

Local-global conjectures and blocks of finite simple groups

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Abstract

We give an expanded treatment of our lecture series at the 2017 Groups St Andrews conference in Birmingham on local-global conjectures and the block theory of finite reductive groups.

1 Introduction

The aim of these notes is to describe recent progress in the ordinary and modular representation theory of finite groups, in particular pertaining to the local-global counting conjectures. As this relies heavily on having sufficient knowledge about the representation theory of finite simple groups, we will also highlight the major advances and results obtained in the block theory of finite groups of Lie type.

The general setup will be as follows: G will be a finite group,

$$\text{Irr}(G) = \{\text{trace functions of irreducible representations } G \rightarrow \text{GL}_n(\mathbb{C})\}$$

its set of irreducible complex characters. For $\chi \in \text{Irr}(G)$, the value $\chi(1)$ at the identity element of G is called its *degree*; it is the degree of any representation affording this character.

We also choose a prime p (with the interesting case being the one when p divides the group order $|G|$).

The aim is now to link, as much as possible, aspects of the representation theory of G , like its set of irreducible characters $\text{Irr}(G)$, their degrees, and so on, to those of local subgroups of G . Here a subgroup N of G is called *p-local* if $N = N_G(Q)$ for some p -subgroup $1 \neq Q \leq G$. An important example of local subgroups is given by the normalisers $N_G(P)$ of Sylow p -subgroups $P \in \text{Syl}_p(G)$.

2 The fundamental conjectures

The character theory of finite groups was invented by G. Frobenius more than a hundred years ago. But still there are many basic open questions that remain unsolved to the present day. We present some of these in this section.

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2.1 The McKay conjecture

John McKay in the beginning of the 1970s counted irreducible characters of odd degree of the newly discovered sporadic simple groups; here are some such numbers:

$$M_{11} : 4, \quad M_{12} : 8, \quad C_{01}, Fi_2 : 32, \quad B, M : 64.$$

Strikingly, all of these are 2-powers. In 1971 Ian Macdonald [53] showed that the number of odd degree irreducible characters is a power of 2 for all symmetric groups \mathfrak{S}_n . (But obviously this statement cannot be true for all groups, think of the cyclic group of order 3.) The right generalisation seems to be as follows: let

$$\text{Irr}_{p'}(G) := \{\chi \in \text{Irr}(G) \mid \chi(1) \not\equiv 0 \pmod{p}\}$$

be the subset of irreducible characters of G of degree prime to p , then the following should be true [61]:

Conjecture 2.1 (McKay (1972)) *Let G be a finite group, p a prime and $P \in \text{Syl}_p(G)$. Then*

$$|\text{Irr}_{p'}(G)| = |\text{Irr}_{p'}(N_G(P))|.$$

This conjecture does indeed predict global data in terms of local information. Since it proposes an explicit formula for $|\text{Irr}_{p'}(G)|$ it is sometimes also called a counting conjecture. Note that for G a sporadic group as above, or a symmetric group, Sylow 2-subgroups are self-normalising and then $\text{Irr}_{2'}(N_G(P)) = \text{Irr}_{2'}(P) = \text{Irr}(P/P')$ is an abelian 2-group and hence of order a power of 2.

Marty Isaacs [42] showed in 1973 that Conjecture 2.1 is true for all groups of odd order, using the Glauberman correspondence, before even having been aware of McKay's paper.

Example 2.2 Let $G = \mathfrak{S}_n$ the symmetric group of degree n . Frobenius showed how the irreducible characters of G can be naturally labelled by partitions $\lambda \vdash n$ of n . Let us write χ^λ for the irreducible character labelled by λ . The degree of χ^λ is given by the well-known hook formula

$$\chi^\lambda(1) = \frac{n!}{\prod_h \ell(h)},$$

where the product runs over all *hooks* of λ , that is, all boxes (i, j) of the Young diagram of λ , and $\ell(h)$ denotes the *length* of the hook in that diagram starting at box (i, j) . Macdonald [53] determined when this expression is an odd number, and obtained the following result: write $n = 2^{k_1} + 2^{k_2} + \dots$ with $k_1 < k_2 < \dots$. Then $|\text{Irr}_{2'}(\mathfrak{S}_n)| = 2^{k_1 + k_2 + \dots}$, which is indeed a power of 2. Formulas for general $p \geq 2$ can be given in terms of the p -adic expansion of n using generating functions (see [53]).

On the other hand a Sylow 2-subgroup P of \mathfrak{S}_n is a direct product $P = P_1 \times P_2 \times \dots$ with $P_i = C_2 \wr C_2 \wr \dots$ (k_i factors), an iterated wreath product, and it

is self-normalising. Now as already pointed out above, the only p' -characters of p -groups are the linear characters, so that $\text{Irr}_{p'}(P) = \text{Irr}(P/P')$. But

$$|\text{Irr}(P/P')| = |\text{Irr}(P_1/P'_1)| \times |\text{Irr}(P_2/P'_2)| \times \cdots,$$

and $|\text{Irr}(P_i/P'_i)| = |P_i/P'_i| = 2^{k_i}$, so indeed we find $|\text{Irr}_{2'}(P)| = 2^{k_1+k_2+\cdots} = |\text{Irr}_{2'}(G)|$ as predicted by McKay's Conjecture 2.1.

While McKay's conjecture is still open, various refinements and extensions have been proposed; as one example let us mention [45]:

Conjecture 2.3 (Isaacs–Navarro (2002)) *In the situation of Conjecture 2.1 there exists a bijection $\Omega : \text{Irr}_{p'}(G) \rightarrow \text{Irr}_{p'}(N_G(P))$ such that $\Omega(\chi)(1) \equiv \pm\chi(1) \pmod{p}$.*

Paul Fong [34] showed that this refinement holds for $G = \mathfrak{S}_n$ and all primes (note that it is stronger than the original McKay conjecture only when $p \geq 5$). Alexandre Turull [83] showed that it holds for all solvable groups. This was shown to hold for alternating groups by Nath [65]

Another such refinement, which is currently being studied quite intensely, was proposed by Navarro [66]; it proposes that the bijection Ω should also be equivariant with respect to $\text{Gal}(\overline{\mathbb{Q}}_p/\mathbb{Q}_p)$; see Brunat and Nath [16] for the case of alternating groups, and Ruhstorfer [74] for groups of Lie type in their defining characteristic.

2.2 The local-global conjectures

The McKay conjecture is concerned with the characters of p' -degree. Now what about characters of degree divisible by p ? How to relate these to local data? There is a natural extension of McKay's conjecture, but in order to formulate this, we need to introduce p -blocks. Let $\mathcal{O} \geq \mathbb{Z}_p$ be a big enough extension, for example containing all $|G|$ th roots of unity, and decompose the group ring of G over \mathcal{O} into a direct sum of minimal 2-sided ideals

$$\mathcal{O}G = B_1 \oplus \cdots \oplus B_r,$$

called the p -blocks of G . It is easily seen that this induces a partition

$$\text{Irr}(G) = \text{Irr}(B_1) \sqcup \cdots \sqcup \text{Irr}(B_r),$$

by decreeing that $\chi \in \text{Irr}(G)$ lies in $\text{Irr}(B_i)$ if and only if $\chi|_{B_i} \neq 0$. This block subdivision can in fact be read off from the character table of G : $\chi, \psi \in \text{Irr}(G)$ lie in the same p -block if and only if

$$\frac{|x^G|\chi(x)}{\chi(1)} \equiv \frac{|x^G|\psi(x)}{\psi(1)} \pmod{\mathfrak{P}} \quad \text{for all } x \in G,$$

where $\mathfrak{P} \trianglelefteq \mathcal{O}$ is the maximal ideal containing p .

Richard Brauer showed how to associate to any p -block B of G a p -subgroup $D \leq G$ of G , unique up to conjugation, called *defect group* of B . This can be

defined as follows: D is minimal amongst p -subgroups P of G for which there exists a p' -element x of G such that P is a Sylow p -subgroup of $C_G(x)$ and

$$\frac{|x^G|\chi(x)}{\chi(1)} \not\equiv 0 \pmod{\mathfrak{P}} \quad \text{for all } \chi \in \text{Irr}(B).$$

Brauer also constructed a block b of $N_G(D)$ called *Brauer correspondent* of B . The Brauer correspondent of B in $N_G(D)$ is the unique p -block b of $N_G(D)$ with defect group D such that

$$\frac{|x^G|\chi(x)}{\chi(1)} \equiv \frac{|x^G|\theta^G(x)}{\theta(1)} \pmod{\mathfrak{P}} \quad \text{for all } x \in G, \chi \in \text{Irr}(B) \text{ and } \theta \in \text{Irr}(b).$$

Example 2.4 (a) Let G be a p -group. Then $\mathcal{O}G$ is a single block, with defect group $D = G$ maximal possible.

(b) Let $\chi \in \text{Irr}(G)$ with $\chi(1)_p = |G|_p$, then the corresponding block B of G has $\text{Irr}(B) = \{\chi\}$ and is called *of defect zero*. Here $D = 1$. For example, if p does not divide $|G|$, then every p -block of G is of defect zero. All other blocks contain at least two characters. For $G = \mathfrak{S}_n$, it is clear from the hook formula in Example 2.2 that χ^λ is of defect zero if and only if λ has no p -hook, that is, if and only if λ is a p -core.

(c) The block B_0 of G containing the trivial character 1_G is called the principal block of G . It always has defect group $D \in \text{Syl}_p(G)$.

With blocks now at our disposal, McKay's Conjecture 2.1 can be naturally refined and generalised as follows (see [1]):

Conjecture 2.5 (Alperin–McKay (1976)) *Let B be a p -block of G with defect group D and Brauer correspondent b in $N_G(D)$. Then*

$$|\text{Irr}_0(B)| = |\text{Irr}_0(b)|,$$

where $\text{Irr}_0(B) := \{\chi \in \text{Irr}(B) \mid \chi(1)_p = |G : D|_p\}$.

The characters in $\text{Irr}_0(B)$ are called *characters of height zero*. Let us point out that for blocks with defect group $D \in \text{Syl}_p(G)$ we have

$$\text{Irr}_0(B) = \text{Irr}(B) \cap \text{Irr}_{p'}(G),$$

i.e., $\chi \in \text{Irr}(B)$ lies in $\text{Irr}_0(B)$ if and only if $\chi \in \text{Irr}_{p'}(G)$. It follows that the Alperin–McKay conjecture implies the McKay conjecture, by just summing over all blocks of full defect.

Again, the Alperin–McKay conjecture gives a local answer to a global question. It has been proved for all p -solvable groups by Okuyama–Wajima and Dade in 1980 [25, 71], for the symmetric groups, the alternating groups and their covering groups by Olsson [72] and Michler–Olsson [62].

In view of this conjecture it is of interest to know when it will provide information on all of $\text{Irr}(B)$, that is, when all characters in $\text{Irr}(B)$ are of height 0. This is the subject of another even older conjecture by Brauer [12]:

Conjecture 2.6 (Brauer (1955)) *Let B be a block with defect group D . Then*

$$\text{Irr}(B) = \text{Irr}_0(B) \iff D \text{ is abelian.}$$

A consequence of this so-called Brauer's Height Zero Conjecture would be an easy criterion to decide from the character table of a finite group whether its Sylow p -subgroups are abelian: indeed, any Sylow p -subgroup $D \in \text{Syl}_p(G)$ is a defect group of the principal block B_0 , and both $\text{Irr}(B_0)$ and $\text{Irr}_0(B_0)$ are encoded in the character table.

Conjecture 2.6 has been proved for p -solvable groups by Gluck and Wolf [37], and for 2-blocks with defect group $D \in \text{Syl}_2(G)$ much more recently by Navarro and Tiep [69] using, among other ingredients, Walter's classification of groups with abelian Sylow 2-subgroup.

Let us introduce a further fundamental conjecture in this subject. This purports to count irreducible characters in positive characteristic. For this let $\text{IBr}(G)$ denote the set of irreducible p -Brauer characters of G ; these are lifts to characteristic zero, constructed by Brauer, of the trace functions of irreducible representations $G \rightarrow \text{GL}_n(\overline{\mathbb{F}}_p)$. Again these are partitioned according to the p -blocks of G , so that $\text{IBr}(G) = \text{IBr}(B_1) \sqcup \dots \sqcup \text{IBr}(B_r)$. A *weight* of G is a pair (Q, ψ) consisting of a p -subgroup $Q \leq G$ and an irreducible character $\psi \in \text{Irr}(N_G(Q)/Q)$ of defect zero. Clearly, G acts on its set of weights by conjugation. Any weight is naturally attached to a well-defined p -block of G . Then the Alperin Weight Conjecture [2] proposes:

Conjecture 2.7 (Alperin (1986)) *Let B be a block of G . Then*

$$|\text{IBr}(B)| = |\{\text{weights of } G \text{ attached to } B\} \sim_G|.$$

So, $|\text{IBr}(B)|$ should be determined locally, or more precisely this is the case whenever B is not of defect zero, since for blocks B of defect zero, with $\text{Irr}(B) = \{\chi\}$ (see Example 2.4(b)), the corresponding weight is just $(1, \chi)$. A proof of the Alperin Weight Conjecture 2.7 for solvable groups was given by Okuyama [70], for p -solvable groups by Isaacs and Navarro [44], for $\text{GL}_n(q)$ and \mathfrak{S}_n by Alperin and Fong [4], and for groups of Lie type when p is their defining prime by Cabanes [17]. For blocks with abelian defect groups (and hence in particular for groups with abelian Sylow p -subgroups), the weight conjecture has the following nice consequence:

Theorem 2.8 (Alperin (1986)) *Let B be a block with abelian defect groups satisfying the Alperin weight conjecture, and b its Brauer correspondent. Then:*

$$|\text{Irr}(B)| = |\text{Irr}(b)| \quad \text{and} \quad |\text{IBr}(B)| = |\text{IBr}(b)|.$$

Knörr and Robinson [49] have given reformulations of Conjecture 2.7 in terms of chains of p -subgroups of G . They also showed the following connection between the conjectures introduced above:

Theorem 2.9 (Knörr–Robinson (1989)) *The following are equivalent for a prime p :*

- (i) *The Alperin–McKay Conjecture 2.5 holds for all p -blocks with abelian defect;*
- (ii) *the Alperin Weight Conjecture 2.7 holds for all p -blocks with abelian defect.*

While all of the above conjectures are open in general, they have been shown to hold for special types of defect groups. By results of Dade they hold whenever the defect group D is cyclic, and by Sambale [75] when D is metacyclic.

Let us mention some further directions which we shall not go into here: several refinements of the above conjectures have been put forward, like the Isaacs–Navarro Conjecture 2.3 introduced above. Further, Dade’s conjecture [26] from 1992 simultaneously generalises the Alperin–McKay conjecture and the Alperin weight conjecture by making predictions on characters of arbitrary height. A recent conjecture of Eaton and Moretó [30] extends Brauer’s Height Zero Conjecture 2.6 to characters of the first positive height.

2.3 The reduction approach

In recent years a new approach for tackling the local-global counting conjectures has emerged: one tries to study a minimal counterexample by making use of the classification of the finite simple groups. A first such reduction in fact dates back quite a while [7]:

Theorem 2.10 (Berger–Knörr (1988)) *The “if” direction of Brauer’s Height Zero Conjecture 2.6 holds if it holds for all blocks of all quasi-simple groups.*

Recall here that a finite group G is *quasi-simple* if G is perfect and moreover $G/Z(G)$ is simple. It took 25 years until the necessary statement for quasi-simple groups could finally be verified, thus giving:

Theorem 2.11 (Kessar–Malle (2013)) *The “if” direction of Brauer’s Height Zero Conjecture 2.6 holds.*

This result is the outcome of work of many mathematicians on determining all p -blocks of all quasi-simple groups, the case of groups of Lie type being by far the most challenging. Major contributions are due to Fong–Srinivasan [35], Broué–Malle–Michel [15], Cabanes–Enguehard [20], Blau–Ellers [8], Bonnafé–Rouquier [11] and Enguehard [32], before the final case, the so-called quasi-isolated blocks of exceptional groups of Lie type at bad primes was settled by Kessar and Malle [46].

About 15 years after Berger–Knörr the issue of reductions of the long-standing conjectures was again taken up by Gabriel Navarro, which led to the following [43]:

Theorem 2.12 (Isaacs–Malle–Navarro (2007)) *The McKay Conjecture 2.1 holds for a prime p if all finite simple groups are McKay good for p .*

Here, the reduction is not as clean as for Brauer’s height zero conjecture. The condition of a simple group being McKay good is stronger and more complicated than just asking that it satisfies the McKay conjecture. We say that a simple group S of order divisible by p is *McKay good at p* if the following conditions hold, where

G denotes a *universal covering group* of S (that is, G is maximal with respect to being quasi-simple with simple quotient S):

Fix $P \in \text{Syl}_p(G)$. There exists a proper subgroup $M < G$ of G with $N_G(P) \leq M$ such that

- (1) there is a bijection $\Omega : \text{Irr}_{p'}(G) \rightarrow \text{Irr}_{p'}(M)$, such that
- (2) Ω respects central characters, that is, if $\chi \in \text{Irr}_{p'}(G)$ lies above $\nu \in \text{Irr}(Z(G))$, then so does $\Omega(\chi)$ (note that $Z(G) \leq N_G(P) \leq M$);
- (3) Ω is equivariant with respect to $\text{Aut}(G)_P = \{\alpha \in \text{Aut}(G) \mid \alpha(P) = P\}$; and
- (4) the Clifford theories in $\text{Aut}(G)_P$ above χ and $\Omega(\chi)$ agree (for example, the corresponding 2-cocycles are the same).

The last condition is often the most difficult to check, but it is satisfied automatically for example if $\text{Out}(G)$ is cyclic; the latter property holds, for example, for sporadic simple groups, for alternating groups \mathfrak{A}_n with $n \neq 6$, but also for the exceptional groups of Lie type $E_8(q)$.

Let us give some indications on how such a reduction theorem might be arrived at. The first and crucial step is to generalise the desired assertion:

Conjecture 2.13 (Relative McKay Conjecture) *Let G be a finite group, $L \trianglelefteq G$, $P/L \in \text{Syl}_p(G/L)$ and suppose that $\nu \in \text{Irr}(L)$ is P -invariant. Then*

$$|\text{Irr}_{p'}(G|\nu)| = |\text{Irr}_{p'}(N_G(P)|\nu)|.$$

The original McKay conjecture is recovered from this as the special case when $L = 1$, $\nu = 1$. But, despite of seeming to be stronger, the relative version is much more accessible to an inductive approach. Indeed, Wolf [84] showed that Conjecture 2.13 holds for p -solvable groups.

To prove the relative conjecture, let (G, L, ν) be a minimal counterexample with respect to $|G/L|$.

Step 1: We may assume that ν is G -invariant:

Let $L \leq T := G_\nu$ be the stabiliser of ν in G . By assumption $P \leq T$, so $|G : T|$ and $|N : N \cap T|$ are prime to p . Clifford theory now yields bijections $\text{Irr}(T|\nu) \rightarrow \text{Irr}(G|\nu)$ and $\text{Irr}(T \cap N|\nu) \rightarrow \text{Irr}(N|\nu)$ preserving the sets of p' -degrees. Thus, if $T < G$ then G cannot be a minimal counterexample.

Step 2: We may assume $L \leq Z(G)$ is a cyclic p' -group and ν is faithful:

This uses the well-established theory of *character triples*: there exists a triple (G^*, L^*, ν^*) with $L^* \leq Z(G^*)$ cyclic, $\nu^* \in \text{Irr}(L^*)$ faithful and $G/L \cong G^*/L^*$ such that the Clifford theories in G above ν and in G^* above ν^* agree (see e.g. [42, p. 186]).

Step 3: We may assume that G/L has a unique minimal normal subgroup K/L , of order divisible by p :

Let K/L be a minimal normal subgroup of G/L . Then $|G/K| < |G/L|$. Set

$M := N_G(KP)$. If $M < G$ then we may conclude by induction. So we have $M = G$, and hence $KP \trianglelefteq G$. But then G/K is p -solvable. If G/L is p -solvable, then so is G , and we may conclude by the theorem of Wolf. Hence, G/L is not p -solvable but G/K is. From this it easily follows that K/L is the unique minimal normal subgroup.

Step 4: We are done if the simple composition factors of K/L are McKay good for p : This is by far the most difficult part of the argument in [43], and we will not go into it here.

A streamlined version of the arguments for this and more general reductions has been published by Späth [82].

Since the publication of Theorem 2.12 all conjectures introduced above have been shown to reduce to properties of simple groups:

1. the Alperin–McKay Conjecture 2.5 holds if all simple groups are *AMcK good* (Späth [79]);
2. the Alperin Weight Conjecture 2.7 holds if all simple groups are *AWC good* (Navarro–Tiep [68], and Späth [80] for the blockwise version);
3. the “only if” direction of Brauer’s Height Zero Conjecture 2.6 holds if all simple groups are AMcK good and moreover it holds for all quasi-simple groups (Navarro–Späth [67]).

The assertion on quasi-simple groups necessary for BHZ was subsequently shown by Kessar–Malle [48].

Moreover, for blocks B with abelian defect groups, Koshitani and Späth [50] show that being AWC good is implied by being AMcK good, if moreover the p -modular decomposition matrix of G is lower uni-triangular with respect to an $\text{Aut}(G)$ -stable subset of characters. Thus the reductions have also uncovered some remarkably strong connections between the various conjectures.

We will not endeavour to spell out the somewhat technical conditions for being good in the various cases, let us just say that they are similar to the one of being McKay good explained above. See [82], for example. Späth [81] has also succeeded in reducing Dade’s conjecture to a property of simple groups.

So now all of the conjectures have been reduced to questions on finite simple groups, can we solve them? Well, it turns out that our knowledge on the representation theory of quasi-simple groups is not yet well-developed enough to really answer these questions. Roughly speaking, the alternating groups can be treated combinatorially (see also Example 3.7 for an illustration in symmetric groups), extending the aforementioned results of Olsson and Alperin–Fong to accommodate the stronger inductive conditions (see Denoncin [28]), the sporadic simple groups can be treated by ad hoc case-by-case methods (and this has been completed by various authors, except for the Alperin weight conjecture for the very largest sporadic groups, see e.g. An and Dietrich [6] and Breuer [14]). Similarly, the case of the finitely many exceptional covering groups of the simple groups of Lie type have been settled. Thus we are left with the by far biggest class of examples: the 16

infinite families of simple groups of Lie type. Here, the case when p is the defining prime has been shown to hold by Späth for all conjectures [78, 79, 80] building on previous work of Maslowski [60].

For the rest of these lectures we will concentrate on the McKay conjecture for groups of Lie type. First we need to understand the sets $\text{Irr}(G)$ and $\text{Irr}(N_G(P))$, or $\text{Irr}(M)$ for a suitable proper subgroup $N_G(P) \leq M < G$.

2.4 McKay’s conjecture for $\text{GL}_n(q)$

Let’s take a look at the case of $G = \text{GL}_n(q)$, $q = p^f$ a prime power. Here, the ordinary character table was determined by Green [39] in 1955. We need two ingredients. First let

$$B = \left\{ \begin{pmatrix} * & \cdots & * \\ & \ddots & \\ 0 & & * \end{pmatrix} \right\} \leq G$$

be the Borel subgroup of G consisting of upper triangular invertible matrices, and consider the induced character 1_B^G (the permutation character of G on the cosets of B).

Theorem 2.14 (Green (1955)) *The constituents of 1_B^G are in bijection with partitions $\lambda \vdash n$ such that*

$$1_B^G = \sum_{\lambda \vdash n} \chi^\lambda(1) \rho_q^\lambda,$$

where $\rho_q^\lambda \in \text{Irr}(\text{GL}_n(q))$ is the character labelled by λ , and $\chi^\lambda \in \text{Irr}(\mathfrak{S}_n)$ is as in Example 2.2.

Thus, the permutation character of G on B decomposes similarly to the regular character of \mathfrak{S}_n . This result is one reason why $\text{GL}_n(q)$ is sometimes called a “quantisation of \mathfrak{S}_n ”, or “ $\mathfrak{S}_n = \text{GL}_n(1)$ ”. The proof of Theorem 2.14 rests on the fact that the endomorphism algebra $\text{End}_{\mathbb{C}G}(1_B^G)$ is an Iwahori–Hecke algebra $\mathcal{H}(\mathfrak{S}_n, q)$ at the parameter q , which by Tits’ deformation theorem is isomorphic to the complex group algebra $\mathbb{C}\mathfrak{S}_n$ of \mathfrak{S}_n . The constituents ρ_q^λ , $\lambda \vdash n$, of 1_B^G occurring in Theorem 2.14 were later called the *unipotent characters* of G .

Now let $s \in \text{GL}_n(q)$ be a p' -element; then s is diagonalisable over a finite extension of \mathbb{F}_q (it is a *semisimple element* of G). Its characteristic polynomial has the form $\prod_{i=1}^r f_i^{n_i}$ with suitable irreducible polynomials $f_i \in \mathbb{F}_q[X]$ of degrees $d_i = \deg(f_i)$ such that $\sum_i n_i d_i = n$. Then

$$C_{\text{GL}_n(q)}(s) \cong \text{GL}_{n_1}(q^{d_1}) \times \cdots \times \text{GL}_{n_r}(q^{d_r}).$$

Now for partitions $\lambda_i \vdash n_i$, $1 \leq i \leq r$, and $\rho_{q^{d_i}}^{\lambda_i}$ the corresponding unipotent characters of the factors $\text{GL}_{n_i}(q^{d_i})$, we have an irreducible character $\rho_{q^{d_1}}^{\lambda^1} \otimes \cdots \otimes \rho_{q^{d_r}}^{\lambda^r}$ of $C_G(s)$. Let us write \mathcal{S} for the set of all pairs $(s, \underline{\lambda})$, where $s \in \text{GL}_n(q)$ is a semisimple element up to conjugation with characteristic polynomial $\prod f_i^{n_i}$, and $\underline{\lambda} = (\lambda^1, \dots, \lambda^r) \vdash (n_1, \dots, n_r)$ is an r -tuple of partitions.

Theorem 2.15 (Green (1955)) *There is a natural bijection*

$$\mathcal{S} \longrightarrow \text{Irr}(\text{GL}_n(q)), \quad (s, \underline{\lambda}) \mapsto \rho^{s, \underline{\lambda}},$$

such that

$$\rho^{s, \underline{\lambda}}(1) = |\text{GL}_n(q) : C_{\text{GL}_n(q)}(s)|_{p'} \cdot \prod_{i=1}^r \rho_{q^{d_i}}^{\lambda^i}(1).$$

The sets $\mathcal{E}(G, s) := \{\rho^{s, \underline{\lambda}}\} \subseteq \text{Irr}(G)$ are called *Lusztig series*. Observe that by its definition $\mathcal{E}(G, s)$ is in bijection with the set of r -tuples $\{(\lambda^1, \dots, \lambda^r) \vdash (n_1, \dots, n_r)\}$ of partitions, which in turn parametrise the unipotent characters of $C_G(s) = \text{GL}_{n_1}(q^{d_1}) \times \dots \times \text{GL}_{n_r}(q^{d_r})$. This is called the *Jordan decomposition* of the characters in $\text{Irr}(\text{GL}_n(q))$.

Thus, in order to verify for example the McKay conjecture we need to know the unipotent character degrees. These turn out to be given by a quantisation of the hook formula that we already saw for the character degrees of \mathfrak{S}_n in Example 2.2

$$\rho_q^\lambda(1) = q^{a(\lambda)} \frac{[n]_q!}{\prod_h [\ell(h)]_q},$$

where the product runs again over all hooks h of λ . Here, $[m]_q := (q^m - 1)/(q - 1)$ for $m \geq 1$, $[n]_q! := [1]_q \cdots [n]_q$, and $a(\lambda) := \sum_i (i - 1)\lambda_i$ when $\lambda = (\lambda_1 \geq \lambda_2 \geq \dots)$.

There are two cases to discuss for the McKay conjecture: either the relevant prime equals p , or it is different from p , in which case we will call it ℓ . Let us first prove the following:

Proposition 2.16 *We have*

$$\text{Irr}_{p'}(\text{GL}_n(q)) = \{\rho^{s, \underline{\lambda}} \mid \underline{\lambda} = ((n_1), \dots, (n_r)), s \in \text{GL}_n(q) \text{ semisimple}\},$$

so $\text{Irr}_{p'}(\text{GL}_n(q))$ is in bijection with the semisimple conjugacy classes of $\text{GL}_n(q)$.

Proof By the degree formula given above, p does not divide $\rho^{s, \underline{\lambda}}(1)$ if and only if $\sum a(\lambda^i) = 0$, that is, if and only if all partitions λ^i are of the form $\lambda^i = (n_i)$. \square

Let us now consider the local side. Here

$$P = \left\{ \begin{pmatrix} 1 & & * \\ & \ddots & \\ 0 & & 1 \end{pmatrix} \right\} \leq G$$

is a Sylow p -subgroup of G , and $N_G(P) = B = P.T$ with

$$T = \left\{ \begin{pmatrix} * & & 0 \\ & \ddots & \\ 0 & & * \end{pmatrix} \right\} \leq G$$

an abelian subgroup (a so-called *maximally split maximal torus* of G). Thus

$$\text{Irr}_{p'}(N_G(P)) = \text{Irr}_{p'}(B) = \text{Irr}_{p'}(P.T) = \text{Irr}_{p'}(P/P'.T).$$

Maslowski [60] showed that $\text{Irr}_{p'}(P/P'.T)$ can be parametrised by $\mathbb{F}_q^\times \times (\mathbb{F}_q)^{n-1}$, and thus by the set of monic polynomials of degree n over \mathbb{F}_q with non-vanishing constant coefficient, which are exactly the possible characteristic polynomials of semisimple elements in $\text{GL}_n(q)$. By Proposition 2.16 this establishes McKay's conjecture for $\text{GL}_n(q)$ and the prime p . He also showed that the constructed bijection is equivariant with respect to $\text{Aut}_P(G)$ (the relevant outer automorphisms are field automorphisms and the transpose-inverse automorphism when $n > 2$), and also compatible with respect to central characters.

Now what about primes $\ell \neq p$? Here we have the following observation, which is again immediate from the degree formula:

Proposition 2.17 *Let $\ell \neq p$. Then $\rho^{s,\lambda} \in \text{Irr}_{\ell'}(\text{GL}_n(q))$ if and only if s centralises a Sylow ℓ -subgroup of $\text{GL}_n(q)$ and $\rho_{q^{d_i}}^{\lambda^i} \in \text{Irr}_{\ell'}(\text{GL}_{n_i}(q^{d_i}))$ for all $i = 1, \dots, r$.*

That is, in order to prove the McKay conjecture in this case we are reduced to understanding the unipotent characters ρ^{λ^i} , and for this, to determine when ℓ does divide a factor $q^m - 1$. Let Φ_d denote the d th cyclotomic polynomial over \mathbb{Q} , so that $q^m - 1 = \prod_{d|m} \Phi_d(q)$. We also write $d_\ell(q)$ for the order of q in \mathbb{F}_ℓ^\times , that is, its order modulo ℓ . We have the following elementary criterion:

Lemma 2.18 *Let $d = d_\ell(q)$. Then ℓ divides $\Phi_m(q)$ if and only if $m \in \{d, d\ell, d\ell^2, \dots\}$.*

Then the degree formula implies:

Corollary 2.19 *Let $\ell > 2$. For $\lambda \vdash n$ we have $\rho^\lambda \in \text{Irr}_{\ell'}(\text{GL}_n(q))$ if and only if λ has exactly w hooks of length d , where $d = d_\ell(q)$ and $n = wd + r$ with $0 \leq r < d$.*

Let's now turn to the local situation.

Proposition 2.20 *Let $d = d_\ell(q)$ and write $n = wd + r$ with $0 \leq r < d$. The normaliser $N_G(P)$ of a Sylow ℓ -subgroup P of $G := \text{GL}_n(q)$ is contained in*

$$N_G(\text{GL}_1(q^d)^w) = ((\text{GL}_1(q^d).C_d) \wr \mathfrak{S}_w) \times \text{GL}_r(q) \cong (\text{GL}_1(q^d)^w.G(d, 1, w)) \times \text{GL}_r(q),$$

where $G(d, 1, n) = C_d \wr \mathfrak{S}_w$ is an imprimitive complex reflection group.

The structure of a Sylow ℓ -normaliser is quite complicated in general, but by the Reduction Theorem 2.12 we can instead consider the intermediate group $M := (\text{GL}_1(q^d) \wr G(d, 1, w)) \times \text{GL}_r(q)$ which is much closer to being a finite reductive group like $\text{GL}_n(q)$ itself. Now taking together [54] and the result of Späth [77] show:

Theorem 2.21 (Malle, Späth (2010)) *Let $G = \text{GL}_n(q)$ and M as above. There is a bijection*

$$\text{Irr}_{\ell'}(G) \rightarrow \text{Irr}_{\ell'}(M).$$

The proof relies on combinatorial descriptions of the sets on both sides that coincide.

This sketch shows how to find a McKay bijection in the case of $\mathrm{GL}_n(q)$. A very similar statement also holds for the general unitary groups, but the proofs are different and more complicated.

Now the general linear groups are in general not quasi-simple; the right groups to consider are the special linear groups $\mathrm{SL}_n(q)$, where unfortunately the situation is much less transparent. Still, Cabanes and Späth [22] showed how to descend the bijection from Theorem 2.21 to $\mathrm{SL}_n(q)$ and thus show that this group is McKay good for all ℓ .

2.5 Groups of Lie type

We now turn to general groups of Lie type. A finite group G is said to be of Lie type if $G = \mathbf{G}^F$, where \mathbf{G} is a connected reductive group over an algebraic closure of a finite field with a Frobenius map $F : \mathbf{G} \rightarrow \mathbf{G}$ (we refer to [59] for an introduction to the structure theory of these groups). A subgroup \mathbf{H} of \mathbf{G} is said to be F -stable if $F(h) \in \mathbf{H}$ for all $h \in \mathbf{H}$. If $\mathbf{H} \leq \mathbf{G}$ is F -stable, then \mathbf{H}^F is a finite group. A *torus* of \mathbf{G} is a closed connected abelian subgroup of \mathbf{G} consisting of semisimple elements. The group \mathbf{G} acts on the set of tori of \mathbf{G} ; maximal tori form a single \mathbf{G} -orbit. The group \mathbf{G}^F acts on the set of F -stable tori of \mathbf{G} but there is in general more than one \mathbf{G}^F -orbit of F -stable maximal tori of \mathbf{G} and F -fixed point subgroups of tori in different \mathbf{G}^F -classes have different orders. The \mathbf{G}^F -classes of maximal tori can be described using Weyl groups. Fix an F -stable maximal torus \mathbf{T} of \mathbf{G} and set $W(\mathbf{T}) = N_{\mathbf{G}}(\mathbf{T})/\mathbf{T}$, the Weyl group of \mathbf{T} . The F -action on \mathbf{T} induces an F -action on $W(\mathbf{T})$. Elements w and w' of $W(\mathbf{T})$ are said to be F -conjugate if $w' = xwF(x)^{-1}$ for some $x \in W(\mathbf{T})$. The \mathbf{G}^F -conjugacy classes of F -stable maximal tori are in bijection with the F -conjugacy classes of W which in turn can be described combinatorially.

Example 2.22 Let $\mathbf{G} = \mathrm{GL}_n(\overline{\mathbb{F}}_q)$ and $F : \mathbf{G} \rightarrow \mathbf{G}$ be the standard Frobenius map which sends every entry of a matrix in \mathbf{G} to its q -th power. Then $G = \mathbf{G}^F = \mathrm{GL}_n(q)$. Let \mathbf{T} be the subgroup of \mathbf{G} consisting of all diagonal matrices. Then \mathbf{T} is an F -stable maximal torus of \mathbf{G} . It is easy to see that $N_{\mathbf{G}}(\mathbf{T})$ consists of all monomial matrices and $W(\mathbf{T}) \cong \mathfrak{S}_n$. Further, since F fixes every permutation matrix the induced action of F on $W(\mathbf{T})$ is trivial. So, the F -conjugacy classes of $W(\mathbf{T})$ are simply the conjugacy classes of $W(\mathbf{T})$. We obtain a bijection

$$\{\text{partitions of } n\} \xrightarrow{1:1} \{F\text{-stable maximal tori of } \mathbf{G}\}/\mathbf{G}^F.$$

If $\lambda = (\lambda_1, \dots, \lambda_s) \vdash n$ corresponds to the \mathbf{G}^F -class of the maximal torus \mathbf{T}_λ then

$$|\mathbf{T}_\lambda^F| = (q^{\lambda_1} - 1) \cdots (q^{\lambda_s} - 1).$$

In their seminal 1976 paper [27], Deligne and Lusztig showed how to construct ordinary \mathbf{G}^F -representations from the ℓ -adic cohomology spaces of certain algebraic

varieties (now called Deligne–Lusztig varieties) on which \mathbf{G}^F acts. For each F -stable maximal torus \mathbf{T} of \mathbf{G} , they constructed a pair of \mathbb{Z} -linear maps

$$R_{\mathbf{T}}^{\mathbf{G}} : \mathbb{Z} \text{Irr}(\mathbf{T}^F) \rightarrow \mathbb{Z} \text{Irr}(\mathbf{G}^F), \quad {}^*R_{\mathbf{T}}^{\mathbf{G}} : \mathbb{Z} \text{Irr}(\mathbf{G}^F) \rightarrow \mathbb{Z} \text{Irr}(\mathbf{T}^F).$$

The maps $R_{\mathbf{T}}^{\mathbf{G}}$ and ${}^*R_{\mathbf{T}}^{\mathbf{G}}$ are called the *Deligne–Lusztig induction and restriction maps* respectively. These maps are adjoint to each other with respect to the standard scalar product on the space of class functions of \mathbf{G}^F , that is for each $\chi \in \text{Irr}(\mathbf{G}^F)$, $\theta \in \text{Irr}(\mathbf{T}^F)$,

$$\langle \chi, R_{\mathbf{T}}^{\mathbf{G}}(\theta) \rangle = \langle {}^*R_{\mathbf{T}}^{\mathbf{G}}(\chi), \theta \rangle.$$

For an F -stable maximal torus \mathbf{T} of \mathbf{G} and θ an irreducible character of \mathbf{T}^F , let $\mathcal{E}(\mathbf{G}^F | (\mathbf{T}, \theta))$ be the subset of $\chi \in \text{Irr}(\mathbf{G}^F)$ consisting of those χ such that $\langle \chi, R_{\mathbf{T}}^{\mathbf{G}}(\theta) \rangle \neq 0$. The group \mathbf{G}^F acts by conjugation on the set of pairs (\mathbf{T}, θ) where \mathbf{T} is an F -stable maximal torus of \mathbf{G} and θ is an irreducible character of \mathbf{T}^F and this action preserves the sets $\mathcal{E}(\mathbf{G}^F | (\mathbf{T}, \theta))$, that is for all $g \in \mathbf{G}^F$, \mathbf{T} an F -stable maximal torus of \mathbf{G} and $\theta \in \text{Irr}(\mathbf{T}^F)$,

$$\mathcal{E}(\mathbf{G}^F | {}^g(\mathbf{T}, \theta)) = \mathcal{E}(\mathbf{G}^F | (\mathbf{T}, \theta)).$$

The virtual characters $R_{\mathbf{T}}^{\mathbf{G}}(\theta)$ as (\mathbf{T}, θ) runs over all pairs of F -stable maximal tori \mathbf{T} of \mathbf{G} and irreducible characters θ of \mathbf{T}^F “trap” all irreducible characters of \mathbf{G}^F :

Theorem 2.23 (Deligne–Lusztig (1976))

$$\text{Irr}(\mathbf{G}^F) = \bigcup_{(\mathbf{T}, \theta)} \mathcal{E}(\mathbf{G}^F | (\mathbf{T}, \theta))$$

as (\mathbf{T}, θ) runs over the \mathbf{G}^F -conjugacy classes of pairs (\mathbf{T}, θ) where \mathbf{T} is an F -stable maximal torus of \mathbf{G} and θ is an irreducible character of \mathbf{T}^F .

2.6 Characters of groups of Lie type

Let \mathbf{G} be connected reductive with a Frobenius map $F : \mathbf{G} \rightarrow \mathbf{G}$. If \mathbf{G} is simple of simply connected type (like, for example, $\mathbf{G} = \text{SL}_n$), then \mathbf{G}^F is, apart from a few exceptions, a finite quasi-simple group of Lie type. Moreover, all such groups, except for the Ree and Suzuki groups for which a slightly more general setup is needed, are obtained in this way. This turns out to be the right setting to study the character theory of the families of groups of Lie type.

Recall that for $\mathbf{T} \leq \mathbf{G}$ an F -stable maximal torus, and $\theta \in \text{Irr}(\mathbf{T}^F)$ there is an associated virtual Deligne–Lusztig character $R_{\mathbf{T}}^{\mathbf{G}}(\theta)$. As for $\text{GL}_n(q)$ the set of irreducible characters of $G = \mathbf{G}^F$ can be partitioned into Lusztig series, as follows. Define a graph on $\text{Irr}(G)$ by connecting two characters $\chi, \chi' \in \text{Irr}(G)$ if there exists a pair (\mathbf{T}, θ) such that $\langle \chi, R_{\mathbf{T}}^{\mathbf{G}}(\theta) \rangle \neq 0 \neq \langle \chi', R_{\mathbf{T}}^{\mathbf{G}}(\theta) \rangle$. The connected components of this graph are the *Lusztig series in $\text{Irr}(G)$* . This also defines an equivalence relation on the set of pairs (\mathbf{T}, θ) which seems a bit mysterious. Lusztig has shown that the Lusztig series can instead also be parametrised by semisimple classes of a

group G^* closely related to G . It is obtained from the Langlands dual group \mathbf{G}^* of \mathbf{G} as fixed points under a Frobenius map that we will also denote by F . Here, the Langlands dual \mathbf{G}^* has root datum obtained from that of \mathbf{G} by exchanging character group and cocharacter group. For example,

$$\mathrm{GL}_n^* = \mathrm{GL}_n, \quad \mathrm{SL}_n^* = \mathrm{PGL}_n, \quad \mathrm{Sp}_{2n}^* = \mathrm{SO}_{2n+1}, \quad E_8^* = E_8, \dots$$

One usually writes $\mathcal{E}(G, s) \subseteq \mathrm{Irr}(G)$ for the Lusztig series indexed by $s \in G^*$.

Example 2.24 Let $G = \mathrm{GL}_n(q)$, $s \in G^* = \mathrm{GL}_n(q)$ semisimple. Let $\mathbf{T} \leq C_{\mathbf{G}}(s)$ be an F -stable maximal torus. Then $s \in T = \mathbf{T}^F$ corresponds to some $\theta \in \mathrm{Irr}(T)$ under the isomorphisms $T \cong \mathrm{Irr}(T)$ induced by the duality between \mathbf{G} and \mathbf{G}^* . Then

$$\mathcal{E}(G, s) = \bigcup_{T, \theta} \mathcal{E}(\mathbf{G}^F | (\mathbf{T}, \theta)),$$

the union running over all such pairs (T, θ) .

In particular when $s = 1$ then all tori T contain s , and s corresponds to the trivial character 1_T of T , so

$$\mathcal{E}(G, 1) = \bigcup_T \mathcal{E}(\mathbf{G}^F | (\mathbf{T}, 1_T))$$

and these are the *unipotent characters of G* . Lusztig has shown that they are parametrised independently of q by suitable combinatorial data only depending on the complete root datum (the type) of (\mathbf{G}, F) . For example, we had already seen that for $G = \mathrm{GL}_n(q)$, the unipotent characters are parametrised by partitions of n , independently from q . As for $\mathrm{GL}_n(q)$ there is a Jordan decomposition, which we state here only in a special situation, see [51]:

Theorem 2.25 (Lusztig (1984)) *Assume that $s \in G^*$ is such that $C_{\mathbf{G}^*}(s)$ is connected. Then there is a bijection*

$$J_s : \mathcal{E}(G, s) \longrightarrow \mathcal{E}(C_{G^*}(s), 1),$$

with

$$\chi(1) = |G^* : C_{G^*}(s)|_{p'} \cdot J_s(\chi)(1).$$

Lusztig called this bijection the *Jordan decomposition* of irreducible characters. The assumption on s is satisfied for example for all semisimple elements in $\mathrm{GL}_n(q)$, and more generally for all semisimple elements in groups \mathbf{G} with connected centre, like, for example, PGL_n or E_8 .

Example 2.26 Let $s \in G^*$ be such that $C_{\mathbf{G}^*}(s) = \mathbf{T}^*$ is a maximal torus of \mathbf{G}^* . The element s is then called *regular*. Regular semisimple elements are dense in \mathbf{G}^* , so this assumption is satisfied for “most” elements. In this case $|\mathcal{E}(G, s)| = 1$, and $\chi(1) = |G^* : T^*|_{p'}$ for $\{\chi\} = \mathcal{E}(G, s)$.

If $C_{\mathbf{G}^*}(s)$ is disconnected, the situation is considerably more complicated, but still Lusztig obtained an analogue of Jordan decomposition [52].

Example 2.27 Let $G = \mathrm{SL}_2(q)$ with q odd, so $G^* = \mathrm{PGL}_2(q)$. The semisimple elements in G^* are: the trivial element, two classes of elements of order 2 with disconnected centraliser (one lying inside $\mathrm{PSL}_2(q)$, one outside), and all other semisimple elements are regular with centraliser of order either $q - 1$ or $q + 1$. Letting s_1, s_2 denote representatives of the two classes of involutions we thus have

$$\mathrm{Irr}(G) = \mathcal{E}(G, 1) \cup \mathcal{E}(G, s_1) \cup \mathcal{E}(G, s_2) \cup \bigcup_{s: s^2 \neq 1} \mathcal{E}(G, s),$$

where $|\mathcal{E}(G, 1)| = |\mathcal{E}(G, s_i)| = 2$, $|\mathcal{E}(G, s)| = 1$; here s_1, s_2 are representatives of the two classes of involutions.

2.7 Towards McKay's conjecture for groups of Lie type

Again it is straightforward from the Jordan decomposition to classify the characters in $\mathrm{Irr}_{\ell'}(G)$:

Proposition 2.28 *Let $\chi \in \mathcal{E}(G, s)$. Then $\chi \in \mathrm{Irr}_{\ell'}(G)$ if and only if s centralises a Sylow ℓ -subgroup of G^* and moreover $J_s(\chi) \in \mathcal{E}(C_{G^*}(s), 1)$ is contained in $\mathrm{Irr}_{\ell'}(C_{G^*}(s))$.*

So again our question is reduced to studying unipotent characters. Their degrees are given by polynomial expressions in the field size q , as we already saw for $\mathrm{GL}_n(q)$ with the hook formula. It is combinatorially easy to determine the ℓ' -degrees from this for classical types; for exceptional types this is just a finite task.

Now let's turn again to the local picture. We set $d = d_{\ell}(q)$, where we recall that $d_{\ell}(q)$ denotes the order of ℓ modulo q . Assume for simplicity that $\ell \neq 2$. We describe the picture for \mathbf{G} of classical type, that is $G = G_n(q) = \mathrm{Sp}_{2n}(q)$, $\mathrm{SO}_{2n+1}(q)$ or $\mathrm{SO}_{2n}^{\pm}(q)$. First assume that d is odd and write $n = ad + r$ with $0 \leq r < d$. Then there is a torus $T_d = \mathrm{GL}_1(q^d) \times \cdots \times \mathrm{GL}_1(q^d)$ (a factors) of G such that

$$N_G(P) \leq N_G(T_d) = T_d.(C_{2d} \wr \mathfrak{S}_w) \times G_r(q)$$

contains the normaliser of a Sylow ℓ -subgroup P of $G_n(q)$ (see [15, §3A]).

Example 2.29 Assume $d = 1$ and $G = G_n(q) \neq \mathrm{SO}_{2n}^-(q)$. Then $T_d \cong C_{q-1}^n$ is a maximally split torus, with $N_G(T_d) = T_d.W$, with W the Weyl group of G .

If instead $d = 2e$ is even, then the same type of result holds, we just have to replace the cyclic group $\mathrm{GL}_1(q^d) = C_{q^d-1}$ by the cyclic group $\mathrm{GU}_1(q^e) = C_{q^e+1}$.

Theorem 2.30 (Malle, Späth (2010)) *Let \mathbf{G} be simple of simply connected type, $\ell \neq p$ a prime, and $d = d_{\ell}(q)$. Then there exists a bijection $\Omega : \mathrm{Irr}_{\ell'}(G) \rightarrow \mathrm{Irr}_{\ell'}(N_G(T_d))$ with $\Omega(\chi)(1) \equiv \pm\chi(1) \pmod{\ell}$ for all χ , unless one of*

- $\ell = 3$, $G = \mathrm{SL}_3(q)$, $\mathrm{SU}_3(q)$, or $G_2(q)$ with $q \equiv 2, 4, 5, 7 \pmod{9}$, or

- $\ell = 2$, $G = \mathrm{Sp}_{2n}(q)$ with $q \equiv 3, 5 \pmod{8}$.

This bijection can be chosen to preserve central characters.

The proof is obtained by parametrising both sides by the same combinatorial data.

For the listed exceptions $N_G(T_d)$ does not even contain a Sylow ℓ -subgroup; for example when $G = \mathrm{Sp}_2(q) \cong \mathrm{SL}_2(q)$ with $q \equiv 3, 5 \pmod{8}$ a Sylow 2-subgroup is quaternion and thus cannot be contained in $N_G(T_d)$ which is an extension of a cyclic group of order $q \pm 1$ by a group of order 2.

Still the exceptions were shown to be McKay good [56]. Note that the group $N_G(T_d)$ only depends on d , but not on ℓ . Theorem 2.30 also gives the Isaacs–Navarro refinement from Conjecture 2.3.

Now, what's missing for proving McKay goodness? Equivariance and Clifford theory! Recall: for G quasi-simple of Lie type, $\mathrm{Out}(G)$ is made up of diagonal, graph and field automorphisms (see e.g. [59, Thm. 24.24]):

1. diagonal automorphisms are induced e.g. by the embedding of $\mathrm{SL}_n(q)$ into $\mathrm{GL}_n(q)$, or $\mathrm{Sp}_{2n}(q)$ into $\mathrm{CSp}_{2n}(q)$,
2. graph automorphisms come from the Dynkin diagram (e.g., the transpose-inverse automorphism for $\mathrm{SL}_n(q)$, $n \geq 3$, or triality on $D_4(q)$),
3. field automorphisms come from the field \mathbb{F}_q over which G is defined.

Example 2.31 The worst case, in the sense that the structure of the outer automorphism group is most complicated, occurs for $G = \mathrm{Spin}_8^+(q)$ with $q \equiv 1 \pmod{2}$; here $\mathrm{Out}(G) = 2^2.\mathfrak{S}_3.C_f$, where $q = p^f$.

Nice cases (with small outer automorphism group) are, by contrast, $G = E_8(q)$ or $G = \mathrm{Sp}_{2n}(q)$ with q even; here $\mathrm{Out}(G) = C_f$ is cyclic.

Theorem 2.32 (Cabanes–Späth (2013)) *Let S be simple of Lie type such that $\mathrm{Out}(S)$ is cyclic, then the bijection in Theorem 2.30 can be made equivariant. In particular, S is then McKay good.*

In general, we need to solve the following hard problem:

Problem 2.33 For G quasi-simple of Lie type, determine the action of $\mathrm{Aut}(G)$ on $\mathrm{Irr}(G)$.

Partial results are available: Lusztig determined the action of diagonal automorphisms: they leave $\chi \in \mathcal{E}(G, s)$ invariant unless possibly when $C_{\mathbf{G}^*}(s)$ is disconnected.

Also, the action of all automorphisms is known on Lusztig series where $C_{\mathbf{G}^*}(s)$ is connected.

Example 2.34 (Lusztig) Consider the case of unipotent characters $\chi \in \mathcal{E}(G, 1)$. If G is quasi-simple then any automorphism of G fixes all unipotent characters, unless G is of type D_{2n} , or B_2 , F_4 in characteristic 2, or type G_2 in characteristic 3 (see e.g. [56]).

Let $\mathbf{G} \hookrightarrow \tilde{\mathbf{G}}$ be a *regular embedding*, that is $\tilde{\mathbf{G}}$ is a connected reductive group with a connected center and the same derived subgroup as \mathbf{G} . For example, the inclusion of SL_n in GL_n is a regular embedding. We assume that the Frobenius endomorphism F extends to a Frobenius morphism, also denoted F , of $\tilde{\mathbf{G}}$. The action of $\tilde{G} = \tilde{\mathbf{G}}^F$ on G is by inner-diagonal automorphisms. Denote by D the group of graph and field automorphisms of \tilde{G} . Then Späth [78] showed the following:

Theorem 2.35 (Criterion of Späth) *Assume there is an $\mathrm{Aut}(G)_P$ -equivariant bijection $\tilde{\Omega} : \mathrm{Irr}_{\ell'}(\tilde{G}) \rightarrow \mathrm{Irr}_{\ell'}(\tilde{M})$ compatible with multiplication by $\mathrm{Irr}(\tilde{G}/G)$. If*

- for every $\tilde{\chi} \in \mathrm{Irr}_{\ell'}(\tilde{G})$ there is $\chi \in \mathrm{Irr}_{\ell'}(G|\tilde{\chi})$ with

$$(\tilde{G} \rtimes D)_{\tilde{\chi}} = \tilde{G}_{\tilde{\chi}} \rtimes D_{\tilde{\chi}}$$

and χ extends to $(G \rtimes D)_{\chi}$, and

- the analogous condition holds on the local side,

then $G/Z(G)$ is McKay good for ℓ .

So, for $G = \mathrm{SL}_n(q)$, for example, one uses the bijection for $\tilde{G} = \mathrm{GL}_n(q)$, then has to check the stabiliser condition and finally prove extendibility. This leads to the following situation at the time of writing: S simple group is McKay good for all primes, unless possibly when S is of type $B_n(q)$, ${}^{(2)}D_n(q)$, ${}^{(2)}E_6(q)$ or $E_7(q)$.

There is one prime for which more can be said: $\ell = 2$.

Theorem 2.36 (Malle–Späth (2016)) *Let G be quasi-simple of Lie type, not of type A , and $\chi \in \mathrm{Irr}_{2'}(G)$. Then there exists a linear character $\theta \in \mathrm{Irr}(B)$ where $B \leq G$ is a Borel subgroup, such that χ is a constituent of $\mathrm{Ind}_B^G(\theta)$, unless some cases when $G = \mathrm{Sp}_{2n}(q)$ with $q \equiv 3 \pmod{4}$.*

The proof, which is not too hard, uses Lusztig’s Jordan decomposition and the degree formulas.

But now $B = U.T$, with $T \leq G$ a maximal torus, and $U \leq \ker(\theta)$, so in fact $\theta \in \mathrm{Irr}(T)$, and

$$\mathrm{Ind}_B^G(\theta) = \mathrm{Ind}_B^G(\mathrm{Infl}_T^B(\theta)) = R_{\mathbf{T}}^{\mathbf{G}}(\theta).$$

To check the criterion, we need to know the action of $\mathrm{Aut}(G)$ on the constituents of $R_{\mathbf{T}}^{\mathbf{G}}(\theta)$ with $T \leq B$. But the decomposition of $R_{\mathbf{T}}^{\mathbf{G}}(\theta)$ is controlled by the Iwahori–Hecke algebra of the relative Weyl group $W(\theta) = N_G(T, \theta)/T$:

$$\mathrm{End}_{\mathbb{C}G}(R_{\mathbf{T}}^{\mathbf{G}}(\theta)) = \mathcal{H}(W(\theta), q) \cong \mathbb{C}W(\theta).$$

(As seen in the case of $\mathrm{GL}_n(q)$ above). This does allow us to compute the action of $\mathrm{Aut}(G)$ on $\mathrm{Irr}_{2'}(G)$; more considerations are needed on the local side, and extendibility has to be guaranteed.

Theorem 2.37 (Malle–Späth (2016)) *The McKay Conjecture 2.1 holds for the prime $p = 2$.*

In order to generalise this to other primes, one needs to work with more general Levi subgroups: Let $\mathbf{P} \leq \mathbf{G}$ be an F -stable parabolic subgroup, and $\mathbf{L} \leq \mathbf{P}$ an F -stable Levi complement, with finite groups of fixed points $L = \mathbf{L}^F \leq P = \mathbf{P}^F \leq G$. Then there is the functor of Harish-Chandra induction

$$R_{\mathbf{L}}^{\mathbf{G}} : \mathbb{C}L\text{-mod} \rightarrow \mathbb{C}G\text{-mod}, \quad M \mapsto \text{Ind}_P^G \text{Infl}_L^P(M).$$

The special case $L = T \leq P = B$ was considered above.

Now $\lambda \in \text{Irr}(L)$ is called *cuspidal* if it does not occur as constituent of $R_M^{\mathbf{L}}(\mu)$ for any proper Levi subgroup $M < L$, $\mu \in \text{Irr}(M)$. Again the decomposition of $R_{\mathbf{L}}^{\mathbf{G}}(\lambda)$, with λ cuspidal, is controlled by the Iwahori–Hecke algebra of a relative Weyl group (Howlett–Lehrer [40]). To apply the above argument, one needs to understand the action of automorphisms on cuspidal characters.

Theorem 2.38 (Malle (2017)) *Let G be quasi-simple of Lie type. Then the action of $\text{Aut}(G)$ on the cuspidal characters of G lying in quasi-isolated series is known.*

This does, however, not yet solve the extension problem, and moreover the local situation also needs to be studied.

3 Blocks and characters of finite simple groups

As seen in the discussion around the McKay conjecture, in order to make a success of the reduction strategy for the local-global conjectures one needs very detailed knowledge both of the character theory of finite simple groups as well as of their p -local structure. For the block-wise versions of these conjectures we require this information at a yet finer level. A first step would be to obtain workable descriptions of block partitions of irreducible characters and the corresponding defect groups. The block distribution problem for finite (quasi and almost) simple groups falls naturally into four cases:

- sporadic groups
- alternating groups
- finite groups of Lie type in describing characteristic
- finite groups of Lie type in non-describing characteristic

Of these the most difficult case is the last. We will discuss this case at some length. Block distributions in sporadic groups can be worked out through the ATLAS character tables. The third case, namely the blocks of finite groups of Lie type in defining characteristic is in some sense the easiest as there are very few blocks (see Example 3.8). In Example 3.7 we give a flavour of the first case by describing the block distribution for finite symmetric groups.

3.1 Local Block Theory

In order to get started we need to recall some foundational results from local block theory. As in Section 2.2 let $\mathcal{O} \geq \mathbb{Z}_p$ be a large enough extension. Let $k := \mathcal{O}/\mathfrak{P}$

be the residue field of \mathcal{O} and $\bar{\cdot} : \mathcal{O} \rightarrow k, \alpha \mapsto \bar{\alpha} := \alpha + \mathfrak{P}$, the natural epimorphism. The block decomposition

$$\mathcal{O}G = B_1 \oplus \dots \oplus B_r$$

induces the unique decomposition

$$kG = \bar{B}_1 \oplus \dots \oplus \bar{B}_r$$

into a direct sum of minimal two-sided ideals of kG where for any element $a = \sum_{g \in G} \alpha_g g$ of $\mathcal{O}G$ we denote $\bar{a} := \sum_{g \in G} \bar{\alpha}_g g$. These decompositions correspond to unique decompositions

$$1_{\mathcal{O}G} = e_{B_1} + \dots + e_{B_r},$$

$$1_{kG} = e_{\bar{B}_1} + \dots + e_{\bar{B}_r},$$

of $1_{\mathcal{O}G}$ and of 1_{kG} into a sum of central primitive idempotents, called *block idempotents* of kG . The expression for $e_{\bar{B}}$ is obtained by reducing coefficients modulo p . Thus we have bijections $B_i \leftrightarrow \bar{B}_i \leftrightarrow e_{\bar{B}_i}$ between the sets of blocks of $\mathcal{O}G$, blocks of kG and block idempotents of kG . By a defect group of \bar{B}_i , e_{B_i} or $e_{\bar{B}_i}$ we mean a defect group of B_i . Similarly we may denote $\text{Irr}(B_i)$ by $\text{Irr}(\bar{B}_i)$, $\text{Irr}(e_{B_i})$ or $\text{Irr}(e_{\bar{B}_i})$.

Block idempotents can be read off the character table of G . If B is a block of G , then

$$e_B = \sum_{\chi \in \text{Irr}(B)} \frac{\chi(1)}{|G|} \sum_{x \in G_{p'}} \chi(x)x^{-1},$$

is the sum of central idempotents of KG corresponding to the elements of $\text{Irr}(B)$.

Example 3.1 For G a finite group, $O^p(G)$ denotes the smallest normal subgroup of G with quotient a p -group and $O_p(G)$ denotes the largest normal p -subgroup of G .

(a) For any block B of G , $e_B \in \mathcal{O}O^p(G)$.

(b) For any block B of G and any normal subgroup N of G , $e_B \in \mathcal{O}C_G(N)$. In particular, if $C_G(O_p(G)) \leq O_p(G)$, then the principal block is the unique block of G .

(c) If $G = G_1 \times G_2$ is a direct product then the block idempotents of $\mathcal{O}G$ are of the form $e_1 e_2$ where e_i is a block idempotent of $\mathcal{O}G_i$, $i = 1, 2$.

Let Q be a p -subgroup of G . For an element $a = \sum_{g \in G} \alpha_g g$ of kG set

$$\text{Br}_Q(a) = \sum_{g \in C_G(Q)} \alpha_g g \in kC_G(Q).$$

The Brauer map

$$\text{Br}_Q : kG \rightarrow kC_G(Q), \quad a \mapsto \text{Br}_Q(a),$$

restricts to a multiplicative map on $Z(kG)$.

Theorem 3.2 (Brauer's first main theorem) *Let D be a p -subgroup of G . The map*

$$e \mapsto \text{Br}_D(e)$$

induces a bijection between block idempotents of kG with defect group D and block idempotents of $kN_G(D)$ with defect group D . If B is a p -block of G with defect group D , then $\text{Br}_D(e_{\bar{B}})$ is the block idempotent of the Brauer correspondent of B in $kN_G(D)$.

A G -Brauer pair (also known as *subpair*) is a pair (Q, e) where $Q \leq G$ is a p -subgroup of G and e is a block idempotent of $kC_G(Q)$. We denote by $\mathcal{P}(G)$ the set of G -Brauer pairs and for a block B of G we denote by $\mathcal{P}(B)$ the subset of $\mathcal{P}(G)$ consisting of Brauer pairs (Q, e) such that $\text{Br}_Q(e_{\bar{B}})e \neq 0$, the elements of $\mathcal{P}(B)$ are called B -Brauer pairs. It is easily seen that there is a partition

$$\mathcal{P}(G) = \mathcal{P}(B_1) \sqcup \dots \sqcup \mathcal{P}(B_r)$$

where B_1, \dots, B_r are the blocks of G .

We would like to relate the block decomposition of $\text{Irr}(G)$ with the block decomposition of $\mathcal{P}(G)$. Brauer's second main theorem gives us a way of doing this. For x a p -element of G and $\chi \in \text{Irr}(G)$, we let $d^x \chi : C_G(x) \rightarrow \mathcal{O}$ be the function defined by

$$d^x \chi(y) = \begin{cases} \chi(xy) & \text{if } y \in G_{p'}, \\ 0 & \text{if } y \notin G_{p'}. \end{cases}$$

Theorem 3.3 (Brauer's second main theorem) *Let B be a block of G , $x \in G$ a p -element and C a block of $C_G(x)$. Suppose that there exists $\chi \in \text{Irr}(B)$, $\psi \in \text{Irr}(C)$ such that*

$$\langle d^x \chi, \psi \rangle := \sum_{y \in C_G(x)_{p'}} \chi(xy) \psi(y) \neq 0.$$

Then $(\langle x \rangle, e_C) \in \mathcal{P}(B)$. In particular, if $d^x \chi \neq 0$, then $\mathcal{P}(B)$ contains an element of the form $(\langle x \rangle, e)$.

The assignment $\chi \mapsto d^x \chi$ extends by linearity to a map d^x from the set of \mathcal{O} -valued class functions on G to the set of \mathcal{O} -valued class functions on $C_G(x)$. The map d^x is called the *generalised decomposition map* with respect to x . When we want to emphasise the underlying group G , the generalised decomposition map is denoted $d^{x,G}$.

The set $\mathcal{P}(B)$ has a nice description when B is the principal block.

Theorem 3.4 (Brauer's third main theorem) *Let B_0 be the principal block of G and let $(Q, e) \in \mathcal{P}(G)$. Then $(Q, e) \in \mathcal{P}(B_0)$ if and only if e is the idempotent of the principal block of $C_G(Q)$.*

The set $\mathcal{P}(G)$ is a G -set via

$${}^x(Q, e) = ({}^x Q, {}^x e), \quad \text{for all } x \in G, (Q, e) \in \mathcal{P}(G)$$

where ${}^x a := xax^{-1}$ for $x \in G$, $a \in kG$. The subset $\mathcal{P}(B)$ is G -invariant for B a block of G . In [3] Alperin and Broué endowed $\mathcal{P}(G)$ and $\mathcal{P}(B)$ with a G -poset structure. Let $(Q, e), (R, f) \in \mathcal{P}(G)$. We say that (Q, e) is *normal in* (R, f) and write $(Q, e) \trianglelefteq (R, f)$ if $Q \trianglelefteq R$, ${}^x(Q, e) = (Q, e)$ for all $x \in R$ and $\text{Br}_R(e)f \neq 0$. We say that $(Q, e) \leq (R, f)$ if there exists a chain of normal inclusions

$$(Q, e) =: (Q_0, e_0) \trianglelefteq \dots \trianglelefteq (Q_n, e_n) := (R, f)$$

in $\mathcal{P}(G)$ starting at (Q, e) and ending at (R, f) .

Theorem 3.5 (Alperin–Broué (1979)) *$(\mathcal{P}(G), \leq)$ is a G -poset. For any $(R, f) \in \mathcal{P}(G)$ and any $Q \leq R$, there exists a unique block e of $kC_G(Q)$ such that $(Q, e) \leq (R, f)$. The sets $\mathcal{P}(B)$ as B runs over the blocks of G are the connected components of $(\mathcal{P}(G), \leq)$. For a block B of G ,*

- (a) $\mathcal{P}(B)$ is G -invariant and $(1, e_{\bar{B}})$ is the unique minimal element of $\mathcal{P}(B)$.
- (b) G acts transitively on the set of maximal elements of $\mathcal{P}(B)$ and an element (D, d) of $\mathcal{P}(B)$ is maximal if and only if D is a defect group of B .

The following is sometimes known as Brauer’s extended first main theorem.

Theorem 3.6 (Recognition of maximal Brauer pairs) *Let $(Q, e) \in \mathcal{P}(G)$. Then (Q, e) is maximal if and only if there exists $\theta \in \text{Irr}(e)$ such that*

- $Z(Q) \leq \ker(\theta)$,
- as a character of $C_G(Q)/Z(Q)$, θ is of defect 0, and
- $N_G(Q, e)/QC_G(Q)$ is a p' -group.

Example 3.7 As discussed in Example 2.4, an irreducible character χ^λ of \mathfrak{S}_n lies in a p -block of defect zero if and only if λ is a p -core. By the Nakayama conjecture, posed in [64] and proved by Brauer and Robinson [13], given partitions λ, λ' of n , the corresponding characters χ^λ and $\chi^{\lambda'}$ lie in the same p -block of \mathfrak{S}_n if and only if λ and λ' have the same p -core.

Puig [73] showed how the block distribution of irreducible characters matches up with the block distribution of $\mathcal{P}(\mathfrak{S}_n)$. Let Q be a p -subgroup of \mathfrak{S}_n . Then $n = m + pw$, where Q fixes m points and moves pw points in the natural permutation representation of \mathfrak{S}_n , and

$$C_{\mathfrak{S}_n}(Q) = \mathfrak{S}_m \times C_{\mathfrak{S}_{pw}}(Q), \quad N_{\mathfrak{S}_n}(Q) = \mathfrak{S}_m \times N_{\mathfrak{S}_{pw}}(Q).$$

The action of $Q \leq \mathfrak{S}_{pw}$ is fixed-point free and it is not hard to show that this implies that

$$C_{C_{\mathfrak{S}_{pw}}(Q)}(O_p(C_{\mathfrak{S}_{pw}}(Q))) \leq O_p(C_{\mathfrak{S}_{pw}}(Q)).$$

By Example 3.1(b), the principal block is the unique block of $C_{\mathfrak{S}_{pw}}(Q)$. Thus by Example 3.1(c) every Brauer pair with first component Q is of the form ef where e is a block idempotent of $k\mathfrak{S}_m$ and f is the principal block idempotent (in fact the identity element) of $kC_{\mathfrak{S}_{pw}}(Q)$.

Next, we describe the inclusion of Brauer pairs. This is a difficult and subtle step and is carried out inductively — a crucial ingredient is the Murnaghan–Nakayama rule which is an inductive combinatorial rule for calculating values of irreducible characters of symmetric groups. Let Q' be a p -subgroup of \mathfrak{S}_n containing Q and suppose that Q' moves pw' points and fixes m' points. Since $Q \leq Q'$, $w' \geq w$ and $m' \leq m$. Let $e'f'$ be a block of $C_{\mathfrak{S}_n}(Q')$ with e' a block of $\mathfrak{S}_{m'}$ and f' the principal block idempotent of $kC_{\mathfrak{S}_{pw'}}(Q')$. Then one can show that $(Q, ef) \leq (Q, e'f')$ if and only if λ is obtained from μ by adding a sequence of p -hooks for $\chi^\lambda \in \text{Irr}(e)$, $\chi^\mu \in \text{Irr}(e')$.

Finally, we describe the maximal pairs. Applying Theorem 3.6 in both directions, one sees that (Q, ef) is a maximal \mathfrak{S}_n -Brauer pair if and only if $(1, e)$ is a maximal Brauer pair for \mathfrak{S}_m and (Q, f) is a maximal \mathfrak{S}_{pw} -Brauer pair. By the inclusion rule described above, $(1, e)$ is a maximal Brauer pair for \mathfrak{S}_m if and only if e is the block idempotent of a block of \mathfrak{S}_m of defect zero, that is, a block whose unique irreducible character is of the form χ^λ where λ is a p -core. Since f is a principal block idempotent, by Brauer's third main theorem (Q, f) is a maximal \mathfrak{S}_{pw} -Brauer pair if and only if Q is a Sylow p -subgroup of \mathfrak{S}_{pw} . Thus, the G -conjugacy classes of maximal \mathfrak{S}_n -Brauer pairs are in bijection with pairs (μ, w) where w is a non-negative integer such that $pw \leq n$ and μ is a partition of $n - pw$ which is a p -core. By Theorem 3.5 the G -conjugacy classes of maximal G -Brauer pairs are in one-to-one correspondence with the blocks of G . Hence we obtain a bijection between the set of blocks of \mathfrak{S}_n and pairs as above; the character $\chi^\lambda \in \text{Irr}(\mathfrak{S}_n)$ lies in the block indexed by the pair (μ, w) if and only if μ is the p -core of λ . If a block B is indexed by the pair (μ, w) , then w is called the *weight of B* and μ is called the *core of B* .

If B has weight w and core μ , then a Sylow p -subgroup P of \mathfrak{S}_{pw} is a defect group of B and the Brauer correspondent of B in

$$N_{\mathfrak{S}_n}(P) = \mathfrak{S}_m \times N_{\mathfrak{S}_{pw}}(P) \quad \text{where } \mu \vdash m$$

has the form

$$B_\mu B_w$$

where B_μ is the block of \mathfrak{S}_m indexed by the pair $(\mu, 0)$ and B_w is the principal block of $N_{\mathfrak{S}_{pw}}(P)$. The irreducible characters in the Brauer correspondent are of the form

$$\chi^\mu \cdot \eta, \quad \text{where } \eta \in \text{Irr}(C_w).$$

From this, it is easy to check that the map $\chi^\mu \cdot \eta \mapsto \eta$ is a height preserving bijection between the set of irreducible characters of the Brauer correspondent of B and the set of irreducible characters of C_w . Thus we obtain:

- (I) Given any non-negative integer w , there is a height preserving bijection between the irreducible characters of a Brauer correspondent of a weight w -block of a symmetric group and the principal block of $N_{\mathfrak{S}_{pw}}(P)$, where $P \leq \mathfrak{S}_{pw}$ is a Sylow p -subgroup. In particular, there is a height preserving bijection between the irreducible characters of the Brauer correspondents of any two weight w blocks of (possibly different) symmetric groups.

In [31] Enguehard showed that the global analogue of the above statement also holds, namely:

- (II) Given any non-negative integer w , there is a height preserving bijection between the irreducible characters of any two weight w blocks of (possibly different) symmetric groups.

Thus the problem of checking a desired local-global statement for blocks of symmetric groups can often be reduced to checking it for a single block of any given weight w .

Let us consider Brauer's height zero conjecture (Conjecture 2.6) for $p = 2$. Since blocks with the same weight have isomorphic defect groups, in order to prove the height zero conjecture for blocks of symmetric groups it suffices to prove that it holds for the principal block B of \mathfrak{S}_{2w} . The defect groups of B are the Sylow 2-subgroups of S_{2w} . Hence B has abelian defect groups if and only if $w = 1$. On the other hand,

$$\text{Irr}(B) = \{\chi^\lambda \mid \lambda \vdash 2w \text{ and } \lambda \text{ has empty 2-core}\}.$$

Since B is the principal block, $\text{Irr}_0(B) = \text{Irr}(B) \cap \text{Irr}_{2'}(\mathfrak{S}_{2w})$. Thus we are reduced to checking the following statement:

$w \geq 2$ if and only if there is $\lambda \vdash n$ such that λ has empty 2-core and 2 divides $\chi^\lambda(1)$.

The backward implication is immediate as the only partitions of 2 are (2) and (1, 1). Now suppose that $w \geq 2$. Then $\lambda = (2w - 1, 1)$ has empty 2-core — we first remove successively $w - 1$ horizontal hooks of length 2 from the first part of the Young diagram, then remove the remaining vertical 2-hook. The hook length formula (see Example 2.2) easily yields that χ^λ has even degree.

The local-local and global-global bijections described in (I) and (II) above are shadows of deeper categorical equivalences. It is quite easy to deduce from the above discussion that any two Brauer correspondents of blocks of symmetric groups with the same weight are Morita equivalent. Much harder is the analogous global version proven by Chuang and Rouquier [24]: any two p -blocks of symmetric groups of the same weight are derived equivalent.

Example 3.8 Groups of Lie type in characteristic p have very few p -blocks. The main structural reason for this is the Borel–Tits theorem. As in Section 2.5, let \mathbf{G} be a simple algebraic group over $\overline{\mathbb{F}}_p$ with a Frobenius endomorphism $F : \mathbf{G} \rightarrow \mathbf{G}$ and let $G = \mathbf{G}^F$. Suppose that \mathbf{G} is simply connected and $Z(G) = 1$. Then the Borel–Tits theorem [59, Thm. 26.5] implies that if Q is a non-trivial p -subgroup of G , then

$$C_{N_G(Q)}(O_p(N_G(Q))) \leq O_p(N_G(Q)).$$

By Example 3.1(b) applied to $N_G(Q)$, the principal block is the unique block of $N_G(Q)$. Put another way, the identity element is the unique central idempotent of $kN_G(Q)$. Now suppose that f is a block idempotent of $kC_G(Q)$. Then it is easy to see that the sum f' of the distinct $N_G(Q)$ -conjugates of f is a central idempotent

of $kN_G(Q)$. Hence f' is the identity element of $kN_G(Q)$. From this it follows that $f = f'$ is the identity element of $kC_G(Q)$, and consequently f is the principal block of $kC_G(Q)$. In other words, the only G -Brauer pair with first component Q is the pair (Q, f) , where f is the principal block idempotent of $kC_G(Q)$. Now Brauer's third main theorem (Theorem 3.4) gives that if $(Q, e) \in \mathcal{P}(G)$ with $Q \neq 1$, then $(Q, e) \in \mathcal{P}(B_0)$, where B_0 is the principal block of G . We conclude that the irreducible characters χ of G lying outside the principal block are all of defect zero. It turns out that there is only one character of defect zero, namely the Steinberg character [41, Thm. 8.3]. Thus G has precisely two blocks: the principal block and the block containing the Steinberg character.

If the assumption that $Z(G) = 1$ is dropped, then we obtain more blocks, but the extra blocks are in bijection with the non-trivial elements of $Z(G)$. More precisely, we have the following [41, Thm. 8.3]: Suppose that \mathbf{G} is simple and simply connected. The blocks of non-zero defect of G are in bijection with the elements of $Z(G)$ and all have the Sylow p -subgroups of G as defect groups. There is exactly one block of defect zero, namely the block containing the Steinberg character of G .

3.2 Blocks of groups of Lie type in non-defining characteristic.

We continue in the setting and notation of Section 2.5, so \mathbf{G} is a connected reductive group over $\overline{\mathbb{F}}_p$ with a Frobenius endomorphism $F : \mathbf{G} \rightarrow \mathbf{G}$ and $G = \mathbf{G}^F$. Let ℓ be a prime different from p . Our aim is to give a broad idea of how the ℓ -block partition of $\text{Irr}(G)$ can be described in terms of Lusztig's parametrisation of $\text{Irr}(\mathbf{G}^F)$. For notational simplicity for any F -stable subgroup \mathbf{H} of \mathbf{G} or of the dual group \mathbf{G}^* , we will denote by H the F -fixed point subgroup \mathbf{H}^F . For a character χ of G and $x \in G$ an ℓ -element denote by $d^{x,G}\chi : C_G(x) \rightarrow \mathcal{O}$ the function $d^x\chi$ as defined for Brauer's second main theorem (see Section 3.1).

A key starting point is the following result which relates generalised decomposition maps to Deligne–Lusztig induction and restriction. Let \mathbf{T} be an F -stable maximal torus of \mathbf{G} , $T := \mathbf{T}^F$, and let $x \in T_\ell$. Set $\mathbf{H} := C_{\mathbf{G}}^\circ(x)$, the connected component of the centraliser of x in \mathbf{G} . The group \mathbf{H} is again a connected reductive group which is F -stable and $H := \mathbf{H}^F$ is a normal subgroup of $C_G(x)$ which may be proper (equality holds for example if $C_{\mathbf{G}}(x)$ is itself connected). However, since x is an ℓ -element, the general structure theory of connected reductive groups gives that the index of H in $C_G(x)$ is a power of ℓ . By definition, for any character χ of G , $d^{x,G}\chi$ is a function which takes zero values on ℓ -singular elements. Since H contains all ℓ -regular elements of $C_G(x)$, we may regard $d^{x,G}\chi$ as a function from H to \mathcal{O} . Note that \mathbf{T} is a maximal torus of \mathbf{H} so $R_{\mathbf{T}}^{\mathbf{H}}$ and $*R_{\mathbf{T}}^{\mathbf{H}}$ are defined.

Theorem 3.9

$$*R_{\mathbf{T}}^{\mathbf{H}}(d^{x,G}\chi) = d^{x,T}(*R_{\mathbf{T}}^{\mathbf{G}}(\chi)) \quad \text{for all } \chi \in \text{Irr}(G),$$

that is, Deligne–Lusztig restriction commutes with generalised decomposition maps.

For an F -stable maximal torus \mathbf{T} of \mathbf{G} let $\text{Irr}(T)_{\ell'}$ denote the subset of irreducible characters of T of ℓ' -order, that is $\text{Irr}(T)_{\ell'}$ consists of those irreducible characters θ such that $T_{\ell} \leq \ker(\theta)$. Let $\mathcal{E}(G, \ell')$ denote the subset of $\text{Irr}(G)$ consisting of those χ such that $\langle R_{\mathbf{T}}^{\mathbf{G}}(\theta), \chi \rangle \neq 0$ for some F -stable maximal torus \mathbf{T} of \mathbf{G} and some $\theta \in \text{Irr}(T)_{\ell'}$. The following theorem illustrates how Brauer's local block theory and the theory of Deligne–Lusztig characters come together.

Theorem 3.10 *Let \mathbf{T} be an F -stable maximal torus of \mathbf{G} , $\theta \in \text{Irr}(T)_{\ell'}$. Suppose that*

$$C_{\mathbf{G}}(T_{\ell}) = \mathbf{T}. \quad (*)$$

Then all elements of $\mathcal{E}(G|(\mathbf{T}, \theta))$ lie in the same ℓ -block B of G . Further, if e is the block idempotent of $kT = kC_{\mathbf{G}}(T_{\ell})$ containing θ , then (T_{ℓ}, e) is a B -Brauer pair.

The proof of the above theorem goes the following way (details may be found in [46, Props. 2.12, 2.13, 2.16]). By general structure theory the group $C_{\mathbf{G}}(Q)$ is a reductive algebraic group for any $Q \leq T_{\ell}$. For simplicity we will assume that $C_{\mathbf{G}}(Q)$ is also connected.

Step 1: Let $\chi \in \mathcal{E}(G|(\mathbf{T}, \theta))$. By the adjointness of Deligne–Lusztig induction and restriction, θ is a constituent of the virtual character $*R_{\mathbf{T}}^{\mathbf{G}}(\chi)$. Another key property of these maps is that since χ is a constituent of $R_{\mathbf{T}}^{\mathbf{G}}(\theta)$ and $\theta \in \text{Irr}(T)_{\ell'}$, all irreducible constituents of $*R_{\mathbf{T}}^{\mathbf{G}}(\chi)$ belong to $\text{Irr}(T)_{\ell'}$. Write

$$*R_{\mathbf{T}}^{\mathbf{G}}(\chi) = a_{\theta}\theta + \sum_{\tau \in \text{Irr}(T)_{\ell'} \setminus \{\theta\}} a_{\tau}\tau, \quad \text{with } a_{\theta}, a_{\tau} \in \mathbb{Z}, \text{ and } a_{\theta} \neq 0.$$

Let $x \in T_{\ell}$. Since T is an abelian group, it follows easily from the definition of generalised decomposition maps that if $\theta_1, \theta_2 \in \text{Irr}(T)_{\ell'}$ then

$$\langle d^{x, T}\theta_1, \theta_2 \rangle = \frac{1}{|T_{\ell}|} \langle \theta_1, \theta_2 \rangle.$$

Applying this to the above expression for $*R_{\mathbf{T}}^{\mathbf{G}}(\chi)$ gives

$$\langle d^{x, T}(*R_{\mathbf{T}}^{\mathbf{G}}(\chi)), \theta \rangle = \frac{1}{|T_{\ell}|} a_{\theta} \neq 0.$$

By the commutation property in Theorem 3.9,

$$\langle d^{x, T}(*R_{\mathbf{T}}^{\mathbf{G}}(\chi)), \theta \rangle = \langle *R_{\mathbf{T}}^{\mathbf{H}}(d^{x, G}\chi), \theta \rangle = \langle d^{x, G}\chi, R_{\mathbf{T}}^{\mathbf{H}}(\theta) \rangle$$

where $\mathbf{H} = C_{\mathbf{G}}(x)$ and where the second equality holds by adjointness. Now Brauer's Second Main Theorem 3.3 implies that there exists a B -Brauer pair $(\langle x \rangle, f)$ such that

$$\mathcal{E}(C_G(x)|(\mathbf{T}, \theta)) \cap \text{Irr}(f) \neq 0.$$

Step 2: Let $\{x_1, \dots, x_m\}$ be a generating set of T_ℓ and let $Q_i = \langle x_1, \dots, x_i \rangle$, $1 \leq i \leq m$. Applying Step 1 repeatedly with G replaced by $C_G(Q_i)$, we obtain a sequence of inclusions of B -Brauer pairs

$$(Q_1, f_1) \leq \dots \leq (Q_m, f_m)$$

such that

$$\mathcal{E}(C_G(Q_i)|(T, \theta)) \cap \text{Irr}(f_i) \neq 0 \quad \text{for all } i.$$

Since $Q_m = T_\ell$, the hypothesis (*) implies that $C_G(Q_m) = T$. Since θ is in $\text{Irr}(e)$ as well as in $\text{Irr}(f_m)$, $f_m = e$, and the uniqueness of inclusion of Brauer pairs allows us to conclude.

Example 3.11 (a) Let $\mathbf{G} = \text{GL}_n$, $G = \text{GL}_n(q)$. If ℓ divides $q - 1$, then the F -stable maximal torus \mathbf{T} of diagonal matrices of \mathbf{G} satisfies the hypothesis (*) of Theorem 3.10. Thus, for any $\theta \in \text{Irr}(T)_{\ell'}$, all constituents of $R_{\mathbf{T}}^{\mathbf{G}}(\theta)$ lie in the same ℓ -block. By contrast, if ℓ does not divide $q - 1$, then \mathbf{T}^F has trivial ℓ -part.

(b) Suppose that \mathbf{G} is simple of classical type A , B , C , or D and $\ell = 2$. For any F -stable maximal torus \mathbf{T} of \mathbf{G} , all elements of $\mathcal{E}(G|(T, 1))$ lie in the principal 2-block of G [18]. The key property is that T_2 is non-trivial for all F -stable maximal tori \mathbf{T} (of all F -stable Levi subgroups) of \mathbf{G} . One applies Step (1) of the proof of Theorem 3.10 to some non-trivial x in T_2 and then proceeds by induction on the dimension (as algebraic group) of $C_{\mathbf{G}}(x)$.

The hypothesis (*) of Theorem 3.10 does not hold often enough to obtain satisfactory control of block distribution of characters. The strategy to get around this is to replace F -stable maximal tori by a certain class of well behaved F -stable Levi subgroups.

Let $\mathbf{P} \leq \mathbf{G}$ be a parabolic subgroup and let $\mathbf{L} \leq \mathbf{P}$ be a Levi complement. If \mathbf{L} is F -stable, then we have a pair of mutually adjoint linear maps, called Lusztig induction and restriction,

$$R_{\mathbf{L}}^{\mathbf{G}} : \mathbb{Z} \text{Irr}(\mathbf{L}^F) \rightarrow \mathbb{Z} \text{Irr}(\mathbf{G}^F), \quad {}^*R_{\mathbf{L}}^{\mathbf{G}} : \mathbb{Z} \text{Irr}(\mathbf{G}^F) \rightarrow \mathbb{Z} \text{Irr}(\mathbf{L}^F),$$

enjoying many of the same properties as the maps $R_{\mathbf{T}}^{\mathbf{G}}$ and ${}^*R_{\mathbf{T}}^{\mathbf{G}}$. The construction involves the parabolic subgroup \mathbf{P} , and hence strictly speaking the notation for $R_{\mathbf{L}}^{\mathbf{G}}$ should include \mathbf{P} . However, we take the liberty of omitting this as in almost all situations it is known that the construction is independent of the choice of \mathbf{P} . If \mathbf{P} is also F -stable then Lusztig induction of an irreducible character χ of L is the same as Harish-Chandra induction of χ as considered in the previous section.

For $\lambda \in \text{Irr}(L)$ we let $\mathcal{E}(G|(\mathbf{L}, \lambda))$ denote the set of irreducible constituents of $R_{\mathbf{L}}^{\mathbf{G}}(\lambda)$. We have an analogue of Theorem 3.10 which we state under an assumption on the prime ℓ being ‘‘large enough’’. This assumption can be replaced by other conditions, e.g. $|\mathcal{E}(L, \ell') \cap \text{Irr}(e)| = 1$.

Theorem 3.12 *Suppose that $\ell \geq 7$. Let \mathbf{L} be an F -stable Levi subgroup of \mathbf{G} and let $\lambda \in \mathcal{E}(L, \ell')$. Suppose that*

$$C_{\mathbf{G}}(Z(L)_\ell) = \mathbf{L}. \tag{**}$$

Then all elements of $\mathcal{E}(G|(\mathbf{L}, \lambda))$ lie in the same ℓ -block B of G . Further, if e is the block idempotent of kL containing θ , then $(Z(L)_\ell, e)$ is a B -Brauer pair.

The advantage of Theorem 3.12 over Theorem 3.10 is that there are many Levi subgroups satisfying Condition (**)—the disadvantage is that $R_{\mathbf{L}}^{\mathbf{G}}$ and $*R_{\mathbf{L}}^{\mathbf{G}}$ are harder to work with than $R_{\mathbf{T}}^{\mathbf{G}}$ and $*R_{\mathbf{T}}^{\mathbf{G}}$. It is known that every Levi subgroup of \mathbf{G} is of the form $\mathbf{L} = C_{\mathbf{G}}(\mathbf{S})$ where $\mathbf{S} \leq \mathbf{G}$ is a (not necessarily maximal) torus S . Clearly, if \mathbf{S} is F -stable then so is \mathbf{L} . The class of Levi subgroups that is well adapted to Condition (**) are centralisers of particular F -stable tori which we now describe.

To every F -stable torus \mathbf{S} of \mathbf{G} is associated a monic polynomial $P_{\mathbf{S}}(x)$ with integer coefficients called the *polynomial order* of \mathbf{S} such that

$$|\mathbf{S}^{F^m}| = P_{\mathbf{S}}(q^m) \quad \text{for infinitely many integers } m.$$

The polynomial order of \mathbf{S} is uniquely defined and is a product of cyclotomic polynomials $\Phi_d(x)$, $d \in \mathbb{N}$. If the polynomial order of \mathbf{S} is a power of $\Phi_d(x)$ for a single integer d , then we say that \mathbf{S} is a Φ_d -torus. If \mathbf{L} is the centraliser in \mathbf{G} of a Φ_d -torus, then \mathbf{L} is said to be a d -split Levi subgroup of \mathbf{G} .

The following theorem of Cabanes and Enguehard which we state under some simplifying hypotheses shows that the class of d -split Levi subgroups (for a particular d) satisfies (**).

Theorem 3.13 (Cabanes–Enguehard (1999)) *Suppose that $Z(\mathbf{G})$ is connected, $[\mathbf{G}, \mathbf{G}]$ is simply connected and $\ell \geq 7$. Let $d = d_\ell(q)$ be the order of q modulo ℓ . Then every d -split Levi subgroup of \mathbf{G} satisfies condition (**) of Theorem 3.12.*

Example 3.14 Let $\mathbf{G} = \mathrm{GL}_n(\overline{\mathbb{F}}_q)$, $G = \mathrm{GL}_n(\mathbb{F}_q)$. If \mathbf{L} is an F -stable Levi subgroup of \mathbf{G} , then

$$L \cong \mathrm{GL}_{a_1}(q^{m_1}) \times \cdots \times \mathrm{GL}_{a_r}(q^{m_r})$$

for some positive integers a_i and m_i , $1 \leq i \leq r$, such that $\sum_i a_i m_i = n$. The group \mathbf{L} is d -split if and only if $m_i = d$ for all $1 \leq i \leq r$.

3.3 Lusztig series and Bonnafé–Rouquier reduction.

Another key feature of block theory in non-defining characteristic is that the subset $\mathcal{E}(G, \ell')$ controls the ℓ -block distribution of irreducible characters. This is made precise in the following theorem. Note that $\mathcal{E}(G, \ell')$ is the union of the Lusztig series $\mathcal{E}(G, s)$ as s runs over conjugacy classes of semisimple elements of ℓ' -order in the dual group G^* .

Theorem 3.15 (Hiss (1989), Broué–Michel (1988)) *Let B be an ℓ -block of G . There exists a semisimple ℓ' -element s of G^* , unique up to conjugacy in G^* such that*

$$\mathrm{Irr}(B) \cap \mathcal{E}(G, s) \neq \emptyset.$$

If t is a semisimple element of G^ such that $\mathcal{E}(G, t) \cap \mathrm{Irr}(B) \neq \emptyset$, then $t_{\ell'}$ is conjugate in G^* to s .*

For a semisimple ℓ' -element s of G^* , let $\mathcal{E}_\ell(G, s)$ be the union of Lusztig series $\mathcal{E}(G, t)$ where t runs over all semisimple elements of G^* whose ℓ' -part is G^* -conjugate to s . The above theorem implies that $\mathcal{E}_\ell(G, s)$ is a union of ℓ -blocks of G . Thus, the ℓ -block distribution problem can be broken down as follows. For each (conjugacy class of) semi-simple ℓ' -element s of G^* describe:

- (I) The ℓ -block distribution of $\mathcal{E}(G, s)$.
- (II) For each non-trivial semisimple ℓ -element t in $C_{\mathbf{G}^*}(s)$ describe the ℓ -block distribution of $\mathcal{E}(G, st)$.

This approach is compatible with Theorem 3.12 since if s is a semisimple ℓ' -element of L^* for some F -stable Levi subgroup \mathbf{L} of \mathbf{G} , and $\lambda \in \mathcal{E}(L, s)$, then all elements of $\mathcal{E}(G | (\mathbf{L}, \lambda))$ lie in $\mathcal{E}(G, s)$.

The following powerful theorem of Bonnafé and Rouquier [11] allows for a dramatic shrinking of the magnitude of the problem.

Theorem 3.16 (Bonnafé–Rouquier (2003)) *Let $s \in G^*$ be semisimple such that $C_{\mathbf{G}^*}(s) \leq \mathbf{L}^*$ for some F -stable Levi subgroup \mathbf{L}^* of \mathbf{G}^* . Then the product of ℓ -block algebras in $\mathcal{E}_\ell(G, s)$ is Morita equivalent to the product of ℓ -block algebras in $\mathcal{E}(L, s)$.*

So, inductively, we only need to study the blocks in Lusztig series $\mathcal{E}_\ell(G, s)$ such that s is *quasi-isolated* in \mathbf{G} , that is, such that $C_{\mathbf{G}^*}(s)$ is not contained in any proper Levi subgroup of \mathbf{G} . The blocks in non-quasi-isolated series can be "recovered" from blocks of groups of Lie type where the underlying algebraic group is of smaller dimension than that of \mathbf{G} . There is a price to be paid here: since Levi subgroups of a simple algebraic group are not simple, while working in the inductive set-up we cannot restrict ourselves to only considering simple algebraic groups. We need to take into account all Levi subgroups as well.

Recently, Bonnafé, Dat and Rouquier [9] have given an improvement of Theorem 3.16 which in most situations reduces the set of semisimple elements that need to be considered even further, namely to *isolated* elements. These are elements s such that the connected component of $C_{\mathbf{G}^*}(s)$ is not contained in a proper Levi subgroup of \mathbf{G} .

3.4 Unipotent blocks and d -Harish-Chandra theory.

The best understood class of blocks are the *unipotent* blocks. These are the blocks in $\mathcal{E}_\ell(G, 1)$. By Theorem 3.15, the unipotent blocks are precisely the blocks which contain a unipotent character. In [15], Broué, Malle and Michel generalised the Harish-Chandra theory of Howlett and Lehrer to the context of d -split Levi subgroups (see the discussion before Theorem 2.38). This d -Harish-Chandra theory is an important ingredient in the solution of the block distribution problem.

For d a positive integer, let $\mathcal{U}_d(G)$ denote the set of all pairs (\mathbf{L}, λ) such that \mathbf{L} is a d -split Levi subgroup of \mathbf{G} and λ is an irreducible unipotent character of L . We

regard \mathbf{G} as a d -split Levi subgroup of itself, so $(\mathbf{G}, \chi) \in \mathcal{U}_d(G)$ for any irreducible unipotent character χ of G . The set $\mathcal{U}_d(G)$ is a G -set via

$${}^g(\mathbf{L}, \lambda) = ({}^g\mathbf{L}, {}^g\lambda), \quad \text{for } g \in G, (\mathbf{L}, \lambda) \in \mathcal{U}_d(G).$$

There is also an inclusion relation on $\mathcal{U}_d(G)$ which is defined as follows:

$$(\mathbf{L}, \lambda) \leq (\mathbf{M}, \mu) \quad \text{if } \mathbf{L} \leq \mathbf{M} \text{ and } \lambda \text{ is a constituent of } R_{\mathbf{L}}^{\mathbf{M}}(\mu).$$

The pair (\mathbf{M}, μ) is said to be a *unipotent d -cuspidal pair of G* if there does not exist a unipotent d -cuspidal pair $(\mathbf{L}, \lambda) \leq (\mathbf{M}, \mu)$ with \mathbf{L} proper in \mathbf{M} .

For $(\mathbf{L}, \lambda) \in \mathcal{U}_d(G)$ and \mathbf{M} an F -stable Levi subgroup of \mathbf{G} containing \mathbf{L} , we denote by $N_M(\mathbf{L}, \lambda)$ the stabiliser in M of the pair (\mathbf{L}, λ) and we denote by $W_M(\mathbf{L}, \lambda)$ the group $N_M(\mathbf{L}, \lambda)/L$, the *relative Weyl group of (\mathbf{L}, λ)* .

Theorem 3.17 (Broué–Malle–Michel (1993)) *Let d be a positive integer.*

- (a) $(\mathcal{U}_d(G), \leq)$ is a G -poset. The connected components of $(\mathcal{U}_d(G), \leq)$ are precisely the sets $\mathcal{E}(G|(\mathbf{L}, \lambda))$ as (\mathbf{L}, λ) runs over a set of representatives of G -conjugacy classes of d -cuspidal pairs of G .
- (b) Let (\mathbf{L}, λ) be a d -cuspidal pair of G and let \mathbf{M} be a d -split Levi subgroup of \mathbf{G} containing \mathbf{L} . There exists an isometry

$$\mathbb{Z}\mathcal{E}(M|(\mathbf{L}, \lambda)) \cong \mathbb{Z}\text{Irr}(W_{\mathbf{M}}(\mathbf{L}, \lambda))$$

intertwining $R_{\mathbf{M}}^{\mathbf{G}}$ with $\text{Ind}_{W_{\mathbf{M}}(\mathbf{L}, \lambda)}^{W_G(\mathbf{L}, \lambda)}$.

The proof of Theorem 3.17 is on a case by case basis and relies heavily on the combinatorics associated to unipotent characters. An especially delicate point is the transitivity of \leq since Lusztig induction sends characters to virtual characters. A by-product of part (b) of the theorem is an explicit description of d -cuspidal pairs (\mathbf{L}, λ) and the sets $\mathcal{E}(G|(\mathbf{L}, \lambda))$. For classical groups, this description is in terms of the combinatorial “yoga” associated to partitions and symbols labelling unipotent characters and is in terms of tables for exceptional groups.

The set $\mathcal{E}(G|(\mathbf{L}, \lambda))$ for a given d -cuspidal pair (\mathbf{L}, λ) is called the *d -Harish-Chandra series above (\mathbf{L}, λ)* . Thus, for any $d \geq 1$, Theorem 3.17 provides a partition of the set of unipotent characters into d -Harish Chandra series. It turns out that when d is the order of q modulo ℓ and provided that ℓ is sufficiently large the partition into d -Harish-Chandra series coincides with the block partition of $\mathcal{E}(G, 1)$, and is also closely linked with the ℓ -block partition of $\mathcal{P}(G)$. The following theorem, which makes this more precise, was proved by Cabanes and Enguehard [19]. For very large ℓ it is due to Broué, Malle and Michel [15]. The first assertion is covered by Theorem 3.12 and Theorem 3.13.

Theorem 3.18 (Broué–Malle–Michel (1993), Cabanes–Enguehard (1994)) *Suppose that $\ell \geq 7$ and let $d = d_{\ell}(q)$.*

- (a) For any unipotent d -cuspidal pair (\mathbf{L}, λ) of \mathbf{G} there exists a unique ℓ -block $B_G(\mathbf{L}, \lambda)$ of G containing all elements of $\mathcal{E}(G|(\mathbf{L}, \lambda))$.
- (b) The map $(\mathbf{L}, \lambda) \mapsto B_G(\mathbf{L}, \lambda)$ induces a bijection between the G -classes of unipotent d -cuspidal pairs and the set of unipotent blocks of G .
- (c) $\text{Irr}(B_G(\mathbf{L}, \lambda)) \cap \mathcal{E}(\mathbf{G}, 1) = \mathcal{E}(G|(\mathbf{L}, \lambda))$ for all unipotent d -cuspidal pairs (\mathbf{L}, λ) of \mathbf{G} .
- (d) Suppose that $[\mathbf{G}, \mathbf{G}]$ is simply connected. The map $(\mathbf{M}, \mu) \mapsto (Z(M)_\ell, e(\mu))$ is an order reversing bijection from $(\mathcal{U}_d(G), \leq)$ onto a subset of $(\mathcal{P}(G), \leq)$ where $e(\mu)$ denotes the block idempotent of $kC_G(Z(M)_\ell)$ associated to μ .
- (e) There exists a maximal $B_G(\mathbf{L}, \lambda)$ -Brauer pair (D, e) such that
 - $(Z(\mathbf{L})_\ell, e(\lambda)) \leq (D, e)$;
 - $C_D(Z(\mathbf{L})_\ell) \leq Z(L)_\ell$; and
 - $D/Z(L)_\ell$ is isomorphic to a subgroup of $W_G(\mathbf{L}, \lambda)$.

Theorem 3.18 provides a complete solution to the block distribution] problem for unipotent characters. In other words, it completes Part (I) of the programme outlined in Section 3.3 for $s = 1$. In fact, Cabanes and Enguehard also give a solution to Part (II). We describe this briefly. For simplicity, we assume that $Z(\mathbf{G})$ is connected. Let r be an ℓ -element of G^* . The assumption that $\ell \geq 7$ implies that the centraliser of r in \mathbf{G}^* is a Levi subgroup of \mathbf{G}^* , necessarily F -stable. Duality between \mathbf{G} and \mathbf{G}^* yields a corresponding F -stable Levi subgroup $\mathbf{C}(r) \leq \mathbf{G}$ and a linear character \hat{r} of $C(r)$. By Lusztig's parametrisation of characters, the elements of $\mathcal{E}(G, r)$, are precisely the characters

$$\epsilon R_{\mathbf{C}(r)}^{\mathbf{G}}(\hat{r} \otimes \eta), \quad \eta \in \mathcal{E}(C(r), 1),$$

for some $\epsilon \in \{\pm 1\}$. Cabanes and Enguehard show that to each unipotent d -cuspidal pair (\mathbf{L}', λ') of $C(r)$ is associated a unipotent d -cuspidal pair (\mathbf{L}, λ) of G such that

$$[\mathbf{L}, \mathbf{L}] = [\mathbf{L}', \mathbf{L}'] \quad \text{and} \quad \text{Res}_{[\mathbf{L}, \mathbf{L}]^F}^{\mathbf{L}} \lambda = \text{Res}_{[\mathbf{L}', \mathbf{L}']^F}^{\mathbf{L}'} \lambda.$$

Then $\epsilon R_{\mathbf{C}(r)}^{\mathbf{G}}(\hat{r} \otimes \eta)$ belongs to the block $B_{(\mathbf{L}, \lambda)}$ if and only if η lies in the d -Chandra series of $C(r)$ above a d -cuspidal pair (\mathbf{L}', λ') associated to (\mathbf{L}, λ) .

The condition $\ell \geq 7$ in the above theorem can be replaced by the weaker condition: ℓ is odd, good for \mathbf{G} and $\ell \geq 5$ if \mathbf{G} has a simple component of type D_4 which contributes the triality group ${}^3D_4(q)$ to \mathbf{G}^F . In [32] Enguehard treated the unipotent ℓ -blocks for the remaining primes. One obtains a slightly weaker analogue of Theorem 3.18. The main difference is that the assignment in part (b) of the theorem, while still onto, is no longer one-to-one. In order to obtain a bijection one replaces the set of unipotent d -cuspidal pairs with a slightly smaller set, namely the set of unipotent d -cuspidal pairs with central ℓ -defect. Enguehard also does Part (II) of the problem but only in the case that the center of \mathbf{G} is connected. For disconnected center groups the problem is still open.

3.5 General Blocks.

There has also been a lot of work done to generalise the results of the previous section to non-unipotent blocks. There are two inter-connected approaches to this generalisation: (i) develop a non-unipotent d -Harish Chandra theory (ii) use Jordan decomposition to carry over unipotent d -Harish Chandra theory to the non-unipotent case.

In [20], using a hybrid of the two approaches, Cabanes and Enguehard proved an analogue of Parts (a) and (b) of Theorem 3.18 for non-unipotent blocks as well as a weak analogue of (c) provided ℓ is odd, good for \mathbf{G} and $\ell \geq 5$ if \mathbf{G} has a simple component of type D_4 . They also describe defect group structure. In [33], Enguehard showed that provided that $\ell \geq 7$ and the center of \mathbf{G} is connected, then block distribution of irreducible characters is highly compatible with Jordan decomposition. In particular, for any semisimple ℓ' -element s of G^* , there is a bijection $B \mapsto B_s$ between the blocks of \mathbf{G} in $\mathcal{E}_\ell(G, s)$ and the unipotent blocks of $C_{\mathbf{G}^*}(s)$ such that there is a height preserving bijection between the irreducible characters of B and those of B_s and such that B and B_s have isomorphic defect groups (in fact B and B_s have isomorphic Brauer categories). In the same paper, Enguehard also describes the blocks for classical groups when $\ell = 2$. The paper [46] described the block distribution of $\mathcal{E}_\ell(G, s)$ for \mathbf{G} simple of exceptional type and ℓ a bad prime. Combining all of the previous results with the theorem of Bonnafé and Rouquier, a uniform parametrisation of blocks for all G such that \mathbf{G} is simple was given in [47].

4 On the other conjectures; open problems

To end this survey, let us briefly comment on the status of the inductive conditions for the other local-global conjectures beyond the McKay conjecture and on some related open problems.

The work on block parametrisation described in these sections has been enough to verify Brauer's height zero conjecture for quasi-simple groups [46], [47]. For the remaining conjectures, we will need to do much more. For the immediate future, the two main open problems which need to be resolved for blocks of finite quasi-simple groups of Lie type are:

Problem 4.1 Complete the description of non ℓ' -characters in ℓ -blocks of finite groups of Lie type for small (bad) primes ℓ .

Problem 4.2 Describe the relationship between the ℓ -block distribution of $\mathcal{P}(G)$ and Theorem 3.18.

For the Alperin–McKay Conjecture 2.5 we still do not have control over the global nor over the local situation in general. The hope is that we can prove a reduction of the necessary conditions to so-called quasi-isolated blocks, in the spirit of the Bonnafé–Rouquier Theorem 3.16: While this result gives some kind of reduction for the global situation, we are still missing an analogous local result.

For the Alperin weight conjecture, the following cases have been dealt with so far, see [57] and [76]:

Theorem 4.3 (Malle (2014), Schulte (2016)) *The groups \mathfrak{A}_n , ${}^2B_2(q^2)$, ${}^2G_2(q^2)$, ${}^2F_4(q^2)$, ${}^3D_4(q)$ and $G_2(q)$ are AWC good for all primes ℓ .*

The proof requires the determination of all weights of all radical subgroups; for exceptional groups of larger rank that seems quite challenging at the moment. For classical types, it might be possible to use results of An [5].

Here we hope for

1. a generic description of weights in terms of d -tori and their normalisers
2. a Bonnafé–Rouquier type reduction to a few special situations.

Another ingredient might be the following:

Problem 4.4 Show that the ℓ -modular decomposition matrices of blocks of quasi-simple groups of Lie type are unitriangular.

This statement might follow, at least in good characteristic, from properties of generalised Gelfand–Graev characters. If this were true, one could make use of the result of Koshitani–Späth [50] mentioned before. The unitriangularity will be with respect to a suitable subset of $\text{Irr}(B)$: A linearly independent subset $X \subseteq \text{Irr}(B)$ is called a *basic set for B* if every Brauer character $\varphi \in \text{IBr}(B)$ is an integral linear combination of the elements of X . So in particular $|X| = |\text{IBr}(B)|$. The following is folklore:

Conjecture 4.5 *Any ℓ -block of a quasi-simple group of Lie type has a “natural” basic set.*

Geck and Hiss [36] exhibited such a basic set when ℓ is good for the underlying algebraic group \mathbf{G} and does not divide the order of $Z(G)$: in this case $\mathcal{E}(G, s)$ is a basic set for the union of blocks $\mathcal{E}_\ell(G, s)$. This is yet another situation in which the theories of Brauer and of Lusztig fit together perfectly. It is known that this statement can no longer hold when either ℓ divides $|Z(G)|$, or when ℓ is bad for \mathbf{G} . In some cases, replacements have been found, but this is still open in general.

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