Computing Hodge-theoretic invariants of singularities

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Abstract. We describe the V-filtration on the Brieskorn lattice of an isolated hypersurface singularity. We give an overview of invariants which can be extracted from it and describe an algorithm to compute it.

1. Introduction

Let $Y = V(F) \subset I\!P^{n+1}$ be a smooth hypersurface of degree d. By the exact sequence

$$0 \to H^{n+1}(I\!P^{n+1} \setminus Y, C) \to H^n(Y, C) \to H^{n+2}(I\!P^{n+1}, C) \to 0$$

the primitive cohomology of Y in degree n is identified with the cohomology of its complement in projective space. On the other hand this group can be described as the space of rational differential $n+1$-forms on $I\!P^{n+1}$ with poles only along Y modulo exact forms, by Grothendieck's algebraic de Rham theorem [5]. According to Griffiths [4], this space is filtered by the order of pole of representatives along X and the resulting filtration on $H^n(Y, C)$ is its *Hodge filtration.*

Varying the equation of Y we obtain a holomorphic vector bundle $\mathcal{H}^n$ on the space U of nonsingular hypersurfaces, together with a holomorphically varying filtration $\{\mathcal{F}^p\}$ by the Hodge bundles. The bundle $\mathcal{H}^n$ is equipped with the *Gauss-Manin connection*

$$\nabla : \mathcal{H}^n \to \Omega^1_U \otimes \mathcal{H}^n$$

which is integrable and satisfies the *Griffiths transversality condition*

$$\nabla(\mathcal{F}^p) \subset \Omega^1_U \otimes \mathcal{F}^{p-1}.$$

Now consider a hypersurface Y_0 which may be singular. We put it into a family $f : \mathcal{Y} \to S$ where S is the unit disc in the complex plane, and $Y_0 = f^{-1}(0)$. We assume that 0 is the only critical value of f. Then the cohomology groups of the fibers of f form a local system over the punctured disc, and we have a monodromy transformation which is quasi-unipotent by the monodromy theorem. By passing to a ramified covering of the disc we may assume that the monodromy of the family is in fact unipotent. Let $u_1, \ldots, u_m$ be a basis for the group of multivalued horizontal sections of $R^n f_* \mathbb{Z}_{\mathcal{Y}}$ (modulo torsion). Write $N = \log T$, and let $\tilde{u}_i = \exp(-2\pi i N) u_i$. Then $\tilde{u}_1, \ldots, \tilde{u}_m$ is a basis of sections for $\mathcal{H}^n$ over $S \setminus \{0\}$ and we can extend $\mathcal{H}^n$ to a vectorbundle over the whole of S by taking these sections as a frame.

For a complex vector space V of finite dimension equipped with a nilpotent endomorphism N and a given integer n one defines the *weight filtration of N centered at n* as the unique increasing filtration $\{W_k V\}$ of V with the properties that $N(W_k) \subset W_{k-2}$ for each k and the induced maps $N^k : \mathrm{Gr}^W_{n+k} V \to \mathrm{Gr}^W_{n-k} V$ are isomorphisms for all $k > 0$.

The results of W. Schmid [14] imply that

1. The Hodge bundles extend to holomorphic subbundles $\tilde{\mathcal{F}}^p$ of $\tilde{\mathcal{H}}^n$;

2. the connection extends to a logarithmic connection

$$\nabla : \tilde{\mathcal{H}}^n \to \Omega^1_S(\log 0) \otimes \tilde{\mathcal{H}}^n$$

   whose residue at 0 is nilpotent (and can be identified with N);

3. the vector space $\tilde{\mathcal{H}}^n(0)$ with its Hodge filtration $\{\tilde{\mathcal{F}}^p(0)\}$, its integral lattice given by the values at 0 of the sections $\tilde{u}_1, \ldots, \tilde{u}_m$ and the filtration $W(N, n)$ is a mixed Hodge structure, polarized by N in a suitable sense;

4. the semisimple part T_s of the monodromy (before we had passed to a finite covering) acts as an automorphism of this mixed Hodge structure.

The latter mixed Hodge structure is called the *limit mixed Hodge structure* of the family.

Suppose that we have an isolated hypersurface singularity

$$f : (\mathbb{C}^{n+1}, 0) \to (\mathbb{C}, 0).$$

The coordinates on $\mathbb{C}^{n+1}$ are denoted as $(z_0, \ldots, z_n)$ and t is the coordinate on the target $\mathbb{C}$. As f is finitely determined, we may represent it by a polynomial of arbitrarily high degree d. By results of Brieskorn [2] and Scherk [13] we may assume that f has moreover the following properties:

1. Let $F(Z_0, \ldots, Z_{n+1}) = Z^d_{n+1} f(Z_0/Z_{n+1}, \ldots, Z_n/Z_{n+1})$ be the usual homogenization of f. Then the hypersurface $Y_0 = V(F)$ in $\mathbb{P}^{n+1}$ has a unique singular point at $x = [0, \ldots, 0, 1]$.

2. Let $Y_t = V(F - tZ_{n+1}^d)$, and let X_t be the intersection of Y_t with a sufficiently small ball around x, i.e. a Milnor fibre of f at x. Then the restriction map $H^n(Y_t, C) \to H^n(X_t, C)$ is surjective.

One may consider this restriction map "in the limit", i.e. with the limit mixed Hodge structure of the family as its source. Then under the given conditions, this identifies the cohomology of the Milnor fibre, now denoted by $H^n(X_\infty, C)$, with the quotient of the limit mixed Hodge structure, which we denote by $H^n(Y_\infty, C)_0$, by the part which is invariant under the monodromy. This equips $H^n(X_\infty, C)$ with a mixed Hodge structure, such that its weight filtration is $W(N, n)$ on the part on which T acts with eigenvalues different from 1 and is $W(N, n+1)$ on the unipotent part.

Let us explain this in some more detail. One has the exact *specialization sequence*

$$0 \to H^n(Y_0) \to H^n(Y_t) \to H^n(X_t) \to 0$$

where the first map is the composition of the isomorphism $H^n(Y_0) \simeq H^n(\mathcal{Y})$ (valid because the inclusion of Y_0 into $\mathcal{Y}$ is a homotopy equivalence) with the restriction map $H^n(\mathcal{Y}) \to H^n(Y_t)$. By the *local invariant cycle theorem* the image of $H^n(Y_0) \to H^n(Y_t)$ is equal to the part fixed by the monodromy.

It is this mixed Hodge structure on $H^n(X_t)$ with its action of T_s which contains a wealth of invariants of the singularity. It has first been introduced by the second author in [17].

The discrete invariants of the mixed Hodge structures are coded into the spectrum (more strongly: the spectral pairs). The computation of this spectrum was possible in many cases where one disposed of an embedded resolution of the singularity. A first description of a mixed Hodge structure without reference to a resolution was given by Varchenko [19]. This description was translated into the language of $\mathcal{D}$-modules by Scherk and the second author [13] and was the starting point of M. Saito's theory of mixed Hodge modules [10, 11]. In this paper we describe a genuine algorithm for the computation of the spectrum and the spectral pairs, based on the approach via $\mathcal{D}$-modules. The algorithm can also be used to calculate invariants which are closely related to the spectrum, such as the geometric genus [9] and the irregularity [18]. It has been implemented by the first author in SINGULAR.

Hertling has recently formulated an intriguing conjecture concerning the *variance* of the spectrum, see his paper [6] in this volume. It has been verified by Dimca [3] for weighted homogeneous polynomials, and by M. Saito (unpublished) for irreducible plane curve singularities.

There exists also a spectrum for polynomials, as contrasted to germs. The contribution of Dimca to this volume [3] gives interesting analogues between the local and global cases.

2. The Brieskorn lattice and its V-filtration

We consider an isolated hypersurface singularity $f : (\mathbb{C}^{n+1}, 0) \to (\mathbb{C}, 0)$. We abbreviate

$$\mathcal{O} := \mathcal{O}_{\mathbb{C},0},\ \mathcal{D} := \mathcal{D}_{\mathbb{C},0} = \mathcal{O}_{\mathbb{C},0}[\partial_t],\ \Omega^p := \Omega^p_{\mathbb{C}^{n+1},0}.$$

Since f is an isolated singularity, the *Milnor module*

$$\Omega_f := \Omega^{n+1}/\mathrm{d}f \wedge \Omega^n$$

has finite $\mathbb{C}$-dimension equal to the *Milnor number*

$$\mu := \mu_f := \dim_{\mathbb{C}} \Omega_f.$$

2.1. The Gauss-Manin system

By [12], the *Brieskorn lattices*

$$\begin{aligned} \mathcal{H}' &:= \mathcal{H}'_f := \mathrm{d}f \wedge \Omega^n / \mathrm{d}f \wedge \mathrm{d}\Omega^{n-1}, \\ \mathcal{H}'' &:= \mathcal{H}''_f := \Omega^{n+1}/\mathrm{d}f \wedge \mathrm{d}\Omega^{n-1} \end{aligned}$$

are free $\mathcal{O}$-modules of rank μ and $\mathcal{H}' \subset \mathcal{H}''$. Note that $\Omega_f = \mathcal{H}''/\mathcal{H}'$.

We have an operator $\partial_t : \mathcal{H}' \to \mathcal{H}''$ which is defined as follows. For $x \in \mathcal{H}'$ choose $\eta \in \Omega^n$ such that $\mathrm{d}f \wedge \eta$ represents x, and define $\partial_t(x)$ to be the class of $\mathrm{d}\eta$ in $\mathcal{H}''$. Then $\partial_t(gx) = g'x + g\partial_t(x)$ for $g \in \mathcal{O}$ so ∂_t is a differential operator of degree one.

The modules $\mathcal{H}'$ and $\mathcal{H}''$ are lattices in the so-called *Gauss-Manin system* of f. This is a $\mathcal{D}$-module which is defined in the following way.

By [13, p. 645], the complex $\Omega[D]$ with differential $\mathbf{d}$ defined by

$$\mathbf{d}(\omega D^i) := \mathrm{d}\omega D^i - \mathrm{d}f \wedge \omega D^{i+1}$$

is a complex of $\mathcal{D}$-modules with $\mathcal{D}$-action

$$\begin{aligned} \partial_t \omega D^i &:= \omega D^{i+1}, \\ t\omega D^i &:= f\omega D^i - i\omega D^{i-1}. \end{aligned}$$

The $\mathcal{D}$-module

$$\mathcal{H} := \mathcal{H}_f := \mathrm{H}^{n+1}(\Omega[D], \mathbf{d}) = \Omega^{n+1}[D]/\mathbf{d}\Omega^n[D].$$

is called the *Gauss-Manin system of f*.

2.2. Monodromy

Let $k : C \to C$ be given by $k(s) = \exp(s)$. We have the inclusion $\mathcal{O} \hookrightarrow (k_*\mathcal{O}_C)_0 =: \mathcal{A}$, and the $\mathcal{D}$-action on $\mathcal{O}$ extends to a $\mathcal{D}$-action on $\mathcal{A}$ by

$$\partial_t g := \exp(-s)\partial_s g, \ tg := \exp(s)g$$

and induces an action of $\mathcal{D}$ on the space of *microfunctions*

$$\mathcal{M} := \mathcal{A}/\mathcal{O}.$$

We have an automorphism M of $\mathcal{A}$ given by $\mathrm{M}g := g(s - 2\pi i)$ and which is called the *monodromy operator*. It is the identity on $\mathcal{O}$ and hence induces an automorphism of $\mathcal{M}$.

For any $\mathcal{D}$-module $\mathcal{E}$ we have its *solution space* $E(\mathcal{E}) := \mathrm{Hom}_{\mathcal{D}}(\mathcal{E}, \mathcal{A})$ and its *microsolution space* $F(\mathcal{E}) := \mathrm{Hom}_{\mathcal{D}}(\mathcal{E}, \mathcal{M})$. These carry also monodromy operators.

The analytic monodromy operator M_f on $E(\mathcal{H}_f)$ is called the *monodromy operator of f*. One has a natural identification of $E(\mathcal{H}_f)$ with $H^n(X_\infty, \mathbb{C})$ such that M_f corresponds to the monodromy T from the introduction, so by the monodromy theorem, the eigenvalues of M_f are roots of unity and the Jordan blocks of M_f have size at most $n + 1$.

2.3. Regularity

A finitely generated $\mathcal{D}$-module $\mathcal{E}$ is called *regular* if there exists a free $\mathcal{O}$-submodule $\mathcal{F}$ of $\mathcal{E}$ of finite rank which generates $\mathcal{E}$ as a $\mathcal{D}$-module (i.e. an *$\mathcal{O}$-lattice in $\mathcal{E}$*) and which is stable under the operator $t\partial_t$. This is the case if there exists a free finite rank $\mathcal{O}$-submodule $\mathcal{F}_0$ of $\mathcal{E}$ which generates $\mathcal{E}$ as a $\mathcal{D}$-module and such that the sequence $\mathcal{F}_0 \subset \mathcal{F}_1 \subset \cdots$ defined by $\mathcal{F}_k := \mathcal{F}_{k-1} + t\partial_t \mathcal{F}_{k-1}$ becomes stationary after a finite number of steps. The $\mathcal{O}$-module $\mathcal{F}_k$ for k sufficiently large is then called the *saturation* of $\mathcal{F}_0$.

By [2], the module $\mathcal{H}$ is regular. Moreover the operator ∂_t is invertible on $\mathcal{H}$ by [13, 3.5]. By the classification of regular $\mathcal{D}$-modules (cf. [1]) this implies

PROPOSITION 2.1. *There is an isomorphism of $\mathcal{D}$-modules*

$$\mathcal{H} \cong \bigoplus_{j=1}^{s} \mathcal{D}/\mathcal{D}(t\partial_t - a_j)^{r_j},$$

where $-1 < a_j \le 0$ and $\left(\exp(-2\pi \mathrm{i} a_j)\right)_{1 \le j \le s}$ are the eigenvalues and $(r_j)_{1 \le j \le s}$ the corresponding Jordan block sizes of the monodromy operator M_f.

Note that $a_j \in Q$ and $r_j \leq n+1$ for $1 \leq j \leq s$ by the monodromy theorem.

2.4. V-FILTRATION

For a regular $\mathcal{D}$-module $\mathcal{E}$ with quasi-unipotent monodromy there exists a unique collection $(V^a\mathcal{E})_{a\in Q}$ of lattices in $\mathcal{E}$ such that

1. $V^a\mathcal{E} \subset V^b\mathcal{E}$ if $a > b$;
2. each $V^a\mathcal{E}$ is stable by $t\partial_t$;
3. the induced action of $t\partial_t - a$ on $\mathrm{Gr}_a^V\mathcal{E} := V^a\mathcal{E}/\bigcup_{b>a} V^b\mathcal{E}$ is nilpotent.

We use the notation $V^{>a}\mathcal{E} = \bigcup_{b>a} V^b\mathcal{E}$. If $C^b := \bigcup_{r>0} \ker(t\partial_t - b)^r$, then

$$V^a\mathcal{E} = \sum_{b\geq a} \mathcal{O}C^b \text{ and } V^a\mathcal{E} = C^a \oplus V^{>a}\mathcal{E}.$$

In this way we get a V-filtration on $\mathcal{H}$, which induces one on $\mathcal{H}'$ and $\mathcal{H}''$ by intersection. Note that ${\partial_t}^k V^a\mathcal{H} = V^{a-k}\mathcal{H}$ for all $k \in \mathbb{Z}$ and $tV^a\mathcal{H} \subset V^{a+1}\mathcal{H}$. The induced V-filtration on the subquotient $\mathcal{H}''/{\partial_t}^{-1}\mathcal{H}'' = \Omega_f$ of $\mathcal{H}$ is given by

$$\begin{aligned} V^a\Omega_f &= (V^a\mathcal{H} \cap \mathcal{H}'' + \mathcal{H}')/\mathcal{H}', \\ V^{>a}\Omega_f &= (V^{>a}\mathcal{H} \cap \mathcal{H}'' + \mathcal{H}')/\mathcal{H}'. \end{aligned}$$

2.5. THE HODGE FILTRATION

Following [8, p. 160], we give the following definition. For $k \geq 0$, let

$$F_k\Omega^{n+1}[D] := \bigoplus_{i=0}^{k} \Omega^{n+1}D^i$$

and $F_k\mathcal{H}$ be the image of $F_k\Omega^{n+1}[D]$ under the canonical map

$$\Omega^{n+1}[D] \to \Omega^{n+1}[D]/\mathbf{d}\Omega^n[D] = \mathcal{H}.$$

Moreover we put $F_k\mathcal{H} = 0$ for $k < 0$. This defines a filtration F on $\mathcal{H}$ called the *Hodge filtration*. It is a *good filtration* in the sense that it turns $\mathcal{H}$ into a graded $\mathcal{D}$-module. Here $\mathcal{D}$ carries the filtration, also denoted by F, by the order of the differential operators:

$$F_k\mathcal{D} = \{\sum_{i=0}^{k} g_i(t){\partial_t}^i\}.$$

The associated graded ring $\mathrm{Gr}^F\mathcal{D}$ is isomorphic to $\mathcal{O}[\xi]$ where ξ is the symbol of ∂_t.

We summarize properties of the Hodge filtration in the

PROPOSITION 2.2.

1. *$F_k\mathcal{H} = {\partial_t}^k F_0\mathcal{H}$ for $k \geq 0$. [13, 3.4]*
2. *$\mathcal{H}' = {\partial_t}^{-1}F_0\mathcal{H} \subset F_0\mathcal{H} = \mathcal{H}''$ [8, p. 160-161]*

The induced Hodge filtration on the subquotient $\mathrm{Gr}_V^a\mathcal{H} = C^a$ of $\mathcal{H}$ is given by

$$F_kC^a = (F_k\mathcal{H} \cap V^a\mathcal{H} + V^{>a}\mathcal{H})/V^{>a}\mathcal{H}.$$

2.6. Singularity spectrum

The singularity spectrum is defined in terms of the V-filtration on Ω_f as follows: for $b \in \mathbb{Q}$ let $d(b) := \dim_{\mathbb{C}} \mathrm{Gr}_V^b\Omega_f$. We put

$$\mathrm{Sp}(f) := \sum_{b\in\mathbb{Q}} d(b)(b) \in \mathbb{Z}[\mathbb{Q}]$$

where the latter is the integral group ring of the additive group of the rational numbers. It is called the *singularity spectrum of f*.

By [13, 7.3 (i)], we have the

PROPOSITION 2.3.

1. *$d(b) \neq 0$ implies that $-1 < b < n$.*
2. *$d(n-1-b) = d(b)$*

Proposition 2.3 leads to a description of the singularity spectrum in terms of the Hodge filtration on C^a.

COROLLARY 2.4. *For $-1 < a \leq 0$ and $b = a + k$, ${\partial_t}^k$ induces an isomorphism*

$$\mathrm{Gr}_V^b\Omega_f \simeq \mathrm{Gr}_k^F C^a.$$

Proof. For $k \geq 0$, we have

$$\begin{aligned}
\mathrm{Gr}_k^F C^a &= (V^a\mathcal{H} \cap F_k\mathcal{H} + V^{>a}\mathcal{H})/(V^a\mathcal{H} \cap F_{k-1}\mathcal{H} + V^{>a}\mathcal{H}) \\
&= (V^a\mathcal{H} \cap F_k\mathcal{H})/(V^a\mathcal{H} \cap F_{k-1}\mathcal{H} + V^{>a}\mathcal{H} \cap F_k\mathcal{H}) \\
&= {\partial_t}^k(V^b\mathcal{H} \cap F_0\mathcal{H})/{\partial_t}^k(V^b\mathcal{H} \cap {\partial_t}^{-1}F_0\mathcal{H} + V^{>b}\mathcal{H} \cap F_0\mathcal{H})
\end{aligned}$$

by Proposition 2.2 and

$$\begin{aligned}\mathrm{Gr}_V^b\Omega_f &= (F_0\mathcal{H}\cap V^b\mathcal{H}+\partial_t{}^{-1}F_0\mathcal{H})/(F_0\mathcal{H}\cap V^{>b}\mathcal{H}+\partial_t{}^{-1}F_0\mathcal{H})\\ &= (V^b\mathcal{H}\cap F_0\mathcal{H})/(V^b\mathcal{H}\cap\partial_t{}^{-1}F_0\mathcal{H}+V^{>b}\mathcal{H}\cap F_0\mathcal{H}).\end{aligned}$$

For $k<0$, we have $\mathrm{Gr}_k^F C^a=0$ by definition and $\mathrm{Gr}_V^b\Omega_f=0$ by proposition 2.3.

Remark. In [13], the space $\mathrm{Gr}_k^F C^a$ is identified with the eigenspace of the semisimple monodromy for the eigenvalue $\exp(-2\pi i a)$ inside the p-th graded part for the Hodge filtration on the cohomology of the Milnor fibre of f.

The following corollary will be the key to compute the singularity spectrum.

COROLLARY 2.5. $V^{>-1}\mathcal{H}\supset\mathcal{H}''\supset V^{n-1}\mathcal{H}$

Proof. Since $d(b)=0$ for $b\geq n$, we have $V^n\mathcal{H}\subset\partial_t{}^{-1}\mathcal{H}''$ and hence

$$V^{n-1}\mathcal{H}=\partial_t V^n\mathcal{H}\subset\mathcal{H}''.$$

Since $d(b)=0$ for $b\leq -1$, we also have $\mathcal{H}''\subset V^{>-1}\mathcal{H}$.

2.7. Microlocalization

We define

$$C\{\{s\}\}:=\Big\{\sum_{i\geq 0}a_is^i\in C[\![s]\!]\Big|\sum_{i\geq 0}\frac{a_i}{i!}t^i\in C\{t\}\Big\}.$$

Then $C\{\{\partial_t{}^{-1}\}\}$ is the ring of *microdifferential operators* with constant coefficients. By [13, Lem. 8.3], we have the

LEMMA 2.6. *For $a\neq -1,-2,\ldots$, $([(t\partial_t-a)^j])_{0\leq j<r}$ is a basis of $\mathcal{D}/\mathcal{D}(t\partial_t-a)^r$ as a $C\{\{\partial_t{}^{-1}\}\}[\partial_t]$-module.*

According to [7, Prop. 2.5] we have

PROPOSITION 2.7. *$\mathcal{H}''$ is a free $C\{\{\partial_t{}^{-1}\}\}$-module of rank μ and*

$$\mathcal{H}=\mathcal{H}''\otimes_{C\{\{\partial_t{}^{-1}\}\}}C\{\{\partial_t{}^{-1}\}\}[\partial_t].$$

From now on we abbreviate $s:=\partial_t{}^{-1}$. Note that $\mathcal{H}'=\partial_t{}^{-1}\mathcal{H}''=s\mathcal{H}''$ so $\Omega_f=\mathcal{H}''/s\mathcal{H}''$. Moreover, the operator $s^{-2}t$ satisfies

$$[s^{-2}t,s]=\partial_t{}^2t\partial_t{}^{-1}-\partial_t t=1$$

so $\mathcal{H}$ becomes a $C\{\{s\}\}[\partial_s]$-module if ∂_s acts as $s^{-2}t = \partial_t{}^2 t$. Note that

$$s\partial_s = s^{-1}t = \partial_t t$$

hence we see that the saturation sequence of $\mathcal{H}''$ as an $\mathcal{O}$-module coincides with the saturation sequence of $\mathcal{H}''$ as a $C\{\{s\}\}$-module! However, the latter is much easier to compute, for the following reason.

By a result of Briançon, $f^{n+1}\Omega^{n+1} \subset \mathrm{d}f \wedge \Omega^n$, and this exponent is sometimes really the minimal one. This implies that in general

$$\partial_t \mathcal{H}'' \subset t^{-n-1}\mathcal{H}''$$

i.e. ∂_t has a pole of order at most $n+1$ on $\mathcal{H}''$. However, if we consider the microlocal structure on $\mathcal{H}''$, then we observe that $\partial_s \mathcal{H}'' = s^{-2}t\mathcal{H}'' \subset s^{-2}\mathcal{H}''$, so ∂_s has at most a pole of order two on $\mathcal{H}''$!

3. Algorithms

For $g = \sum_{i\in\mathbb{Z}} g_i s^i \in C\{\{s\}\}[s^{-1}]$ and integers n_1, n_2 we put

$$\mathrm{jet}_{n_1}^{n_2}(g) = \sum_{i=n_1}^{n_2} g_i s^i$$

where $s = \partial_t{}^{-1}$. We abbreviate

$$\mathrm{jet}_{n_1} := \mathrm{jet}_{n_1}^{\infty},\ \ \mathrm{jet}^{n_2} := \mathrm{jet}_{-\infty}^{n_2}.$$

3.1. Operator t

In this subsection, we explain how to compute the operator t in a $C\{\{\partial_t{}^{-1}\}\}$-basis of $\mathcal{H}''$.

Let $m = (m_1, \ldots, m_\mu)^t$ represent a section $v \in \mathrm{Hom}_C(\Omega_f, \mathcal{H}'')$ of $\pi : \mathcal{H}'' \to \Omega_f$. It induces an isomorphism $C\{\{s\}\}^\mu \cong \mathcal{H}''$ by Nakayama's lemma. We consider m as a $C\{\{s\}\}$-basis of $\mathcal{H}''$ and a $C\{\{s\}\}[s^{-1}]$-basis of $\mathcal{H}$. We define the matrix

$$A = \sum_{k\geq 0} A_k s^k \in \mathrm{Mat}(\mu, C\{\{s\}\})$$

of the operator t with respect to m by

$$tm =: Am.$$

Since $[t, s] = s^2$, we have

$$tg = gt + s^2\partial_s(g)$$

for $g = \sum_{k=0}^{\infty} g^k s^k \in C\{\{s\}\}[s^{-1}]$. Hence, for $g \in C\{\{s\}\}[s^{-1}]^\mu$, we have

$$\begin{aligned} tgm &= gtm + s^2\partial_s(g)m \\ &= (gA + s^2\partial_s(g))m \end{aligned}$$

and

$$A + s^2\partial_s : C\{\{s\}\}[s^{-1}]^\mu \to C\{\{s\}\}[s^{-1}]^\mu$$

is the basis representation of t with respect to m.

If $U \in \mathrm{GL}(\mu, C\{\{s\}\}[s^{-1}])$ is a change of basis for the module $\mathcal{H}''$ and $A' \in \mathrm{Mat}(\mu, C\{\{s\}\})$ the matrix of the operator t with respect to $m' := Um$ then

$$\begin{aligned} Am' &= tm' \\ &= tUm \\ &= Utm + s^2\partial_s Um \\ &= UAm + s^2\partial_s Um \\ &= (UA + s^2\partial_s(U))U^{-1}m' \end{aligned}$$

and hence

$$A' = (UA + s^2\partial_s(U))U^{-1}.$$

Since f is an isolated singularity, we may assume $\deg f < \infty$ by the finite determinacy theorem and replace $C\{z\}$ by $C[z]_{(z)}$ for the computation. By definition, we have

$$\Omega_f = \Omega^{n+1}/\mathrm{d}f \wedge \Omega^n$$

and $\mathrm{d}f \wedge \Omega^n = J(f)\Omega^{n+1}$ where $J(f)$ is the Jacobian ideal of f. Hence, one can compute a monomial $C\{\{s\}\}$-basis m of $\mathcal{H}''$ using standard basis methods.

In the following Lemma 3.1 and Proposition 3.2, we show how to compute basis representations with respect to m.

LEMMA 3.1. *For $\omega \in \Omega^{n+1}$ with $[\omega] \in s\mathcal{H}''$, one can compute $\omega' \in \Omega^{n+1}$ such that $[\omega] = s[\omega']$.*

Proof. We have

$$\mathcal{H}''/s\mathcal{H}'' = \Omega^{n+1}/\mathrm{d}f \wedge \Omega^n = \Omega^{n+1}/J(f)\Omega^{n+1}.$$

Since $[\omega] \in s\mathcal{H}''$, we have $\omega \in J(f)\Omega^{n+1}$ and one can compute $\omega'' \in \Omega^n$ such that

$$\omega = \mathrm{d}f \wedge \omega''$$

using standard basis methods. Then we have

$$[\omega] = [\mathrm{d}f \wedge \omega'] = s[\mathrm{d}\omega''] = s[\omega']$$

where $\omega' := \mathrm{d}\omega'' \in \Omega^{n+1}$.

PROPOSITION 3.2. *For a $C\{\{s\}\}$-basis m of $\mathcal{H}''$, one can compute basis representations with respect to m up to arbitrarily high order.*

Proof. Let $h \in \mathcal{H}''$ and $K \geq 0$. By induction, one can compute $v_k \in C^\mu$, $0 \leq k < K$, and $h_K \in \mathcal{H}''$ such that

$$h - \sum_{k=0}^{K-1} v_k s^k m = s^K h_K.$$

Since m represents a monomial C-basis of

$$\mathcal{H}''/s\mathcal{H}'' = \Omega^{n+1}/\mathrm{d}f \wedge \Omega^n = \Omega^{n+1}/(\partial_z f)\Omega^{n+1},$$

one can compute $v_K \in C^\mu$ such that

$$h_K - v_K m \in s\mathcal{H}''$$

using standard basis methods. By Lemma 3.1, one can compute $h_{K+1} \in \mathcal{H}''$ such that

$$h_K - v_K m = s h_{K+1}.$$

Then

$$h - \sum_{k=0}^{K} v_k s^k m = s^{K+1} h_{K+1}$$

and $v = \sum_{k=0}^{K} v_k s^k$ is the basis representation of h with respect to m up to order K.

By the proof of Lemma 3.1 and Proposition 3.2, one can compute the matrix A of the operator t with respect m up to order K using the

ALGORITHM 1.

```
proc tmat(f, m, K) ≡
  w := fm;
  A := 0;
  k := −1;
  while k < K do
        C(m mod df ∧ Ω^n) := w mod df ∧ Ω^n;
        k := k + 1;
        A := A + Cs^k;
        if k < K then
                  w := d((w − Cm)/df);
        fi
  od;
  A.
```

3.2. Monodromy operator

In this subsection, we give an algorithm to compute the eigenvalues of the monodromy operator M_f.

Let $\mathcal{L} \subset \mathcal{H}$ be a $C\{t\}$-lattice. Then

$$\begin{aligned}\mathcal{L}_0 &:= \mathcal{L},\\ \mathcal{L}_{k+1} &:= \mathcal{L}_k + \partial_t t \mathcal{L}_k = \mathcal{L}_k + s\partial_s \mathcal{L}_k\end{aligned}$$

defines an increasing sequence of $C\{t\}$-lattices

$$\mathcal{L}_0 \subseteq \mathcal{L}_1 \subseteq \mathcal{L}_2 \subseteq \cdots .$$

Note that

$$\mathcal{L}_k = \sum_{j=0}^{k} (\partial_t t)^j \mathcal{L}.$$

If $\mathcal{L}$ is a $C\{\{s\}\}$-lattice then also $\mathcal{L}_k$ is a $C\{\{s\}\}$-lattice. Since $\mathcal{H}$ is regular, the *saturation*

$$\mathcal{L}_\infty := \sum_{j \geq 0} (\partial_t t)^j \mathcal{L}$$

of $\mathcal{L}$ is a $C\{t\}$-lattice. Hence,

$$\mathcal{L}_\infty = \mathcal{L}_k$$

for some $k \geq 0$. If $\mathcal{L} = \mathcal{L}_\infty$ or equivalently

$$\partial_t t \mathcal{L} \subset \mathcal{L}$$

then $\mathcal{L}$ is called *saturated*. Note that $\mathcal{L}_\infty$ is saturated. If $\mathcal{L}$ is saturated then $\partial_t t$ induces an endomorphism

$$\overline{\partial_t t} \in \mathrm{End}_{\mathbb{C}}(\mathcal{L}/t\mathcal{L}).$$

If $\mathcal{L}$ is saturated and a $C\{\{s\}\}$-lattice then

$$t\mathcal{L} \subset \partial_t^{-1} \mathcal{L} = s\mathcal{L}$$

and hence $t\mathcal{L} = s\mathcal{L}$. Note that this holds for $\mathcal{H}_\infty$.

The following well known fact [15] will allow us to compute the eigenvalues of monodromy.

PROPOSITION 3.3. *If $\mathcal{L} \subset \mathcal{H}$ is a saturated $C\{t\}$-lattice and*

$$\overline{\partial_t t} \in \mathrm{End}_{\mathbb{C}}(\mathcal{L}/t\mathcal{L})$$

the endomorphism induced by $\partial_t t$ on $\mathcal{L}/t\mathcal{L}$ then the spectrum of the map $\exp(2\pi \mathrm{i} \overline{\partial_t t})$ is the spectrum of the monodromy operator M_f.

For the computation, we replace $C\{\{s\}\}$ by $C[s]_{(s)}$ and use standard basis methods. First, we compute a monomial C-basis m of Ω_f. The matrix A of t with respect to m given by

$$tm = Am.$$

We replace $\mathcal{H}''$, $\partial_t{}^{-1}$, and t by the basis representations $H := C[s]_{(s)}^{\mu}$, s, and $A + s^2\partial_s$ with respect to m. Since

$$H \subset H_k \subset s^{-k}H,$$

the lattices H_k are given by

$$\begin{aligned} \delta_0 H &:= H_0 = H, \\ \delta_{k+1} H &:= \big((s^{-1}A^{\mathrm{jet}^k} + s\partial_s)\delta_k H\big)^{\mathrm{jet}^{-1}}, \\ H_{k+1} &= H_k + \delta_{k+1} H. \end{aligned}$$

We compute sets of generators of the lattices H_k and check if $H_k = H_{k+1}$ until $H_k = H_\infty$. Then we compute a basis m' of H_∞. The matrix of t with respect to m' is given by

$$(A + s^2\partial_s)m' = A'm'.$$

We define

$$\delta(m') := \max\{\mathrm{ord}\big((m'_{i_1})_{j_1}\big) - \mathrm{ord}\big((m'_{i_2})_{j_2}\big) \,|\, (m'_{i_1})_{j_1} \neq 0 \neq (m'_{i_2})_{j_2}\}$$

and the matrices $\mathrm{jet}^k A'$ by

$$(A^{\mathrm{jet}^{k+\delta(m')}} + s^2\partial_s)m' =: (\mathrm{jet}^k A')m'.$$

Note that $\delta(m') \leq k$ if $H_\infty = H_k$. The matrix of $A^{\mathrm{jet}_{k+\delta(m')}}m'$ with respect to m' has order at least k and hence

$$(A')^{\mathrm{jet}^k} = (\mathrm{jet}^k A')^{\mathrm{jet}^k}.$$

Finally, we compute

$$\overline{s^{-1}A' + s\partial_s} = \big(\mathrm{jet}^1 A'\big)_1 \in \mathrm{End}_{\mathbb{C}}\big(C[s]_{(s)}^{\mu}/sC[s]_{(s)}^{\mu}\big).$$

One can compute the eigenvalues of monodromy using the

ALGORITHM 2.

```
proc monospec(f) ≡
  m := basis(Ω_f);
  w := fm;
```

```
A := 0;
H' := 0;
H := C[s]^μ_(s);
δH := H;
k := −1;
K := 0;
while k < K ∨ H' ≠ H do
        C(m  mod df ∧ Ω^n) := w  mod df ∧ Ω^n;
        k := k + 1;
        A := A + Cs^k;
        if H' ≠ H then
                        H' := H;
                        δH := ((s^{-1}A + s∂_s)δH)^{jet^{-1}};
                        H := H + δH;
                        if H = H' then
                                        m' := basis(H');
                                        K := delta(m') + 1;
                        fi
        fi
        if k < K ∨ H' ≠ H then w := d((w − Cm)/df) fi
od;
A'm' := (A + s^2∂_s)m;
spectrum(A'_1).
```

3.3. V-filtration

In this subsection, we give an algorithm to compute the V-filtration on Ω_f. Since

$$V^{>-1}\mathcal{H} \supset \mathcal{H}'' \supset V^{n-1}\mathcal{H},$$

we have

$$V^{>0}\mathcal{H} \supset {\partial_t}^{-1}\mathcal{H}'' \supset V^n\mathcal{H},$$

$$V^{>-1}\mathcal{H} \supset \mathcal{H}''_\infty \supset V^{n-1}\mathcal{H}.$$

We choose $N \geq 0$ such that

$${\partial_t}^{-N}\mathcal{H}''_\infty \subset V^n\mathcal{H}.$$

It suffices to choose $N \geq n+1$. Then

$$\mathcal{H}''_\infty/{\partial_t}^{-N}\mathcal{H}''_\infty = \bigoplus_{a<n}(C^a \cap \mathcal{H}''_\infty) \oplus (V^n\mathcal{H}/{\partial_t}^{-N}\mathcal{H}''_\infty)$$

and we obtain induced endomorphisms $\overline{\partial_t t}, \overline{t\partial_t} \in \mathrm{End}_C(\mathcal{H}''_\infty/{\partial_t}^{-N}\mathcal{H}''_\infty)$ with

$$\ker\big(\overline{\partial_t t} - (a+1)\big)^{n+1} = \ker\big(\overline{t\partial_t} - a\big)^{n+1} = C^a \cap \mathcal{H}''_\infty$$

for $a < n$ and

$$\ker\big(\overline{\partial_t t} - (a+1)\big)^{n+1} = \ker\big(\overline{t\partial_t} - a\big)^{n+1} \subset V^n\mathcal{H}/{\partial_t}^{-N}\mathcal{H}''_\infty$$

for $a \geq n$. We choose $N \geq k_\infty + 1$ where

$$k_\infty := \min\{k \geq 0 \mid \mathcal{H}''_k = \mathcal{H}''_\infty\}.$$

Then $\partial_t t$ preserves the outer terms of the flag

$${\partial_t}^{-N}\mathcal{H}''_\infty \subset {\partial_t}^{-1}\mathcal{H}'' \subset \mathcal{H}'' \subset \mathcal{H}''_\infty$$

and, since

$${\partial_t}^{-1}\mathcal{H}''/{\partial_t}^{-N}\mathcal{H}''_\infty \supset V^n\mathcal{H}/{\partial_t}^{-N}\mathcal{H}''_\infty,$$

the V-filtration $V_{\overline{t\partial_t}} = V_{\overline{\partial_t t}}^{\bullet+1}$ defined by $\overline{t\partial_t}$ on $\mathcal{H}''_\infty/{\partial_t}^{-N}\mathcal{H}''_\infty$ induces the V-filtration on the subquotient $\mathcal{H}''/{\partial_t}^{-1}\mathcal{H}'' = \Omega_f$.

For the computation, we replace $C\{\{s\}\}$ by $C[s]_{(s)}$ and use standard basis methods. As in subsection 3.2, we compute the matrix $\mathrm{jet}^N A'$ such that

$$\overline{s^{-1}A' + s\partial_s} = \overline{s^{-1}\mathrm{jet}^N A' + s\partial_s} \in \mathrm{End}_{\mathbb{C}}\big(C[s]^\mu_{(s)}/s^N C[s]^\mu_{(s)}\big).$$

Then we compute the V-filtration $V^{\bullet+1}_{\overline{s^{-1}A'+s\partial_s}}$, a basis of the lattice $H' \subset C[s]^\mu_{(s)}$ defined by

$$H'm' := H,$$

and the induced V-filtration on the subquotient H'/sH'.

One can compute the V-filtration on Ω_f using the

ALGORITHM 3.

```
proc vfilt(f) ≡
  m := basis(Ω_f);
  w := fm;
  A := 0;
  H' := 0;
  H := C[s]^μ_(s);
  δH := H;
```

```
k := −1;
K := 0;
while k < K ∨ H' ≠ H do
        C(m mod df ∧ Ω^n) := w mod df ∧ Ω^n;
        k := k + 1;
        A := A + Cs^k;
        if H' ≠ H then
                H' := H;
                δH := ((s^{-1}A + s∂_s)δH)^{jet^{-1}};
                H := H + δH;
                if H = H' then
                        m' := basis(H');
                        K := delta(m') + max{k, n} + 1;
                fi
        fi
        if k < K ∨ H' ≠ H then w := d((w − Cm)/df) fi
od;
A'm' := (A + s^2∂_s)m;
H'm' := C[s]^μ_{(s)};
V^{•+1}_{s^{-1}A'+s∂_s}(H'/sH').
```

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