{H}(W_0, \mathbf{u}_0)) \to \operatorname{Irr}(K \otimes_{\mathcal{O}} \mathcal{H}_{\psi_{s,q}}(W_0, \mathbf{u}_0))$. Also recall from Section 3.2, $\operatorname{Irr}(\tilde{K} \otimes_A \mathcal{H}(W_0, \mathbf{u}_0))$ is identified with $\operatorname{Irr}(W_0)$ via $\psi_1$. Henceforth we identify $\operatorname{Irr}(K \otimes_{\mathcal{O}} \mathcal{H}_{\psi_{s,q}}(W_0, \mathbf{u}_0))$ and $\operatorname{Irr}(W_0)$ via the bijections induced by $\psi_{s,q}$ and $\psi_1$.

Denoting the restriction of $\psi_{s,q}$ to $\hat{A}$ again by $\psi_{s,q}$, set $\mathcal{Y}_q := \mathcal{Y}_{\psi_{s,q}}$. Since $W$ is an $\ell'$-group, $W_\theta$ is a parabolic subgroup of $W$ for all $\theta \in \operatorname{Irr}_K(T)$. Then by Theorem 5.7(a) and the above $K \otimes_{\mathcal{O}} \mathcal{Y}_q$ is split semisimple and $\operatorname{Irr}(K \otimes_{\mathcal{O}} \mathcal{Y}_q)$ is in bijection with pairs $(\theta, \phi)$ as $\theta$ runs over representatives of $W$-orbits of $\operatorname{Irr}(T)$ and $\phi \in \operatorname{Irr}(W_\theta)$. Let $\chi'_{\theta,\phi}$ be the irreducible character corresponding to the pair $(\theta, \phi)$. Then $\chi'_{\theta,\phi}$ is afforded by the simple module $U'_{\theta,\phi} := \operatorname{Ind}_{\widehat{B}_\theta}^{\widehat{B}}(U)$ for $U$ a simple $K \otimes_{\mathcal{O}} \mathcal{H}_{\psi_{s,q}}(W_\theta, \mathbf{u}_\theta)$-module corresponding to $\phi$. Here as in Section 5.4, $\mathbf{u}_\theta = \mathbf{u}_0$ in the notation of Section 3.1 if $W_\theta = W_0$.

We consider the following $K$-linear form on $K \otimes_{\mathcal{O}} \mathcal{Y}_q$:

$$\tau_q := \frac{1}{|T|} \sum_{\theta/W} \sum_{\phi \in \operatorname{Irr}(W_\theta)} \frac{1}{\psi_{s,q}(f_{\theta,\phi})} \chi'_{\theta,\phi}. \tag{2}$$

By Lemma 3.6 this is well defined. Since the coefficient of every irreducible character is non zero, $\tau_q$ is a symmetrising form on $K \otimes_{\mathcal{O}} \mathcal{Y}_\psi$ with Schur elements $|T|\psi_{s,q}(f_{\theta,\phi})$.

Let $\mathbf{W} \subset B(W)$ be as in Definition 3.4. By Theorem 5.9, $\{\tilde{y}_t \tilde{y}_{\mathbf{w}} \mid t \in T,\ \mathbf{w} \in \mathbf{W}\}$ is an $\mathcal{O}$-basis of $\mathcal{Y}_q$. Here, as earlier, for $x \in \mathcal{Y}$, we write $\tilde{x} := 1_{\mathcal{O}} \otimes x \in \mathcal{Y}_q$. As in Proposition 5.17, we have

$$\tau_q(\tilde{y}_t \tilde{y}_{\mathbf{w}}) = \delta_{t,1}\delta_{\mathbf{w},1} \qquad \text{for any } t \in T,\ \mathbf{w} \in \mathbf{W},$$

hence the above gives that the restriction of $\tau_q$ to $\mathcal{Y}_q$ takes values in $\mathcal{O}$.

By the strongly symmetric hypothesis, and by Theorem 5.10 there is an $\mathcal{O}$-algebra isomorphism $\sigma : \mathcal{O}[TW] \to \mathcal{Y}_q$ whose restriction to $T$ is the identity on $T$ (where we identify $T$ with its image in $\mathcal{Y}_q$ via $t \mapsto \tilde{y}_t$, $t \in T$). Denote also by $\sigma$ the extension $K[TW] \to K \otimes_{\mathcal{O}} \mathcal{Y}_q$. Then $\tau_\sigma := \tau_q \circ \sigma : K[TW] \to K$ is a symmetrising form on $K[TW]$, with Schur element $|T|\psi_{s,q}(f_{\theta,\phi})$ at the irreducible character of $K[TW]$ corresponding under $\sigma$ to the character $\chi'_{\theta,\phi}$ of $K \otimes_{\mathcal{O}} \mathcal{Y}_q$. Further, by the above $\tau_\sigma(t) = \delta_{t,1}$ for all $t \in T$ and the restriction of $\tau\sigma$ to $\mathcal{O}[TW]$ takes values in $\mathcal{O}$. Thus, by Lemmas 2.5 and 2.6,

$$\frac{1}{|T|^2} \sum_{\theta/W} \sum_{\phi \in \operatorname{Irr}(W_\theta)} \frac{1}{\psi_{s,q}(f_{\theta,\phi}^2)} = \frac{\alpha}{|TW|},$$

where $\alpha \in \mathcal{O}$ is such that $\alpha \equiv 1 \pmod{\ell}$.

Recall that $S = T$ and as explained in Section 4.2, there exists a $W$-equivariant bijection between $\operatorname{Irr}(T)$ and $T$. Thus the left hand side of the above equals

$$\frac{\psi_{s,q}(d)}{|T|^2 \psi_{s,q}(p_W^2)},$$

where $d := \sum_{s/\mathcal{F}} \sum_{\phi \in \operatorname{Irr}(W(s))} p_W^2 f_{s,\phi}^{-2}$. By Lemma 4.10, $\psi_{s,q}(d) = \dim(B_0)|_{x=q}$. Since $\psi_{s,q}(p_W) \equiv |W| \pmod{\ell}$, we obtain the validity of Conjectures 4.2 and 4.3 from the displayed equation above.

Finally, we prove Conjecture 4.5. First of all note that since $\sigma$ is the identity on $T$, for any pair $(\theta, \phi)$ as above the irreducible character of $K[TW]$ corresponding under $\sigma$ to the character $\chi'_{\theta,\phi}$ of $K \otimes_{\mathcal{O}} \mathcal{Y}_q$ covers $\theta$ and therefore is of the form $\gamma_{\theta,\tilde\phi} := \mathrm{Ind}_{W_\theta}^{W}(\tilde\phi)$ for some $\tilde\phi \in \mathrm{Irr}(W_\theta)$. In particular, $\sigma$ induces a permutation $\phi \mapsto \tilde\phi$ of $\mathrm{Irr}(W_\theta)$. By Lemma 2.7 we have that for any $\nu \in \mathrm{IBr}(TW)$,

$$\sum_{\theta/W} \sum_{\phi \in \mathrm{Irr}(W_\theta)} \frac{d_{\gamma_{\theta,\tilde\phi}\nu}}{\psi_{\mathrm{s},q}(f_{\theta,\phi})}$$

is divisible by $|T|$ in $\mathcal{O}$. Choose a $W$-equivariant bijection between $\mathrm{Irr}(T)$ and $T$ and let $\Theta : \mathrm{Irr}(TW) \to \mathrm{Irr}(B_0)$, $\gamma \mapsto \hat\gamma$, be the bijection such that if $\gamma = \mathrm{Ind}_{W_\theta}^{W}(\tilde\phi)$, then $\hat\gamma$ is the element of $\mathrm{Irr}(B_0)$ labelled by $(x, \phi)$, where the $W$-class of $x \in T$ corresponds to the $W$-class of $\theta$ for the chosen $W$-equivariant bijection between $\mathrm{Irr}(T)$ and $T$. Then $\Theta$ is $W$-equivariant. Moreover, by Lemma 4.10 the above displayed expression equals $\psi_{\mathrm{s},q}(p_W)^{-1}(\deg \Phi_{\hat\nu})|_{x=q}$. The result follows since $\psi_{\mathrm{s},q}(p_W)$ is an invertible element of $\mathcal{O}$. $\square$

**Remark 5.21.** The above holds in a more general setting. Drop the assumption that $W$ is spetsial; so $W$ is an $\ell$-adic reflection group of order prime to $\ell$. Let $\psi_q : \mathcal{O}[\tilde{\mathbf{u}}^{\pm 1}, \mathbf{v}] \to \mathcal{O}$ be any specialisation as in Section 3.2. Suppose that $\tau : \mathcal{H}(W, \mathbf{u}) \to A$ is a symmetrising form such that the following holds:

(1) there is a section $W \to \mathbf{W} \subset B$ of the natural map $B \to W$ containing 1 whose image in $\mathcal{H}(W, \mathbf{u})$ is an $A$-basis of $\mathcal{H}(W, \mathbf{u})$ with $\tau(h_{\mathbf{w}}) = \delta_{\mathbf{w},1}$ for all $\mathbf{w} \in \mathbf{W}$; and
(2) for any parabolic subgroup $W_0 \le W$, $\tau|_{\mathcal{H}(W_0,\mathbf{u}_0)} : \mathcal{H}(\mathbf{W}_0, \mathbf{u}_0) \to A$ is a symmetrising form.

For $s \in S$, $\phi \in \mathrm{Irr}(W(s))$ let $f_{\tau,s,\phi}$ denote the Schur element of $\tau|_{\mathcal{H}(W_0,\mathbf{u}_0)}$ with respect to $\phi$ where $W_0 = W(s)$. Set

$$d := \sum_{s/\mathcal{F}} \sum_{\phi \in \mathrm{Irr}(W(s))} \frac{p_W^2}{f_{\tau,s,\phi}^2}$$

and for $\nu \in \mathrm{IBr}(SW)$ set

$$\Phi_\nu(1) := \sum_{\gamma \in \mathrm{Irr}(SW)} d_{\gamma\nu} \frac{p_W}{f_{\tau,s,\phi}},$$

where $d_{\gamma\nu}$ is the decomposition number in $SW$ with respect to $\gamma$ and $\nu$. Then with the same proof as above we get

(a) $\psi_{\mathrm{s},q}(d)_\ell = |S|$;
(b) $\frac{\psi_{\mathrm{s},q}(d)}{\psi_{\mathrm{s},q}(p_W)\,|S|} \equiv 1 \pmod{\ell}$; and
(c) for each $\nu \in \mathrm{IBr}(SW)$, $|S|$ divides $\psi_{\mathrm{s},q}(\Phi_\nu(1))$.

*Proof of Corollary 2.* By [26, §3] all imprimitive irreducible spetsial reflection groups are either Coxeter groups, of type $G(e, 1, n)$ or of type $G(e, e, n)$ with $n \ge 3$ and therefore strongly symmetric by Proposition 3.5. Therefore Conjectures 1 and 2 hold for these groups by Theorem 5.20. For the primitive groups, Conjecture 1 holds by Proposition 4.8. Conjecture 2 holds when $W$ is primitive and 2-dimensional by Proposition 3.5(c) and Theorem 5.20; for $G_{14}$ Conjecture 2 holds by direct computation using the description of the decomposition matrix provided in the proof of Proposition 4.11. $\square$

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