Article Submitted to Journal of Symbolic Computation
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Abstract

We give an introduction to the theory of the Gauss-Manin connection of an isolated hypersurface singularity and describe an algorithm to compute the V-filtration on the Brieskorn lattice. We use an implementation in the computer algebra system SINGULAR to prove C. Hertling's conjecture about the variance of the spectrum for Milnor number $\mu \leq 16$.

Algorithms for the Gauss-Manin connection

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*I wish to thank G.-M. Greuel, C. Hertling, H. Schönemann, and J.H.M. Steenbrink for their help and suggestions.

1. Introduction

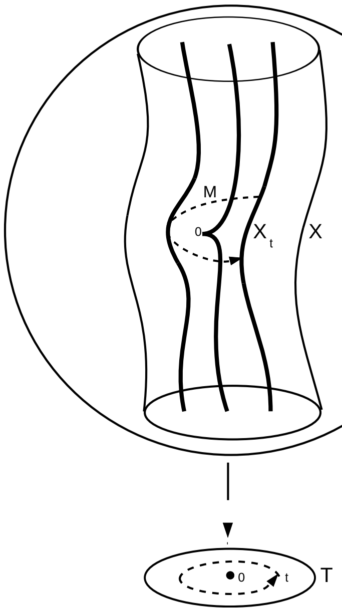
Let $f : U \longrightarrow \mathbb{C}$ be a *holomorphic map* defined in a neighborhood $0 \in U \subset \mathbb{C}^{n+1}$ with *isolated critical point* 0 and *critical value* $f(0) = 0$. By J. Milnor (Mil68), for an appropriately choosen restriction

$$f : X \longrightarrow T$$

of f over a disc $T \subset \mathbb{C}$ around 0, the non-singular fibres are *homotopy equivalent* to a *bouquet* of μ n-spheres and form a $\mathscr{C}^\infty$ *fibre bundle* over the punctured disc $T' := T\backslash\{0\}$ called the *Milnor fibration*. Hence, the cohomology of the general fibre $X_t := f^{-1}(t)$, $t \in T'$, the so-called *Milnor fibre*, is given by

$$\widetilde{\mathrm{H}}^k(X_t, \mathbb{Z}) = \begin{cases} \mathbb{Z}^\mu, & k = n \\ 0, & \text{else.} \end{cases}$$

Figure 1: The Milnor fibration



The local product structure of the Milnor fibration translates to the structure of a *flat complex vector bundle* on the n-th complex cohomology groups of the non singular fibres

$$H^n := \bigcup_{t \in T'} \mathrm{H}^n(X_t, \mathbb{C})$$

called the *cohomology bundle*. The *flatness* of the cohomology bundle means that it can be described by *local frames* with *constant transition functions*. Hence, there is a welldefined notion of a holomorphic section in the cohomology bundle being *constant*. Algebraically, this translates to the existence of a *flat connection* on the cohomology bundle, the so-called *Gauss-Manin connection*, meaning that sections can be differentiated by the *covariant derivative* along vector fields defined on the *base* T. The covariant derivative along ∂_t, where t is the coordinate on T, defines a *differential operator* which we also denote by ∂_t.

Moving an *integer cohomology class* along constant sections once around the critical point in counterclockwise direction, defines an automorhism

$$\mathrm{M} \in \mathrm{Aut}_{\mathbb{Z}}\big(\mathrm{H}(X_t, \mathbb{Z})\big)$$

defined over $\mathbb{Z}$ which is called the *algebraic monodromy*. Since the monodromy is not the identity, flat sections are *multivalued* which means that they are *global flat sections* in the pullback of the cohomology bundle to the *universal covering* of the punctured disc. But one can multiply such a flat multivalued section by appropriate holomorphic *twists*, inverse to the action of the monodromy, in order to obtain a global holomorphic section. The sections arising in this way, defined over arbitary small punctured neighborhoods of $0 \in T$, span a *regular* $\mathbb{C}\{t\}[\partial_t]$*-module* $\mathscr{G}_0$ which we will call the *Gauss-Manin connection*. The *regularity* of the *Gauss-Manin connection* means that the sections have *moderate growth* towards $0 \in T$. As E. Brieskorn (Bri70) has shown, the monodromy of the *Gauss-Manin connection* as $\mathbb{C}\{t\}[\partial_t]$-module coincides with the complex monodromy and its eigenvalues are *roots of unity*. Up to this point, it is totally unclear how to approach this object by methods of computer algebra in order to obtain an algorithm to compute it.

E. Brieskorn (Bri70) gave an algebraic description of the complex monodromy and an algorithm to compute it. Using the *holomorphic De Rham theorem*, the cohomology of the Milnor fibre can be described in terms of integrals of holomorphic differential n-forms over vanishing cycles. The *Gelfand-Leray form* $\frac{\omega}{\mathrm{d}f}$ of a holomorphic differential $(n+1)$-form ω on X defines a holomorphic section in the cohomology bundle. This gives a map $\Omega^{n+1}_{X,0} \longrightarrow \mathscr{G}_0$ which actually factors through an inclusion of the *Brieskorn lattice*

$$\mathscr{H}_0'' = \Omega^{n+1}_{X,0} / \mathrm{d}f \wedge \mathrm{d}\Omega^{n-1}_{X,0}$$

into the Gauss-Manin connection. The *Leray residue formula* gives the formula (Bri70)

$$\partial_t[\mathrm{d}f \wedge \eta] = [\mathrm{d}\eta]$$

for the action of ∂_t. This is the key to an algorithmic approach towards the Gauss-Manin connection. But it is still a non-trivial task to compute the monodromy.

The Brieskorn lattice is a free $\mathbb{C}\{t\}$-module of rank μ (Seb70) and $t^{n+1}\partial_t$ acts on it. E. Brieskorn explained how the computation of this action up to sufficiently high order allows one to compute the complex monodromy. Based on the work of R. Gérard and A.H.M. Levelt (GL73), P.F.M.

Nacken (Nac90) first implemented this algorithm in the computer algebra system MAPLE V. A later implementation by the author in the computer algebra system SINGULAR (GPS01) in the library `mondromy.lib` (Sch01b; Sch99) turned out to be more efficient.

An appropriate restriction of ∂_t is invertible and ∂_t^{-1} acts on the Brieskorn lattice. This extends to a structure over the ring of *microdifferential operators with constant coefficients* $\mathbb{C}\{\{\partial_t^{-1}\}\}$, a power series ring with a certain growth condition. As we will see, the Brieskorn lattice is a free $\mathbb{C}\{\{\partial_t^{-1}\}\}$-module of rank μ (Pha77). We will explain how this structure leads to more efficient algorithms allowing us to compute more than just the monodromy.

The *V-filtration* on the Gauss-Manin system is defined by the *generalized eigenspaces* of $t\partial_t$ which are logarithms of the eigenvalues of the monodromy. The induced V-filtration on the Brieskorn lattice reflects its embedding in the Gauss-Manin connection and defines the *spectrum*, which is an important and deep invariant coming from the *mixed Hodge-structure on the cohomology of the Milnor fibre* (Ste77; Var82; SS85). Based on M. Saito's result (Sai88) saying that, for *Newton non-degenerate* singularities, the V-filtration coincides with the *Newton filtration* defined on $\mathbb{C}\{x_0, \ldots, x_n\}$ by the *Newton polyhedron* of f at 0, S. Endrass (End01) implemented an algorithm for computing the spectrum of Newton non-degenerate singularities in the SINGULAR library `spectrum.lib`. We will present the first algorithm to compute the spectrum of arbitrary singularities.

The weight filtration on the Gauss-Manin connection is defined by the nilpotent part of $t\partial_t$, which is the logarithm of the *unipotent part* of the monodromy. This gives a refinement of the V-filtration defining the *spectral pairs* corresponding to the *Hodge numbers* of the mixed Hodge-structure on the cohomology of the Milnor fibre.

By the methods we are going to explain, one can actually compute all of the above invariants, namely the V- and weight filtration, the spectrum and spectral pairs, and the Hodge numbers, for not necessarily Newton non-degenerate singularities. Most of the algorithms are implemented in the SINGULAR library `gaussman.lib` (Sch01a).

The spectrum consists of μ rational so called *spectral numbers* $\alpha_1, \ldots, \alpha_\mu$ in the interval $(-1, n)$, which are symmetric with mean value $\frac{n-1}{2}$. C. Hertling (Her01) conjectured that their variance is bounded by

$$\gamma := -\frac{1}{4}\sum_{i=1}^{\mu}\Big(\alpha_i - \frac{n-1}{2}\Big)^2 + \frac{\alpha_\mu - \alpha_1}{48}\mu \geq 0,$$

and proved that equality holds for *quasihomogenous singularities*. M. Saito (Sai) proved the conjecture for *irreducible plane curve singularities*. As an application, we use our implementation to prove C. Hertling's conjecture for singularities with *Milnor number* $\mu \leq 16$, which were classified by I.V. Arnold (AGZV85).

This paper is based on the work with J.H.M. Steenbrink (SS01; Sch00).

In addition to (SS01), we give an introduction to the theory of the *Gauss-Manin connection*, a detailed description of the algorithm and its implementation, including a pseudocode, and an application to C. Hertling's conjecture.

The methods presented in this paper are based on the interplay of the $\mathscr{D}$-module structure and the microlocal structure. They may serve as an example for symbolic $\mathscr{D}$-module computations with a computer algebra system.

2. Milnor fibration

We consider an *isolated hypersurface singularity* $f: (\mathbb{C}^{n+1}, 0) \longrightarrow (\mathbb{C}, 0)$ with *Milnor number*

$$\mu := \dim_{\mathbb{C}} \mathbb{C}\{x\}/(\partial_x f) < \infty,$$

where we denote $x = (x_0, \ldots, x_n)$, $\partial_x f = (\partial_{x_0} f, \ldots, \partial_{x_n} f)$. Let

$$X \xrightarrow{f} T$$

be a *good representative* (Loo84) of f. This means that $T \subset \mathbb{C}$ is an open disc around 0 and X is the intersection of $f^{-1}(T)$ with an open ball $B \subset \mathbb{C}^{n+1}$ around 0 such that the *singular fibre* $f^{-1}(0)$ intersects arbitrary small spheres in B around 0 transversally. We denote

$$\begin{aligned} T' &:= T \backslash \{0\}, \\ X' &:= f^{-1}(T') \cap X. \end{aligned}$$

Then $f: X' \longrightarrow T'$ is a $\mathscr{C}^\infty$ fibre bundle with fibres $X_t := f^{-1}(t)$, $t \in T'$ homotopy equivalent to the *bouquet* of μ n-spheres (Mil68). Recall that the bouquet of a set of pointed topological spaces is the topological space which arises from *gluing* these spaces at their base points. Note that this implies that the *cohomology* of the *Milnor fibre* X_t is given by

$$\widetilde{\mathrm{H}}^k(X_t, \mathbb{Z}) = \begin{cases} \mathbb{Z}^\mu, & k = n \\ 0, & \text{else.} \end{cases}$$

3. Gauss-Manin connection

The *cohomology bundle*

$$H^n := \bigcup_{t \in T'} \mathrm{H}^n(X_t, \mathbb{C}) = \bigcup_{t \in T'} \mathrm{H}^n(X_t, \mathbb{Z}) \otimes_{\mathbb{Z}} \mathbb{C} \supset \bigcup_{t \in T'} \mathrm{H}^n(X_t, \mathbb{Z})$$

is a *flat complex vector bundle* of *rank* μ on T'. This means that it can be described by *local frames* with constant *transition functions*. Hence, the sheaf $\mathscr{H}^n$ of *holomorphic sections* in H^n is a *complex local system* in the sense of P. Deligne (Del70). By (Del70, Prop. 2.16), there is a natural *flat* *connection* on H^n and we denote its *covariant derivative* with respect to ∂_t by

$$\mathscr{H}^n \xrightarrow{\partial_t} \mathscr{H}^n\,.$$

It induces a *differential operator* ∂_t on $(i_*\mathscr{H}^n)_0$ where $i : T' \hookrightarrow T$ denotes the inclusion and the lower index 0 denotes germs at 0. Note that an element of $(i_*\mathscr{H}^n)_0$ is represented by a section in a punctured neighborhood of $0 \in T$.

Let $u : T^\infty \longrightarrow T$, $u(\tau) := \exp(2\pi\mathrm{i}\tau)$, be the *universal covering* of T'. By τ we denote the coordinate on T^∞. The *pullback*

$$X^\infty := X' \times_{T'} T^\infty$$

is called the *canonical Milnor fibre*. Since T^∞ is *contractible*, the natural maps $X_{u(\tau)} \cong X^\infty_\tau \hookrightarrow X^\infty$, $\tau \in T^\infty$, are *homotopy equivalences*. Hence, $\mathrm{H}^n(X^\infty, \mathbb{C})$ is a *trivial complex vector bundle* on T^∞. We consider $A \in \mathrm{H}^n(X^\infty, \mathbb{C})$ as a *global flat multivalued section* $A(t)$ in H^n. Note that

$$\partial_t A(t) = 0$$

for $A \in \mathrm{H}^n(X^\infty, \mathbb{C})$.

There is a natural action of the *fundamental group* $\Pi_1(T', t)$, $t \in T'$, on $\mathrm{H}^n(X_t, \mathbb{C}) \cong \mathrm{H}^n(X^\infty, \mathbb{C})$ by *lifting paths* along *flat sections* in the cohomology bundle. A positively oriented generator operates via the *monodromy operator* M defined by

$$(\mathrm{M}\, s)(\tau) := s(\tau + 1)$$

for $s \in \mathrm{H}^n(X^\infty, \mathbb{C})$. Let $\mathrm{M} = \mathrm{M}_s \mathrm{M}_u$ be the decomposition of M into *semisimple* M_s and *unipotent* M_u and

$$N := \log \mathrm{M}_u\,.$$

By the *monodromy theorem* (Bri70), the eigenvalues of M_s are roots of unity and $N^{n+1} = 0$. Let

$$\mathrm{H}^n(X^\infty, \mathbb{C}) \cong \bigoplus_\lambda \mathrm{H}^n(X^\infty, \mathbb{C})_\lambda$$

be the decomposition of $\mathrm{H}^n(X^\infty, \mathbb{C})$ into *generalized eigenspaces*

$$\mathrm{H}^n(X^\infty, \mathbb{C})_\lambda := \ker(\mathrm{M}_s - \lambda)$$

of M and $\mathrm{M}_\lambda := \mathrm{M}|_{\mathrm{H}^n(X^\infty, \mathbb{C})_\lambda}$. For $A \in \mathrm{H}^n(X^\infty, \mathbb{C})_\lambda$, $\lambda = \exp(-2\pi\mathrm{i}\alpha)$, $\alpha \in \mathbb{Q}$, the *elementary section* $s(A, \alpha)$ defined by

$$s(A, \alpha)(t) := t^\alpha \exp\Bigl(-\frac{N}{2\pi\mathrm{i}} \log t\Bigr) A(t)$$

is monodromy invariant and hence $s(A, \alpha)$ defines a holomorphic section in H^n. Note that the *twist* $t^\alpha \exp\bigl(-\frac{N}{2\pi\mathrm{i}} \log t\bigr)$ is inverse to the action of the monodromy on $A(t)$. The elementary sections $i_* s(A, \alpha)$ span a ∂_t-invariant free $\mathscr{O}_T\bigl[t^{-1}\bigr]$-submodule $\mathscr{G} \subset i_*\mathscr{H}^n$ of rank μ. The *Gauss-Manin connection* is the *regular* $\mathscr{D}_0$-module $\mathscr{G}_0$ (Bri70; Pha79) where $\mathscr{D} := \mathscr{O}_T[\partial_t]$ and the lower index 0 denotes germs at 0.

4. V-filtration

We want to use the $\mathscr{D}$-module structure of the Gauss-Manin connection to define the *V-filtration*.

Since the twist $t^\alpha \exp\bigl(-\frac{N}{2\pi\mathrm{i}}\log t\bigr)$ is invertible, $\psi_\alpha(A) := (i_*s(A,\alpha))_0$ defines an inclusion

$$\mathrm{H}^n(X^\infty,\mathbb{C})_\lambda \overset{\psi_\alpha}{\hookrightarrow} \mathscr{G}_0$$

which fulfills $t\circ\psi_\alpha = \psi_{\alpha+1}$ and $\partial_t\circ\psi_\alpha = \psi_{\alpha-1}\circ\bigl(\alpha - \frac{N}{2\pi\mathrm{i}}\bigr)$ by definition of $s(A,\alpha)$. Hence,

$$(t\partial_t - \alpha)\circ\psi_\alpha = \psi_\alpha\circ\Bigl(-\frac{N}{2\pi\mathrm{i}}\Bigr), \tag{1}$$

$$\exp(-2\pi\mathrm{i}t\partial_t)\circ\psi_\alpha = \psi_\alpha\circ \mathrm{M}_\lambda\,. \tag{2}$$

Equality (1) implies that the image

$$C_\alpha := \operatorname{im}\psi_\alpha = \ker(t\partial_t - \alpha)^{n+1}$$

of ψ_α is the generalized α-eigenspace of $t\partial_t$, that $t: C_\alpha \longrightarrow C_{\alpha+1}$ is bijective, and that $\partial_t: C_\alpha \longrightarrow C_{\alpha-1}$ is bijective for $\alpha\neq 0$. Equality (2) gives a relation between the Gauss-Manin connection and the monodromy. The *V-filtration* V on $\mathscr{G}_0$ is defined by

$$V^\alpha := V^\alpha\mathscr{G}_0 := \sum_{\alpha\le\beta}\mathbb{C}\{t\}C_\beta,$$

$$V^{>\alpha} := V^{>\alpha}\mathscr{G}_0 := \sum_{\alpha<\beta}\mathbb{C}\{t\}C_\beta.$$

V^α and $V^{>\alpha}$ are free $\mathbb{C}\{t\}$-modules of rank μ with $V^\alpha/V^{>\alpha}\cong C_\alpha$.

5. Saturation and non-resonance

We want to use equality (2) to express the monodromy in terms of the Gauss-Manin connection.

A $t\partial_t$-stable $\mathbb{C}\{t\}$-lattice $\mathscr{L}\subset\mathscr{G}_0$ is called *saturated*. The notion of *regularity* is defined by the existence of a saturated $\mathbb{C}\{t\}$-lattice. Note that the V^α (resp. $V^{>\alpha}$) are saturated. Since $\mathbb{C}\{t\}$ is a discrete valuation ring, for any two $\mathbb{C}\{t\}$-lattices $\mathscr{L},\mathscr{L}'\subset\mathscr{G}_0$, there is a $k\in\mathbb{Z}$ such that $t^k\mathscr{L}\subset\mathscr{L}'$. Hence, for any $\mathbb{C}\{t\}$-lattice $\mathscr{L}$, there are $\alpha_1<\alpha_2$ such that

$$V^{\alpha_2}\subset\mathscr{L}\subset V^{>\alpha_1}.$$

Since the V^α (resp. $V^{>\alpha}$) are saturated and *noetherian*, this implies that the *saturation*

$$\mathscr{L}_\infty := \sum_{k=0}^{\infty}(t\partial_t)^k\mathscr{L}$$

of a $\mathbb{C}\{t\}$-lattice $\mathscr{L}$ is itself a $\mathbb{C}\{t\}$-lattice. Note that $\mathscr{L}_\infty$ is saturated. One can actually show that $\mathscr{L}_\infty = \sum_{k=0}^{\mu-1}(t\partial_t)^k\mathscr{L}$. Let $V^{\alpha_2} \subset \mathscr{L} \subset V^{>\alpha_1}$ be a saturated $\mathbb{C}\{t\}$-lattice. Since $t\partial_t$ operates on $\mathscr{L}$, there is a decomposition into generalized eigenspaces

$$\mathscr{L} = \Big(\bigoplus_{\alpha_1<\alpha<\alpha_2} \mathscr{L} \cap C_\alpha\Big) \oplus V^{\alpha_2}$$

and a *residue endomorphism* $\overline{t\partial_t} \in \operatorname{End}(\mathscr{L}/t\mathscr{L})$ induced by $t\partial_t$. If $\overline{t\partial_t}$ has no positive integer differences of eigenvalues in any *Jordan block* of N, $\mathscr{L}$ is called *non-resonant* and equality (2) implies that

$$\mathrm{M} = \exp(-2\pi \mathrm{i}\overline{t\partial_t}).$$

6. D-module structure

We want to describe the $\mathscr{D}$-module structure of the Gauss-Manin connection.

Let

$$\mathrm{H}^n(X^\infty,\mathbb{C})_\lambda \cong \bigoplus_{j=1}^{m_\lambda} \mathrm{H}^n(X^\infty,\mathbb{C})_{\lambda,j}$$

be a decomposition of $\mathrm{H}^n(X^\infty,\mathbb{C})_\lambda$ into Jordan blocks $\mathrm{H}^n(X^\infty,\mathbb{C})_{\lambda,j}$ of M_λ of size $n_{\lambda,j} = \dim_\mathbb{C} \mathrm{H}^n(X^\infty,\mathbb{C})_{\lambda,j}$ and $C_{\alpha,j} := \operatorname{im}\psi_\alpha|_{\mathrm{H}^n(X^\infty,\mathbb{C})_{\lambda,j}}$. Then

$$\mathscr{L}_\alpha := \bigoplus_\lambda \bigoplus_{j=1}^{m_\lambda} \mathbb{C}\{t\}C_{\alpha_{\lambda,j},j}$$

is a non-resonant saturated $\mathbb{C}\{t\}$-lattice. Let $A_{\lambda,j}$ be a $\mathbb{C}[N]$-generator of $\mathrm{H}^n(X^\infty,\mathbb{C})_{\lambda,j}$. Since $\partial_t : C_\alpha \longrightarrow C_{\alpha-1}$ is bijective for $\alpha \neq 0$,

$$\begin{aligned}
\mathscr{G}_0 &\cong \mathbb{C}\{t\}\big[t^{-1}\big]\mathscr{L}_\alpha \\
&= \bigoplus_\lambda \bigoplus_{j=1}^{m_\lambda} \sum_{k=0}^{n_{\lambda,j}-1} \mathbb{C}\{t\}\big[t^{-1}\big] s\Big(\Big(-\frac{N}{2\pi\mathrm{i}}\Big)^k A_{\lambda,j}, \alpha_{\lambda,j}\Big) \\
&= \bigoplus_\lambda \bigoplus_{j=1}^{m_\lambda} \sum_{k=0}^{n_{\lambda,j}-1} \mathbb{C}\{t\}[\partial_t](t\partial_t - \alpha_{\lambda,j})^k s(A_{\lambda,j}, \alpha_{\lambda,j}) \\
&\cong \bigoplus_\lambda \bigoplus_{j=1}^{m_\lambda} \mathscr{D}_0/\mathscr{D}_0(t\partial_t - \alpha_{\lambda,j})^{n_{\lambda,j}}
\end{aligned}$$

for $\alpha_{\lambda,j} < 0$ or $\alpha_{\lambda,j} \notin \mathbb{Z}$.

7. Brieskorn lattice

We want to define the *Brieskorn lattice* on which the action of ∂_t can be computed. The basic idea is to describe the cohomology of the Milnor fibre in terms of holomorphic differntial forms via the *De Rham isomorphism*.

Since the Milnor fibre X_t, $t \in T'$, is a *Stein* complex manifold, the *De Rham* homomorphism $\rho : \mathrm{H}^n_{DR}\,\Omega_{X_t} \longrightarrow \mathrm{H}^n(X_t, \mathbb{C})$ defined by

$$\rho\big([\omega]\big)(\delta) := \int_\delta \omega$$

is an isomorphism. A *geometrical section* is a holomorphic section $s(\omega)$ in H^n defined by

$$s(\omega)(t) := \left[\frac{\omega}{\mathrm{d}f}\,\Big|_{X_t}\right]$$

where $\frac{\omega}{\mathrm{d}f}$ is the *Gelfand-Leray form* of the holomorphic differential form ω in $f_*\Omega^{n+1}_X$. The map $s : f_*\Omega^{n+1}_X \longrightarrow i_*\mathscr{H}^n$ factors through the *Brieskorn lattice*

$$\mathscr{H}'' := f_*\Omega^{n+1}_X/\mathrm{d}f \wedge \mathrm{d}\big(f_*\Omega^{n-1}_X\big)$$

with image in $\mathscr{G}$ inducing an isomorphism $\mathscr{H}''|_{T'} \cong \mathscr{H}^n$ (Bri70). The *Leray residue formula* implies that

$$\partial_t s\big([\mathrm{d}f \wedge \eta]\big) = s\big([\mathrm{d}\eta]\big). \tag{3}$$

This formula will allow us to compute the action of ∂_t on the Brieskorn lattice. Since $\mathscr{H}''$ is a free $\mathscr{O}_T$-module of rank μ (Seb70),

$$\mathscr{H}''_0 = \Omega^{n+1}_{X,0}/\mathrm{d}f \wedge \mathrm{d}\Omega^{n-1}_{X,0}$$

is a torsion free $\mathbb{C}\{t\}$-module and hence $s : \mathscr{H}''_0 \longrightarrow \mathscr{G}_0$ is an inclusion. We identify $\mathscr{H}''_0$ with its image in $\mathscr{G}_0$. By B. Malgrange (Mal74, Lem. 4.5), the growth of geometrical sections towards 0 is bounded by

$$\mathscr{H}''_0 \subset V^{>-1}. \tag{4}$$

This will lead to estimations necessary for the computation.

8. Microlocal structure

The isomorphism $\partial_t : V^{>0} \xrightarrow{\sim} V^{>-1}$ induces an action of ∂_t^{-1} on the Brieskorn lattice. This action extends to the *microlocal structure* of the Brieskorn lattice and will be the key to an efficient computation.

The ring of *microdifferential operators* with constant coefficients

$$\mathbb{C}\{\{\partial_t^{-1}\}\} := \Big\{\sum_{k\geq 0} a_k\partial_t^{-k} \in \mathbb{C}[[\partial_t^{-1}]]\,\Big|\,\sum_{k\geq 0}\frac{a_k}{k!}t^k \in \mathbb{C}\{t\}\Big\}$$

is a discrete valuation ring and $t^\alpha\mathbb{C}\{t\}$, $\alpha \in \mathbb{Q}$, are free $\mathbb{C}\{\{\partial_t^{-1}\}\}$-modules of rank 1. For $\alpha > -1$, we identify $\partial_t t : C_\alpha \longrightarrow C_\alpha$ with $(\alpha+1) - \frac{N}{2\pi\mathrm{i}}$ via ψ_α. Then $\partial_t t \circ t^{\frac{N}{2\pi\mathrm{i}}} = (\alpha+1)t^{\frac{N}{2\pi\mathrm{i}}}$ and $\det t^{\frac{N}{2\pi\mathrm{i}}} = t^{\mathrm{tr}\,\frac{N}{2\pi\mathrm{i}}} = 1$. Hence,

$$\mathbb{C}\{t\}C_\alpha \cong t^\alpha\mathbb{C}\{t\}^{\dim_{\mathbb{C}} C_\alpha}$$

as $\mathbb{C}\{t\}[\partial_t t]$-modules and $\mathbb{C}\{t\}C_\alpha$ is a free $\mathbb{C}\{\{\partial_t^{-1}\}\}$-module of rank $\dim_{\mathbb{C}} C_\alpha$. In particular, V^α (resp. $V^{>\alpha}$) is a free $\mathbb{C}\{\{\partial_t^{-1}\}\}$-module of rank μ for $\alpha > -1$ (resp. $\alpha \geq -1$). Since $\partial_t^{-1}\mathscr{H}_0'' \subset \mathscr{H}_0''$ and $\mathscr{H}_0'' \subset V^{>-1}$,

$$\mathscr{H}_0'' \cong \mathbb{C}\{\{\partial_t^{-1}\}\}^\mu \tag{5}$$

is a free $\mathbb{C}\{\{\partial_t^{-1}\}\}$-module of rank μ. Note that

$$\mathscr{H}''/\partial_t^{-1}\mathscr{H}'' = \Omega_f := \Omega^{n+1}/\mathrm{d}f \wedge \Omega^n \cong \mathbb{C}\{x\}/(\partial_x f)$$

is the *Jacobian algebra*.

9. Singularity spectrum

We want to define the *singularity spectrum* which is an important invariant of the singularity coming from the *mixed Hodge structure* on the cohomology of the Milnor fibre.

The *Hodge filtration* F on $\mathscr{G}_0$ is defined by $F_k := F_k\mathscr{G}_0 := \partial_t^k\mathscr{H}''$ and ∂_t induces isomorphisms

$$\mathrm{Gr}_V^{\alpha+k}\mathscr{H}''/\partial_t^{-1}\mathscr{H}'' = \mathrm{Gr}_V^{\alpha+k}\mathrm{Gr}_0^F\mathscr{G}_0 \xrightarrow[\sim]{\partial_t^k} \mathrm{Gr}_k^F\mathrm{Gr}_V^\alpha\mathscr{G}_0 \cong \mathrm{Gr}_k^F C_\alpha.$$

The *singularity spectrum* $\mathrm{Sp} : \mathbb{Q} \longrightarrow \mathbb{N}$ defined by

$$\mathrm{Sp}(\alpha) := \dim_{\mathbb{C}}\mathrm{Gr}_V^\alpha\mathrm{Gr}_0^F\mathscr{G}_0$$

reflects the embedding of $\mathscr{H}_0''$ in $\mathscr{G}_0$ and has the *symmetry* property

$$\mathrm{Sp}(n-1-\alpha) = \mathrm{Sp}(\alpha). \tag{6}$$

Since $\mathscr{H}_0'' \subset V^{>-1}$, this implies that

$$V^{>-1} \supset \mathscr{H}_0'' \supset V^{n-1}$$

or equivalently that $\mathrm{Sp}(\alpha) = 0$ for $\alpha \leq -1$ or $\alpha \geq n$. This fact will be essential for the computation of the V-filtration on the Brieskorn lattice.

The *spectral numbers* $\alpha_1 \leq \cdots \leq \alpha_\mu$ are those α with *multiplicity* $\mathrm{Sp}(\alpha) > 0$ and their mean value is $\frac{n-1}{2}$. C. Hertling (Her01) conjectured that their *variance* is bounded by

$$\gamma := -\frac{1}{4}\sum_{i=1}^{\mu}\Big(\alpha_i - \frac{n-1}{2}\Big)^2 + \frac{\alpha_\mu - \alpha_1}{48}\mu \geq 0,$$

proved that $\gamma = 0$ for *quasihomogeneous singularities*, and gave the explicit formula

$$\gamma(T_{p,q,r}) = \frac{1}{24}\Big(1 - \frac{1}{p} - \frac{1}{q} - \frac{1}{r}\Big) \geq 0$$

for singularities of type $T_{p,q,r}$. M. Saito (Sai) proved the conjecture for *irreducible plane curve singularities*.

10. Algorithm

Based on E. Brieskorn's algebraic description of the Gauss-Manin connection (3), the microlocal structure of the Brieskorn lattice (5), B. Malgrange's result (4), and the symmetry of the spectral numbers (6), we describe an algorithm to compute the V-filtration on the Brieskorn lattice. We abbeviate $\Omega := \Omega_{X,0}$, $\mathscr{H}'' := \mathscr{H}''_0$, $\mathscr{G} := \mathscr{G}_0$, and $s := \partial_t^{-1}$.

10.1. Idea

First, note that the commutator $[s^{-2}t, s] = \partial_t^2 t \partial_t^{-1} - \partial_t t = 1$ and hence $\mathscr{G}$ is a $\mathbb{C}\{\{s\}\}[\partial_s]$-module with ∂_s-action defined by

$$\partial_s := s^{-2}t = \partial_t^2 t.$$

Let us indicate the advantages of the $\mathbb{C}\{\{s\}\}[\partial_s]$-structure compared to the $\mathbb{C}\{t\}[\partial_t]$-structure in E. Brieskorn's algorithm. Since $f^{n+1} \in \langle \partial_x f \rangle$ or equivalently $f^{n+1}\Omega^{n+1} \subset \mathrm{d}f \wedge \Omega^n$,

$$t^{n+1}\partial_t \mathscr{H}'' \subset \mathscr{H}''$$

and hence ∂_t has a t-pole of order of at most $n+1$ on $\mathscr{H}''$. But $s^2\partial_s = \partial_t^{-2}\partial_t^2 t = t$ implies that

$$s^2\partial_s \mathscr{H}'' \subset \mathscr{H}''$$

and hence ∂_s has only an s-pole of at most 2 on $\mathscr{H}''$. But actually the lower pole order does not simplify the computation since

$$\partial_t t = s^{-1}t = s^{-1}s^2\partial_s = s\partial_s.$$

The important point is that, in order to compute $\mathscr{H}''/t^K\mathscr{H}''$, one has to use (Bri70, Prop. 3.3) saying that for each K there is an N such that

$$\langle x \rangle^N \Omega^{n+1} \subset f^K \Omega^{n+1} + \mathrm{d}f \wedge \mathrm{d}\Omega^{n-1}.$$

An estimation for N in terms of K in not known to the author and we can only use linear algebra to compute $\mathscr{H}''/t^K\mathscr{H}''$. But

$$\mathscr{H}''/s\mathscr{H}'' = \Omega_f = \Omega^{n+1}/\mathrm{d}f \wedge \Omega^n \cong \mathbb{C}\{x\}/(\partial_x f)$$

is the Jacobian algebra which can be computed by standard basis methods.

We consider the $\mathbb{C}\{t\}$-lattices and $\mathbb{C}\{\{s\}\}$-lattices

$$\mathscr{H}''_k := \sum_{0 \le j \le k} (\partial_t t)^j \mathscr{H}'' = \sum_{0 \le j \le k} (s\partial_s)^j \mathscr{H}''.$$

Since $V^{>-1} \supset \mathscr{H}'' \supset V^{n-1}$, there is a minimal $k_\infty \le \mu n$ such that $\mathscr{H}''_k = \mathscr{H}''_{k_\infty} = \mathscr{H}''_\infty$ is the saturation of $\mathscr{H}''$ for all $k > k_\infty$. As remarked before, one can actually show that $k_\infty \le \mu - 1$.

For $N \geq n+1$, $V^{>-1} \supset \mathscr{H}'' \supset V^{n-1}$ implies that

$$V^{>-1} \supset \mathscr{H}''_\infty \supset \mathscr{H}'' \supset s\mathscr{H}'' \supset V^n \supset s^N V^{>-1} \supset s^N \mathscr{H}''_\infty$$

and $\mathscr{H}''_\infty$ and $s^N\mathscr{H}''_\infty$ are $\partial_t t$-invariant. Hence, $\partial_t t$ and $t\partial_t$ induce endomorphisms $\overline{\partial_t t}, \overline{t\partial_t} \in \operatorname{End}_\mathbb{C}\big(\mathscr{H}''_\infty / s^N \mathscr{H}''_\infty\big)$ such that the V-filtration $V_{\overline{t\partial_t}} = V^{\bullet+1}_{\overline{\partial_t t}}$ defined by $\overline{t\partial_t}$ on $\mathscr{H}''_\infty/s^N\mathscr{H}''_\infty$ induces the V-filtration on the subquotient $\mathscr{H}''/s\mathscr{H}'' = \operatorname{Gr}^F_0 \mathscr{G}$.

10.2. Computation

By the *finite determinacy theorem*, we may assume that $f \in \mathbb{C}[x]$ is a polynomial. Since $\mathbb{C}[x]_{(x)} \subset \mathbb{C}\{x\}$ is faithfully flat and all data will be defined over $\mathbb{C}[x]_{(x)}$, we may replace $\mathbb{C}\{x\}$ by $\mathbb{C}[x]_{(x)}$ and similarly $\mathbb{C}\{t\}$ by $\mathbb{C}[t]_{(t)}$ and $\mathbb{C}\{\{s\}\}$ by $\mathbb{C}[s]_{(s)}$ for the computation. With the additional assumption $f \in \mathbb{Q}[x]$, all data will be defined over $\mathbb{Q}$ and we can apply methods of computer algebra.

The computer algebra system Singular (GPS01; Sch96) provides *standard basis methods* with respect to *local monomial orderings* for computations over localizations of polynomial rings over $\mathbb{Q}$.

From a *standard basis* of a zero-dimensional ideal, one can compute a monomial $\mathbb{C}$-basis of the quotient by the ideal. In Singular, this is done by the commands `std` and `kbase`. Hence, one can compute a monomial $\mathbb{C}$-basis $m = (m_1, \ldots, m_\mu)^t$ of

$$\Omega_f = \Omega^{n+1}/\mathrm{d}f \wedge \Omega^n \cong \mathbb{C}\{x\}/(\partial_x f).$$

Since $\mathscr{H}''/s\mathscr{H}'' \cong \Omega_f$, m represents a $\mathbb{C}\{\{s\}\}$-basis of $\mathscr{H}''$ and a $\mathbb{C}(s)$-basis of $\mathscr{G}$ by *Nakayama's lemma*.

The matrix $A = \sum_{k\geq 0} A_k s^k$ of the operator t with respect to m is defined by $tm =: Am$. Note that t is not $\mathbb{C}\{\{s\}\}$-linear and A does not define the basis representation of t with respect to m just by matrix multiplication. But t is a differential operator and $t = s^2\partial_s$ implies that the basis representation of t with respect to m is given by

$$tgm = \big(gA + s^2\partial_s(g)\big)m$$

for $g = (g_1, \ldots, g_\mu) \in \mathbb{C}(s)^\mu$. If U is a $\mathbb{C}(s)$-basis transformation and A' the matrix of t with respect to $m' := Um$ then

$$A' = \big(UA + s^2\partial_s(U)\big)U^{-1}$$

is the basis transformation formula with respect to U.

A *reduced normal form* allows us to compute the projection to the quotient by a zero-dimensional ideal. In Singular, this is done by the command `reduce`. Hence, one can compute the projection to the upper summand in

$$\begin{matrix} \mathbb{C}\{x\}/(\partial_x f) \\ \oplus \\ (\partial_x f) \end{matrix} \cong \begin{matrix} \Omega_f \\ \oplus \\ \mathrm{d}f\wedge\Omega^n \end{matrix} \twoheadrightarrow \begin{matrix} \Omega_f \\ \oplus \\ \mathrm{d}f\wedge\Omega^n/\mathrm{d}f\wedge\mathrm{d}\Omega^{n-1} \end{matrix} \cong \begin{matrix} \mathscr{H}''/s\mathscr{H}'' \\ \oplus \\ s\mathscr{H}'' \end{matrix}.$$

A *syzygy computation* allows us to express elements as a linear combination of generators of a module. In SINGULAR, this is done by the command `division`. Hence, one can compute $\eta = \sum_{i=0}^{n}(-1)^i\eta_i \mathrm{d}x_0 \wedge \cdots \wedge \widehat{\mathrm{d}x_i} \wedge \cdots \wedge \mathrm{d}x_n$ from $\mathrm{d}f \wedge \eta = \sum_{i=0}^{n} \partial_{x_i}(f)\eta_i \mathrm{d}x_0 \wedge \cdots \wedge \mathrm{d}x_n$. Since $s^{-1}[\mathrm{d}f \wedge \eta] = \partial_t[\mathrm{d}f \wedge \eta] = [\mathrm{d}\eta]$, one can compute basis representations with respect to m inductively up to arbitrary order. Note that $t[\omega] = [f\omega]$ and A is the basis representation of tm with respect to m.

The basis representation H_k of $\mathscr{H}''_k$ with respect to m defined by $\mathscr{H}''_k =: H_k m$ can be computed inductively by

$$\begin{aligned} H'_0 &:= H_0 := \mathbb{C}\{\{s\}\}^\mu, \\ H'_{k+1} &:= \mathrm{jet}_{-1}\big(s^{-1}H'_k \,\mathrm{jet}_k(A) + s\partial_s H'_k\big), \\ H_{k+1} &:= H_k + H'_{k+1}. \end{aligned}$$

Note that H_{k+1} depends only on the k-jet of A and that the coefficients of H_k are in $s^{-k}\mathbb{C}\{\{s\}\}_{\leq k}$. A *normal form* allows us to test for module membership. In SINGULAR, this is done by the command `reduce`. Hence, one can check if $H_k = H_{k+1}$ to find k_∞. A *syzygy computation* gives a minimal set of generators of a module. In SINGULAR, this is done by the command `minbase`. Hence, one can compute a $\mathbb{C}\{\{s\}\}$-basis M of $H_{k_\infty} = H_\infty$ with

$$\delta(M) := \max\big\{\mathrm{ord}\big(M_{i_1,j_1}\big) - \mathrm{ord}\big(M_{i_2,j_2}\big) \big| M_{i_1,j_1} \neq 0 \neq M_{i_2,j_2}\big\} \leq k_\infty.$$

The matrix A' of t with respect to the $\mathbb{C}\{\{s\}\}$-basis Mm of $\mathscr{H}''_\infty$ is defined by $MA + s^2\partial_s M =: A'M$ and $\mathrm{jet}_k(A') = \mathrm{jet}_k(A'_{\leq k})$ for $A'_{\leq k}$ defined by

$$M\,\mathrm{jet}_{k+\delta(M)}(A) + s^2\partial_s M =: A'_{\leq k}M.$$

Note that $A'_{\leq k}$ depends only on a finite jet of A. Hence, one can compute A' up to arbitrary order and the basis representation

$$\overline{s^{-1}A' + s\partial_s} = \overline{s^{-1}A'_{\leq N} + s\partial_s} \in \mathrm{End}_{\mathbb{C}}\big(\mathbb{C}\{\{s\}\}^\mu / s^N\mathbb{C}\{\{s\}\}^\mu\big)$$

of $\overline{\partial_t t} \in \mathrm{End}_{\mathbb{C}}\big(\mathscr{H}''_\infty / \partial_t^{-N}\mathscr{H}''_\infty\big)$ with respect to Mm for arbitrary N.

As before, one can compute the basis representation H'' of $\mathscr{H}''$ with respect to Mm defined by $H_0 =: H''M$. Choosing $N \geq n+1$ as before, the V-filtration $V^{\bullet+1}_{\overline{s^{-1}A'_{\leq N}+s\partial_s}}$ defined by $\overline{s^{-1}A'_{\leq N} + s\partial_s}$ on $\mathbb{C}\{\{s\}\}^\mu / s^N\mathbb{C}\{\{s\}\}^\mu$ induces the V-filtration $V^{\bullet+1}_{\overline{s^{-1}A'_{\leq N}+s\partial_s}}(H''/sH'')$ on the subquotient H''/sH'' and

$$V^{\bullet+1}_{\overline{s^{-1}A'_{\leq N}+s\partial_s}}(H''/sH'')M$$

is the basis representation of the V-filtration $V(\mathscr{H}''/s\mathscr{H}'')$ on $\mathscr{H}''/s\mathscr{H}''$ with respect to m.

The matrix of $\overline{s^{-1}A'_{\leq N} + s\partial_s}$ with respect to the canonical $\mathbb{C}$-basis

$$\begin{pmatrix} 1 & & s & & s^2 & & & s^{N-1} & \\ & \ddots & & \ddots & & \ddots & \cdots & & \ddots & \\ & & 1 & & s & & s^2 & & & s^{N-1} \end{pmatrix}^t$$

of $\mathbb{C}\{\{s\}\}^\mu/s^N\mathbb{C}\{\{s\}\}^\mu$ is given by

$$\begin{pmatrix} A'_1 & A'_2 & A'_3 & A'_4 & \cdots & A'_N \\ & A'_1+1 & A'_2 & A'_3 & \cdots & A'_{N-1} \\ & & A'_1+2 & A'_2 & \cdots & A'_{N-2} \\ & & & A'_1+3 & \ddots & \vdots \\ & & & & \ddots & A'_2 \\ & & & & & A'_1+N-1 \end{pmatrix}$$

where $A' = \sum_{k\geq 0} A'_k s^k$.

Since the eigenvalues of A'_1 are rational by the monodromy theorem, they can be computed using *univariate factorization*. In Singular, this can be done using the commands `det` and `factorize`. From the eigenvalues of $\overline{s^{-1}A'_{\leq N} + s\partial_s}$, one can compute $V^{\bullet+1}_{\overline{s^{-1}A'_{\leq N}+s\partial_s}}(H''/sH'')$ using methods of linear algebra. To compute only the spectrum, one can use its symmetry to simplify the computation. In Singular, one can use the command `syz` to compute kernels and hence generalized eigenspaces and the commands `intersect`, `reduce`, and `std` for modules with constant coefficients to compute intersections, quotients, and bases of vector spaces.

10.3. Extensions

We indicate two possible extensions of our algorithm.

The *V-filtration on the Jacobian algebra* is defined by the V-filtration and the action of the Jacobian algebra $\mathbb{C}\{x\}/(\partial_x f)$ on Ω_f by multiplication and can be computed from the V-filtration on Ω_f.

After a Jordan decomposition of the residue on H_∞, one can use the basis transformation formula to replace H_∞ by a non-resonant lattice with the same properties. Then $\exp(-2\pi \mathrm{i} A'_1)$ is a monodromy matrix and the *weight filtration* is defined by the nilpotent part of A'_1 on the graded parts of H_∞. Hence, one can compute the monodromy and the *spectral pairs*.

10.4. Implementation

The Singular library `gaussman.lib` (Sch01a) contains an implementation of the algorithm to compute the V-filtration on the Brieskorn lattice based on the following pseudocode:

proc vfiltration($f \in \mathbb{Q}[x]$) $\equiv$
  $m :=$ basis(Ω_f);
  $w := fm$;
  $A := 0$;
  $H'' := 0$;
  $H := \mathbb{C}\{\{s\}\}^\mu$;

$H' := H;$
$k := -1;$
$K := 0;$
while $k < K \ \vee \ H'' \neq H$ do
  $Cm := w \mod \mathrm{d}f \wedge \Omega^n;$
  $k := k + 1;$
  $A := A + Cs^k;$
  if $H'' \neq H$ then
    $H'' := H;$
    $H' := \mathrm{jet}_{-1}\big(s^{-1}H'A + s\partial_s H'\big);$
    $H := H + H';$
    if $H'' = H$ then
      $M :=$ basis$(H'');$
      $K :=$ delta$(M) + n + 1;$
    fi
  fi
  if $k < K \ \vee \ H'' \neq H$ then $w := \mathrm{d}\big((w - Cm)/\mathrm{d}f\big)$ fi
od;
$A'M := MA + s^2\partial_s M;$
$H''M := \mathbb{C}\{\{s\}\}^\mu;$
$V^{\bullet+1}_{s^{-1}A'+s\partial_s}(H''/sH'')M, m.$

10.5. Example

We use the Singular library gaussman.lib (Sch01a) to compute an example. First, we have to load the library:

```
> LIB "gaussman.lib";
```

Then we define the ring $R := \mathbb{Q}[x,y]_{(x,y)}$ and the polynomial $f = x^5 + x^2y^2 + y^5 \in R$:

```
> ring R=0,(x,y),ds;
> poly f=x5+x2y2+y5;
```

Note that f defines a singularity of type $T_{2,5,5}$. Finally, we compute the V-filtration of the singularity defined by f on Ω_f:

```
> list l=vfiltration(f);
> print(matrix(l[1]));
-1/2,-3/10,-1/10,0,1/10,3/10,1/2
> l[2];
1,2,2,1,2,2,1
> l[3];
[1]:
   _[1]=gen(11)
[2]:
   _[1]=gen(10)
```

```
   _[2]=gen(6)
[3]:
   _[1]=gen(9)
   _[2]=gen(4)
[4]:
   _[1]=gen(5)
[5]:
   _[1]=gen(8)
   _[2]=gen(3)
[6]:
   _[1]=gen(7)
   _[2]=gen(2)
[7]:
   _[1]=gen(1)
> print(matrix(l[4]));
y5,y4,y3,y2,xy,y,x4,x3,x2,x,1
```

The result is a list with the following entries: The first contains the spectral numbers, the second, the corresponding multiplicities, the third, $\mathbb{C}$-bases of the graded parts of the V-filtration on Ω_f in terms of the monomial $\mathbb{C}$-basis in the fourth entry. In the third entry, $gen(i)$ repesents the i-th unit vector. A monomial $x^\alpha y^\beta$ in the fourth entry is considered as $[x^\alpha y^\beta \mathrm{d}x \wedge \mathrm{d}y] \in \Omega_f$.

The result is presented in the following table.

Table 1: V-filtration of $f = x^5 + x^2y^2 + y^5$ on Ω_f

α	$-\frac{1}{2}$	$-\frac{3}{10}$	$-\frac{1}{10}$	0	$\frac{1}{10}$	$\frac{3}{10}$	$\frac{1}{2}$
$\mathrm{Gr}_V^\alpha\, \Omega_f/[\mathrm{d}x \wedge \mathrm{d}y]$	$\langle 1 \rangle$	$\langle x, y \rangle$	$\langle x^2, y^2 \rangle$	$\langle xy \rangle$	$\langle x^3, y^3 \rangle$	$\langle x^4, y^4 \rangle$	$\langle y^5 \rangle$

10.6. Application

We use the Singular library `gaussman.lib` (Sch01a) to compute the spectrum and the γ-invariant for Milnor number $\mu \leq 16$ following the classification in (AGZV88). C. Hertling's conjecture, saying that $\gamma \geq 0$, holds for quasihomogeneous singularities and singularities of type $T_{p,q,r}$ (Her01) and for irreducible plane curve singularities (Sai). Our results presented in the following table prove the conjecture for singularities with Milnor number $\mu \leq 16$. Most of the spectra computed occur already in the list of spectra of unimodal and bimodal singularities in (AGZV85).

The computation was done on a Pentium II 350 with Linux operating system. The choice of monomial ordering has a strong influence on the computation time.

Table 2: Spectrum and γ-invariant for Milnor number $\mu \leq 16$

singularity	polynomial	singularity spectrum	γ-invariant	computation time/s
$Z_{1,1}$	$y^8 + x^2y^3 + x^3y$	$-\frac{4}{7}, -\frac{7}{16}, -\frac{5}{16}, -\frac{2}{7}, -\frac{3}{16}, -\frac{1}{7}, -\frac{1}{16}, 0, 0, \frac{1}{16}, \frac{1}{7}, \frac{3}{16}, \frac{2}{7}, \frac{5}{16}, \frac{7}{16}, \frac{4}{7}$	$\frac{1}{384}$	35
$W_{1,1}$	$y^7 + x^2y^3 + x^4$	$-\frac{7}{12}, -\frac{3}{7}, -\frac{1}{3}, -\frac{2}{7}, -\frac{1}{6}, -\frac{1}{7}, -\frac{1}{12}, 0, 0, \frac{1}{12}, \frac{1}{7}, \frac{1}{6}, \frac{2}{7}, \frac{1}{3}, \frac{3}{7}, \frac{7}{12}$	$\frac{1}{336}$	2
$W^{\#}_{1,1}$	$x^4 + 2x^2y^3 + xy^5 + y^6$	$-\frac{7}{12}, -\frac{11}{26}, -\frac{9}{26}, -\frac{7}{26}, -\frac{5}{26}, -\frac{3}{26}, -\frac{1}{12}, -\frac{1}{26}, \frac{1}{26}, \frac{1}{12}, \frac{3}{26}, \frac{5}{26}, \frac{7}{26}, \frac{9}{26}, \frac{11}{26}, \frac{7}{12}$	$\frac{7}{1872}$	8
$Q_{2,1}$	$yz^2 + y^7 + x^2y^2 + x^3$	$-\frac{1}{12}, \frac{1}{14}, \frac{3}{14}, \frac{1}{4}, \frac{1}{3}, \frac{5}{14}, \frac{5}{12}, \frac{1}{2}, \frac{7}{12}, \frac{9}{14}, \frac{2}{3}, \frac{3}{4}, \frac{11}{14}, \frac{13}{14}, \frac{13}{12}$	$\frac{1}{336}$	36
$Q_{2,2}$	$yz^2 + y^8 + x^2y^2 + x^3$	$-\frac{1}{12}, \frac{1}{16}, \frac{3}{16}, \frac{1}{4}, \frac{5}{16}, \frac{1}{3}, \frac{5}{12}, \frac{7}{16}, \frac{9}{16}, \frac{7}{12}, \frac{2}{3}, \frac{11}{16}, \frac{3}{4}, \frac{13}{16}, \frac{15}{16}, \frac{13}{12}$	$\frac{7}{1152}$	10
$S_{1,1}$	$yz^2 + y^6 + x^2z + x^2y^2$	$-\frac{1}{10}, \frac{1}{12}, \frac{1}{5}, \frac{1}{4}, \frac{3}{10}, \frac{2}{5}, \frac{5}{12}, \frac{1}{2}, \frac{7}{12}, \frac{3}{5}, \frac{7}{10}, \frac{3}{4}, \frac{4}{5}, \frac{11}{12}, \frac{11}{10}$	$\frac{1}{288}$	26
$S_{1,2}$	$yz^2 + y^7 + x^2z + x^2y^2$	$-\frac{1}{10}, \frac{1}{14}, \frac{1}{5}, \frac{3}{14}, \frac{3}{10}, \frac{5}{14}, \frac{2}{5}, \frac{1}{2}, \frac{1}{2}, \frac{3}{5}, \frac{9}{14}, \frac{7}{10}, \frac{11}{14}, \frac{4}{5}, \frac{13}{14}, \frac{11}{10}$	$\frac{1}{140}$	31
$S^{\#}_{1,1}$	$x^2z + yz^2 + y^3z + xy^4$	$-\frac{1}{10}, \frac{1}{11}, \frac{2}{11}, \frac{3}{11}, \frac{3}{10}, \frac{4}{11}, \frac{5}{11}, \frac{1}{2}, \frac{6}{11}, \frac{7}{11}, \frac{7}{10}, \frac{8}{11}, \frac{9}{11}, \frac{10}{11}, \frac{11}{10}$	$\frac{1}{220}$	211
$S^{\#}_{1,2}$	$z^2y + zx^2 + zy^3 + y^3x^2$	$-\frac{1}{10}, \frac{1}{12}, \frac{1}{6}, \frac{1}{4}, \frac{3}{10}, \frac{1}{3}, \frac{5}{12}, \frac{1}{2}, \frac{1}{2}, \frac{7}{12}, \frac{2}{3}, \frac{7}{10}, \frac{3}{4}, \frac{5}{6}, \frac{11}{12}, \frac{11}{10}$	$\frac{13}{1440}$	144
$U_{1,1}$	$y^2z^2 + xz^2 + xy^3 + x^3$	$-\frac{1}{9}, \frac{1}{10}, \frac{1}{5}, \frac{2}{9}, \frac{3}{10}, \frac{2}{5}, \frac{4}{9}, \frac{1}{2}, \frac{5}{9}, \frac{3}{5}, \frac{7}{10}, \frac{7}{9}, \frac{4}{5}, \frac{9}{10}, \frac{10}{9}$	$\frac{11}{2160}$	46
$U_{1,2}$	$y^4z + xz^2 + xy^3 + x^3$	$-\frac{1}{9}, \frac{1}{11}, \frac{2}{11}, \frac{2}{9}, \frac{3}{11}, \frac{4}{11}, \frac{4}{9}, \frac{5}{11}, \frac{6}{11}, \frac{5}{9}, \frac{7}{11}, \frac{8}{11}, \frac{7}{9}, \frac{9}{11}, \frac{10}{11}, \frac{10}{9}$	$\frac{1}{99}$	1628
$V_{1,1}$	$yx^2 + z^4 + z^2y^2 + y^5$	$-\frac{1}{8}, \frac{1}{10}, \frac{1}{8}, \frac{1}{4}, \frac{3}{10}, \frac{3}{8}, \frac{1}{2}, \frac{1}{2}, \frac{5}{8}, \frac{7}{10}, \frac{3}{4}, \frac{7}{8}, \frac{9}{10}, \frac{9}{8}$	$\frac{1}{240}$	7
$V^{\#}_{1,1}$	$yx^2 + z^3y + y^4 + z^3x$	$-\frac{1}{8}, \frac{1}{9}, \frac{1}{8}, \frac{2}{9}, \frac{1}{3}, \frac{3}{8}, \frac{3}{8}, \frac{4}{9}, \frac{5}{9}, \frac{5}{8}, \frac{5}{8}, \frac{2}{3}, \frac{7}{9}, \frac{7}{8}, \frac{8}{9}, \frac{9}{8}$	$\frac{5}{864}$	9

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