A COHOMOLOGICAL INTERPRETATION OF DERIVATIONS ON GRADED ALGEBRAS

XIA LIAO AND MATHIAS SCHULZE

Abstract. We trace derivations through Demazure's correspondence between a finitely generated positively graded normal k-algebras A and normal projective k-varieties X equipped with an ample $\mathbb{Q}$-Cartier $\mathbb{Q}$-divisor D. We obtain a generalized Euler sequence involving a sheaf on X whose space of global sections consists of all homogeneous k-linear derivations of A and a sheaf of logarithmic derivations on X.

Contents

1. Introduction

As an elementary motivation for the topic of this article, we recall the Euler sequence on projective space $X = \mathbb{P}^n_k$ over a field k. Setting $\mathscr{M} := \mathscr{O}_X(1)^{n+1}$, it reads

$$0 \to \mathscr{O}_X \to \mathscr{M} \to \Theta_X \to 0. \tag{1.1}$$

For any $d \in \mathbb{Z}$, $H^0(X, \mathscr{M} \otimes \mathscr{O}_X(d))$ computes the homogeneous k-linear derivations of degree d of the section ring $A = \bigoplus_{d\geq 0} H^0(X, \mathscr{O}_X(d))$. In fact, the section ring identifies with the polynomial ring $A = k[x_0, \ldots, x_n]$ and the module of k-linear derivations of A,

$$\mathrm{Der}_k(A) \cong \bigoplus_{i=0}^{n} A\partial_{x_i} \cong A(1)^{n+1},$$

is free of rank $n+1$. Therefore, $\widetilde{\mathrm{Der}_k(A)} \cong \mathscr{M}$ and $H^0(X, \mathscr{M} \otimes \mathscr{O}_X(d)) \cong \mathrm{Der}_k(A)_d$.

Date: September 19, 2017.
1991 *Mathematics Subject Classification.* 13N15, 14N05 (Primary), 14C20 (Secondary).
Key words and phrases. Euler sequence, graded algebra, graded linear series, derivation.
The research leading to these results has received funding from the People Programme (Marie Curie Actions) of the European Union's Seventh Framework Programme (FP7/2007-2013) under REA grant agreement n° PCIG12-GA-2012-334355.

There is the following generalization due to Wahl [Wah83a]. Let k be algebraically closed of characteristic 0, let X be a normal projective k-variety, and let $\mathscr{L} \cong \mathscr{O}_X(D)$ be the sheaf of sections of an ample line bundle L on X. Then

- there exists a coherent $\mathscr{O}_X$-module $\mathscr{M}$ and a short exact sequence (1.1),
- $\pi^*\mathscr{M} \cong \mathscr{D}\mathrm{er}_{L^{-1}}(-\log X)$, where $\pi : L^{-1} \to X$ is the projection, and X is identified with the zero section of the line bundle L^{-1},
- $H^0(X, \mathscr{M} \otimes \mathscr{L}^d)$ computes the homogeneous k-linear derivations of degree d of the graded normal section ring $\bigoplus_{i\geq 0} H^0(X, \mathscr{L}^i)$, and,
- if $\dim X \geq 2$ and $d < 0$, $H^0(X, \mathscr{M} \otimes \mathscr{L}^d) \cong H^0(X, \Theta_X \otimes \mathscr{L}^d)$.

Based on this result and a generalization of Zariski's lemma due to Mori and Sumihiro, Wahl gave a cohomological characterization of projective space: If X is smooth and $H^0(X, \Theta_X \otimes \mathscr{L}^{-1}) \neq 0$, then $(X, \mathscr{L}) \cong (\mathbb{P}^n_k, \mathscr{O}_X(1))$ or $(X, \mathscr{L}) \cong (\mathbb{P}^1_k, \mathscr{O}_X(2))$. From his investigations, Wahl derived the conjecture [Wah83b, Conj. 1.4] that normal graded isolated singularities do not admit negative degree derivations, at least for *some* choice of grading. In case of isolated complete intersection singularities (ICIS), where the grading is essentially unique, Aleksandrov [Ale85, §6] studied this conjecture. His results are affirmative if at least one defining equation has multiplicity at least 3, but he also gives counter-examples.

In this article, we give a cohomological interpretation of homogeneous derivations on an arbitrary finitely generated positively graded normal algebra over an arbitrary field k. Our approach relies on a construction due to Demazure reviewed in §2: To a k-algebra A under consideration, one associates an ample $\mathbb{Q}$-Cartier $\mathbb{Q}$-divisor D on $X = \operatorname{Proj} A$, unique up to adding a principal divisor, such that $\mathscr{O}_X(iD) \cong \mathscr{O}_X(i)$ (see (2.2)) for any $i \in \mathbb{Z}$. Conversely, given a normal projective k-variety X equipped with an ample $\mathbb{Q}$-Cartier $\mathbb{Q}$-divisor D, the section ring $A = \bigoplus_{i\in\mathbb{N}} H^0(X, \mathscr{O}_X(iD))$ is a finitely generated positively graded normal k-algebra. These two constructions are mutually inverse, hence any finitely generated positively graded normal k-algebra A takes the form $\bigoplus_{i\in\mathbb{N}} H^0(X, \mathscr{O}_X(iD))$ for an appropriate $\mathbb{Q}$-Cartier $\mathbb{Q}$-divisor D on $X = \operatorname{Proj} A$. Watanabe [Wat81] described the depth and the Cohen–Macaulay and Gorenstein properties of A in terms of D.

Theorem 1.1 (Generalized Euler Sequence)**.** *Let X be a normal projective k-variety with singular locus $Z := X^{\mathrm{sing}}$, D an ample $\mathbb{Q}$-Cartier $\mathbb{Q}$-divisor and A the section ring.*

(1) For any $d \in \mathbb{Z}$, there is a coherent reflexive $\mathscr{O}_X$-module $\mathscr{M}_{D,d}$ (see (3.2)) on X and divisors $W_{D,d}$ and $L_{D,d}$ (see Definition 4.5). They fit into an exact sequence

$$(1.2)\quad 0 \to \mathscr{O}_X(dD + W_{D,d}) \xrightarrow{\varphi} \mathscr{M}_{D,d} \xrightarrow{\psi} \mathscr{D}\mathrm{er}_X(dD - \log L_{D,d}) \to \mathscr{H}^2_Z(\mathscr{O}_X(dD + W_{D,d})),$$

where $\mathscr{H}^\bullet_Z$ is Grothendieck's local cohomology sheaf with support Z and $\mathscr{D}\mathrm{er}_X(dD - \log L_{D,d})$ is a certain sheaf of logarithmic derivations (see Definitions 3.5 and 4.5).

(2) If $\dim X \geq 1$, then the space of global sections $H^0(X, \mathscr{M}_{D,d})$ identifies with the space $\mathrm{Der}_k(A)_d$ of homogeneous degree-d k-linear derivations on A.

Proof. Proposition 5.1 proves (1) and Lemma 3.3 proves (2). □

Remark 1.2. (1) The hypotheses for §4 show that Z in Theorem 1.1 may be replaced by the locus where some irreducible component of D fails to be Cartier. However, the local cohomology sheaf in (1.2) is only a coarse estimate for the image of ψ in general.

(2) In case A is a graded normal surface singularity, the sequence (1.2) was described by Wahl [Wah88, (4.3) Thm.].

(3) In case D is a Cartier (integer) divisor, the sequence (1.2) reduces to the short exact Euler sequence of Wahl (see [Wah83a, Thm. 1.3]) which takes the form (1.1).

Alternatively it can be obtained by adapting our arguments in §4 (see Remark 4.4). Wahl proves surjectivity of ψ using a smoothness argument limited to the case where D is a Cartier divisor (see Remark 2.1).

(4) A particular case of the sequence (1.2) can be obtained by dualizing the sequence in the proof of [Fle81a, (8.9) Lem.]. There is also short exact Euler sequence for toric varieties (see [CLS11, Thm. 8.1.6]).

(5) There is a vast generalization of the Demazure correspondence due to Altmann, Hausen and Süss [AH06, AHS08].

Corollary 1.3. *Assume that* $\operatorname{ch} k = 0$*, that* A *is* (S_3)*, and that* X *is* (R_2) *of dimension* $\dim X \geq 2$*. Then there is a short exact sequence*

$$0 \longrightarrow \mathscr{O}_X(dD) \longrightarrow \mathscr{M}_{D,d} \longrightarrow \mathscr{D}\mathrm{er}_X(dD - \log L_{D,d}) \longrightarrow 0 \tag{1.3}$$

whose sequence of global sections is short exact as well. In particular, if $d < 0$*, then*

$$\mathrm{Der}_k(A)_d \cong H^0(X, \mathscr{M}_{D,d}) \cong H^0(X, \mathscr{D}\mathrm{er}_X(dD - \log L_{D,d})).$$

Proof. The proof is given in Proposition 5.3. □

Corollary 1.4. *If* X *is regular,* $W_{D,d} = 0$*, and* $L_{D,d}$ *a free divisor in the sense of Saito* [Sai80]*, then the extension class of the short exact sequence* (1.3) *is given by* $\varepsilon_D \in H^1(X, \Omega^1_X(\log L_{D,d}))$ *(see* (5.9)*).*

Proof. The proof is given in Proposition 5.4. □

We present two examples:

In §6.1, we consider an ICIS counter-example to Wahl's conjecture found by Michel Granger and the second author (see [GS14]). We describe the associated projective variety X and its singularities and apply our results to interpret a degree--1 derivation as a twisted vector field on X.

In §6.2, we study the degree--1 derivations on $k[x, y, z]/\langle xy - z^2\rangle$ with x, y, z all of degree 1. Notice that if $\mathrm{ch}(k) \neq 2$ there is no such derivation, while if $\mathrm{ch}(k) = 2$, ∂_z is one such derivation. We will give a cohomological explanation of this phenomenon.

Acknowledgements. We are grateful to Jonathan Wahl for stimulating questions and useful hints.

2. Demazure's construction

Demazure [Dem88] established a correspondence between

(A) finitely generated positively graded normal algebras A over a field k and
(B) normal projective k-varieties X equipped with an ample $\mathbb{Q}$-Cartier $\mathbb{Q}$-divisor defined up to ($\mathbb{Z}$-)linear equivalence.

In the following, we briefly summarize the main features of this correspondence.

(B)⇝(A). We start with an ample $\mathbb{Q}$-Cartier $\mathbb{Q}$-divisor D on a normal projective k-variety X. More specifically, this condition means that we can write D as a finite sum over prime Weil divisors V (codimension-1 subvarieties of X)

$$D = \sum_V r_V \cdot V, \quad r_V = \frac{p_V}{q_V} \in \mathbb{Q}, \tag{2.1}$$

where p_V and $q_V > 0$ are coprime integers for each V, and there exists a positive integer N such that ND is an ample Cartier (integer) divisor. For any k-variety X, let $K(X)$

denote the space of rational functions and $\mathscr{K}_X$ the corresponding constant sheaf. With $\lfloor - \rfloor$ being the coefficient-wise round-down, one defines

$$\mathscr{O}_X(D) := \mathscr{O}_X(\lfloor D \rfloor) \tag{2.2}$$

in the usual way as a coherent reflexive $\mathscr{O}_X$-submodule of $\mathscr{K}_X$. Consider the following quasicoherent graded $\mathscr{O}_X$-algebra

$$\mathscr{A} := \bigoplus_{i\in\mathbb{Z}} \mathscr{A}_i \supset \bigoplus_{i\in\mathbb{N}} \mathscr{A}_i =: \mathscr{A}^+. \tag{2.3}$$

defined by

$$\mathscr{A}_i := \mathscr{O}_X(iD)T^i.$$

The inclusion (2.3) yields an open embedding of cylinders

$$C := \operatorname{Spec}_X \mathscr{A} \hookrightarrow \operatorname{Spec}_X \mathscr{A}^+ =: C^+ \xrightarrow{\pi} X \tag{2.4}$$

over X whose complement is the image of the zero-section of $C^+ \to X$ defined by $\mathscr{A}^+ \to \mathscr{A}_0$, which is isomorphic to X (see [Dem88, 2.2 Lem.]).

Remark 2.1. In case D is a Cartier (integer) divisor, $\mathscr{O}_X(D)$ is the sheaf of sections of an ample line bundle L on X. The cylinder C^+ then becomes the total space of the line bundle L^{-1} (see [Wah83a, (1.2)]) and the morphism π is smooth. This was used by Wahl to construct the short exact sequence (1.1) in this case (see [Wah83a, Proof of Thm. 1.3]).

In general π is smooth if and only if it is flat with geometrically regular fibers. By [Dem88, 2.8 Prop.], the latter enforces $q_V = 1$ in (2.1). Moreover, flatness of $\mathscr{A}$ over $\mathscr{O}_X$ means that $\mathscr{A}_i \cong \mathscr{O}_X(iD)$ is locally free for all $i \in \mathbb{Z}$. Therefore smoothness of π occurs exactly in the case D is a Cartier (integer) divisor.

The desired positively graded k-algebra associated with X and D is

$$A := H^0(X, \mathscr{A}^+) = \bigoplus_{i\in\mathbb{N}} A_i, \quad A_i := H^0(X, \mathscr{O}_X(iD))T^i.$$

The restriction morphisms $A \to H^0(U, \mathscr{A}^+)$ glue to a canonical morphism

$$C^+ \to \operatorname{Spec} A =: \Gamma^+, \tag{2.5}$$

contracting the image of the zero-section of $C^+ \to X$ to the vertex $\mathfrak{m}$ of the cone Γ^+. It induces an isomorphism

$$C \cong \Gamma := \operatorname{Spec} A \setminus \{\mathfrak{m}\} \tag{2.6}$$

of the open cylinder with the open cone and, by passing to the $\mathbb{G}_m$-quotient, an isomorphism

$$X \cong \operatorname{Proj} A,$$

where A is finitely generated (see [Dem88, 3.2 Prop., 3.3 Prop.]). In particular, this gives an identification (see [Dem88, Cor.(3.2)]).

$$K(C) = K(X)(T) = Q(A). \tag{2.7}$$

The normality of Γ^+ and hence A follows using (2.6) from [Dem88, 2.7 Lem.] and [Wat81, Cor. (2.3)], together with Serre's normality criterion. Alternatively, it can be verified directly by checking that $\mathscr{A}$ is a sheaf of integrally closed rings.

(A)$\rightsquigarrow$(B). Assume now conversely given a finitely generated positively graded normal k-algebra A. Then one obtains a normal projective k-variety by setting

$$X := \operatorname{Proj} A.$$

Without loss of generality, we may assume that the degrees of the generators are coprime. For $i \in \mathbb{Z}$, denote as usual by

$$\mathscr{O}_X(i) := \widetilde{A(i)}$$

the coherent sheaf on X associated with the graded module $A(i)$. Taking $\mathscr{A}_i := \mathscr{O}_X(i)$, one obtains quasicoherent graded $\mathscr{O}_X$-algebras $\mathscr{A} \supset \mathscr{A}^+$ and cylinders $C \subset C^+$ as in (2.3) and (2.4). By assumption on the degrees of the generators of A, there is a homogeneous rational function $T \in Q(A)$ of degree 1. By the way C is defined, $T \in K(C)$ and the homogeneity of T means that the associated principal divisor $\operatorname{div}(T)$ on C is $\mathbb{G}_m$-stable. It then comes from a $\mathbb{Q}$-divisor D on X (see [Dem88, 3.5 Thm.]),

$$\operatorname{div}(T) = \pi^*(D).$$

A different choice of T, say T', will result in a different divisor D, say D', on X. However, $T/T' \in K(X)(T)$, being homogeneous of degree 0, implies that T/T' defines an $f \in K(X)$. Thus, D and D' differ by a principal divisor,

$$D = D' + \operatorname{div}(f). \tag{2.8}$$

From the normality of A, one deduces the following equalities (see [Dem88, 3.5 Thm.])

$$A_i = H^0(X, \mathscr{O}_X(iD))T^i, \quad \mathscr{O}_X(i) = \mathscr{O}_X(iD)T^i, \quad \text{for all } i \in \mathbb{Z}, \tag{2.9}$$

of subspaces of $K(C)$ and subsheaves of $\pi_*\mathscr{K}_C$ respectively. We denote the k-th Veronese subring of A by

$$A^{(k)} := \bigoplus_{i \in \mathbb{N}} A_{ik}.$$

Since A is finitely generated there exists an $N \geq 1$ such that, for all $i \geq 1$, $A^{(iN)}$ is generated over A_0 by $A_1^{(iN)} = A_{iN}$ (see [Bou98, Ch. III, §1, Prop. 3]). Combined with the second equality in (2.9), this shows that iND is a very ample divisor for those $i \geq 1$ which clear the denominators of ND.

3. Reflexive sheaves and rational derivations

We now discuss the basic idea of this paper. We adopt the setup of Demazure's construction (see §2) and use the following list of data freely in the sequel:

- a finitely generated positively graded normal k-algebra A with coprime degrees of generators,
- a corresponding ample $\mathbb{Q}$-Cartier $\mathbb{Q}$-divisor D on $X = \operatorname{Proj} A$,
- a homogeneous $T \in K(C)$ of degree 1 with $\operatorname{div}(T) = \pi^{-1}(D)$,
- the graded $\mathscr{O}_X$-algebra $\mathscr{A} = \bigoplus_{i\in\mathbb{Z}} \mathscr{O}_X(iD)T^i = \bigoplus_{i\in\mathbb{Z}} \mathscr{O}_X(i)$,
- the cylinder $\pi : C = \operatorname{Spec}_X(\mathscr{A}) \to X$ over X and the cone $\Gamma^+ = \operatorname{Spec} A$.

Recall that a coherent $\mathscr{O}_X$-module $\mathscr{F}$ is called reflexive if the canonical map

$$\mathscr{F} \to \mathscr{F}^{\vee\vee}$$

to the double-dual is an isomorphism. In other words, reflexive $\mathscr{O}_X$-modules are duals of coherent $\mathscr{O}_X$-modules (see [Har80, Cor. 1.2]). In particular, the tangent sheaf

$$\Theta_X := \mathscr{D}er_k\, \mathscr{O}_X = (\Omega^1_{X/k})^\vee \tag{3.1}$$

is a reflexive $\mathscr{O}_X$-module for any normal k-variety X. An $\mathscr{O}_X$-module $\mathscr{F}$ is called normal if the restriction map

$$\mathscr{F}(U) \to \mathscr{F}(U \setminus Y)$$

is bijective for all $U \subset X$ open and $Y \subset U$ closed of codimension ≥ 2.

Lemma 3.1 ([Har80, Prop. 1.6])**.** *A coherent $\mathscr{O}_X$-module $\mathscr{F}$ is reflexive if and only if it is torsion-free and normal.* □

The grading on $\pi_*\mathscr{O}_C = \bigoplus_{i\in\mathbb{Z}} \mathscr{O}_X(iD)T^i$ induces a grading on the $\mathscr{O}_X$-module

$$\mathscr{M}_D := \pi_*\Theta_C = \mathscr{D}\mathrm{er}_k\, \pi_*\mathscr{O}_C = \bigoplus_{d\in\mathbb{Z}} \mathscr{M}_{D,d}, \quad \mathscr{M}_{D,d} = (\pi_*\Theta_C)_d. \tag{3.2}$$

The following two lemmas establish the properties of $\mathscr{M}_{D,d}$ stated in Theorem 1.1.

Lemma 3.2. *The sheaves $(\pi_*\Theta_C)_d$ are coherent reflexive $\mathscr{O}_X$-modules.*

Proof. Since Θ_C is a coherent $\mathscr{O}_C$-module and π_* is exact (π is an affine morphism), $\pi_*\Theta_C$ is a coherent $\pi_*\mathscr{O}_C$-module. It follows that $(\pi_*\Theta_C)_d$ is coherent over $(\pi_*\mathscr{O}_C)_0 = \mathscr{O}_X$.

Let $y \in X$ be an arbitrary (closed or non-closed) point. By [Dem88, 2.5], $\pi^{-1}(y)^{\mathrm{red}} \cong \operatorname{Spec} \kappa(y)[t, t^{-1}]$ is 1-dimensional. In particular, this implies that $\operatorname{codim}_C \pi^{-1}(Y) = \operatorname{codim}_X Y$ for any closed subset $Y \subset X$. Thus, reflexivity of $(\pi_*\Theta_C)_d$ follows from reflexivity of Θ_C (see 3.1) using Lemma 3.1. □

Lemma 3.3. *If $\dim X \geq 1$, then $H^0(X, (\pi_*\Theta_C)_d) = (\mathrm{Der}_k\, A)_d$.*

Proof. Since A is a normal k-algebra, Γ^+ is normal affine k-variety (see (2.5)) and $\operatorname{codim}_{\Gamma^+}\{\mathfrak{m}\} = \dim(\Gamma^+) = \dim X + 1 \geq 2$ where $\mathfrak{m}$ is the vertex of Γ^+. By (3.1) and Lemma 3.1, Θ_{Γ^+} is normal and, using (2.6), there is a graded identification

$$\mathrm{Der}_k\, A = H^0(\Gamma^+, \Theta_{\Gamma^+}) = H^0(\Gamma, \Theta_\Gamma) = H^0(C, \Theta_C) = H^0(X, \pi_*\Theta_C). \qquad \square$$

The following lemma serves to define the maps in the generalized Euler sequence (1.2) in Theorem 1.1.

Lemma 3.4. *There is a canonically split short exact sequence*

$$\begin{array}{ccccccccc} 0 & \longrightarrow & \mathscr{K}_C & \xrightarrow{\Phi} & \mathscr{D}\mathrm{er}_k\, \mathscr{K}_C & \xrightarrow{\Psi} & \pi^*\, \mathscr{D}\mathrm{er}_k\, \mathscr{K}_X & \longrightarrow & 0 \\ & & 1 & \longmapsto & T\partial_T & & & & \end{array} \tag{3.3}$$

obtained as the localization of an underlying graded sequence.

Proof. Since T in (2.7) is transcendental over $K(X)$, $K(X) \subset K(C)$ a separable field extension. This yields a short exact sequence of $K(C)$-vector spaces (see [Mat89, Thm. 26.6])

$$0 \leftarrow \Omega_{K(C)/K(X)} \leftarrow \Omega_{K(C)/k} \leftarrow K(C) \otimes_{K(X)} \Omega_{K(X)/k} \leftarrow 0.$$

Its $K(C)$-dual reads

$$0 \to \mathrm{Der}_{K(X)}(K(C)) \to \mathrm{Der}_k(K(C)) \to K(C) \otimes_{K(X)} \mathrm{Der}_k(K(X)) \to 0. \tag{3.4}$$

Since $\Omega^1_{K(X)[T]/K(X)} = K(X)[T]dT$ and Kähler differentials commute with localization,

$$\begin{aligned} \mathrm{Der}_{K(X)}(K(C)) &= \mathrm{Hom}_{K(C)}(\Omega^1_{K(C)/K(X)}, K(C)) \\ &= \mathrm{Hom}_{K(C)}(K(C) \otimes_{K(X)[T^{\pm 1}]} \Omega^1_{K(X)[T^{\pm 1}]/K(X)}, K(C)) \\ &= K(C) \otimes_{K(X)[T^{\pm 1}]} \mathrm{Hom}_{K(C)}(\Omega^1_{K(X)[T^{\pm 1}]/K(X)}, K(X)[T^{\pm 1}]) \\ &= K(C) \otimes_{K(X)[T^{\pm 1}]} \mathrm{Der}_{K(X)}(K(X)[T^{\pm 1}]) \end{aligned}$$

is a localization of the graded module $\operatorname{Der}_{K(X)}(K(X)[T^{\pm 1}])$ which is free with with homogeneous basis $T\partial_T$ of degree 0. Similarly the entire sequence (3.4) is obtained by localizing the dual of the graded short exact sequence

$$0 \leftarrow \Omega_{K(X)[T]/K(X)} \leftarrow \Omega_{K(X)[T]/k} \leftarrow K(X)[T] \otimes_{K(X)} \Omega_{K(X)/k}. \tag{3.5}$$

Viewing all terms of the sequence (3.4) as constant sheaves yields the sequence (3.3).

The canonical splitting of the sequence (3.4) is induced the composition of the inclusions

$$\operatorname{Der}_k(K(X)) \hookrightarrow \operatorname{Der}_k(K(X)[T]) \hookrightarrow \operatorname{Der}_k(K(C)),$$

the first one defined by setting $\sigma(T) = 0$ for any $\sigma \in \operatorname{Der}_k(K(X))$, and the second one by extension to the fraction field. □

Now we are ready to explain the idea behind Theorem 1.1: By Lemma 3.3 and (3.2), any $\delta \in (\operatorname{Der}_k A)_d$ can be considered as a rational derivation $\delta \in H^0(X, \mathscr{D}\mathrm{er}_k\, \pi_*\mathscr{K}_C) = K(C)$ with the additional requirement that

$$\delta(\mathscr{O}_X(iD)T^i) \subset \mathscr{O}_X((i+d)D)T^{i+d}. \tag{3.6}$$

More explicitly, for a dense open subset U and its (dense open) preimage $V := \pi^{-1}(U)$, the restriction of δ to U in $H^0(U, \pi_*(\Theta_C)_d) \subset H^0(V, \Theta_C)$ can be viewed as a rational derivation $\delta \in \operatorname{Der}_k(K(C))$. By Lemma 3.4, δ decomposes uniquely as

$$\delta = \sigma + \alpha T\partial_T, \tag{3.7}$$

where $\alpha \in K(C)$ and $\sigma \in K(C) \otimes_{K(X)} \operatorname{Der}_k(K(X))$. The condition (3.6) imposes restrictions on the possible choices of σ and α in (3.7). We will investigate these options via a local computation in §4. The first two arrows in our generalized Euler sequence in Theorem 1.1 will be induced by the simple minded inclusion $\alpha \mapsto \alpha T^d T\partial_T = \pi_*\Phi(\alpha T^d)$ and the projection $\delta \mapsto T^{-d}\sigma = \pi_*\Psi(T^{-d}\delta)$ induced by π_* of (3.3) (see (4.12) and (4.16)). The possible σ in (3.7) will turn out to form a sheaf of certain logarithmic derivations. It is defined in terms of rational derivations in the spirit of Zariski (see [Kni73, Ch. I]).

Definition 3.5. Let S be a $\mathbb{Q}$-divisor and L a reduced effective Weil divisor. Define the sheaf of derivations with poles along S logarithmic along L as the sheaf sections of the constant sheaf $\mathscr{D}\mathrm{er}_k\, \mathscr{K}_X$ of rational k-linear derivations on X which map $\mathscr{O}_X(-L)$ to $\mathscr{O}_X(S-L)$ (see (2.2)),

$$\mathscr{D}\mathrm{er}_X(S - \log L) := \mathscr{O}_X(S-L) :_{\mathscr{D}\mathrm{er}_k \mathscr{K}_X} \mathscr{O}_X(-L).$$

Remark 3.6. For $S = 0$ in Definition 3.5, the sheaf $\mathscr{D}\mathrm{er}_X(S - \log L) = \mathscr{D}\mathrm{er}_X(-\log L)$ is the one defined by Saito [Sai80], as suggested by our notation. However for a (possibly non-reduced) effective divisor S it differs from the multilogarithmic derivations associated with hyperplane arrangements as defined by Ziegler (see [Zie89]).

Lemma 3.7. *(1) The sheaf $\mathscr{D}\mathrm{er}_X(S - \log L)$ is a coherent reflexive $\mathscr{O}_X$-module.*

(2) The canonical map

$$\mathscr{D}\mathrm{er}_X(-\log L) \otimes_{\mathscr{O}_X} \mathscr{O}_X(S) \to \mathscr{D}\mathrm{er}_X(S - \log L)$$

is an isomorphism if $\lfloor S \rfloor$ is a Cartier divisor.

(3) The map in (2) induces an isomorphism

$$(\mathscr{D}\mathrm{er}_X(-\log L) \otimes_{\mathscr{O}_X} \mathscr{O}_X(S))^{\vee\vee} \cong \mathscr{D}\mathrm{er}_X(S - \log L).$$

Proof. The statements are of local nature, so we may assume that $X = \operatorname{Spec} B$ is affine. Then $H^0(X, \mathscr{O}_X(S - L))$ is a fractional ideal. It follows that, for any non-zero divisor $g \in H^0(X, \mathscr{O}_X(-L))$, there is an $f \in B$ such that

$$\mathscr{I} := \mathscr{O}_X(S - \operatorname{div}(f) - L) \subset \mathscr{O}_X(-\operatorname{div}(g)) = \mathscr{O}_X \cdot g \subset \mathscr{O}_X(-L) =: \mathscr{J} \tag{3.8}$$

is a chain of inclusions of (coherent) ideal sheaves. As derivations extend uniquely to fields of fractions, $\mathscr{D}\mathrm{er}_k(\mathscr{K}_X) = \mathscr{D}\mathrm{er}_k(\mathscr{O}_X, \mathscr{K}_X)$. Since multiplication by f defines an automorphism of the latter $\mathscr{O}_X$-module, we obtain

$$f \cdot \mathscr{D}\mathrm{er}_X(S - \log L) \cong \mathscr{D}\mathrm{er}_X(S - \operatorname{div}(f) - \log L). \tag{3.9}$$

After suitably modifying S, we may therefore assume that $f = 0$. By the Leibniz rule, $\mathscr{D}\mathrm{er}_k(\mathscr{O}_X \cdot g) \subset \mathscr{D}\mathrm{er}_k(\mathscr{O}_X)$ and (3.8) implies that

$$\mathscr{D}\mathrm{er}_X(S - \log L) \cong \mathscr{I} :_{\mathscr{D}\mathrm{er}_k \mathscr{O}_X} \mathscr{J}.$$

Writing $\mathscr{J} = \langle f_1, \dots, f_m \rangle_{\mathscr{O}_X}$, another application of the Leibniz rule shows that

$$\mathscr{I} :_{\mathscr{D}\mathrm{er}_k \mathscr{O}_X} \mathscr{J} = \mathscr{D}\mathrm{er}_k(\mathscr{O}_X, \mathscr{I} :_{\mathscr{O}_X} \mathscr{J}) \cap \bigcap_{i=1}^m \mathscr{I} :_{\mathscr{D}\mathrm{er}_k \mathscr{O}_X} f_i.$$

Thus, coherence of $\mathscr{D}\mathrm{er}_X(S - \log L)$ follows from coherence of $\mathscr{D}\mathrm{er}_k(\mathscr{O}_X, \mathscr{M}) = \mathscr{H}\mathrm{om}_{\mathscr{O}_X}(\Omega^1_X, \mathscr{M})$ for any coherent $\mathscr{O}_X$-module $\mathscr{M}$. Since $\mathscr{D}\mathrm{er}_X(S - \log L)$ is defined by codimension-1 conditions, it is a normal sheaf and hence reflexive by Lemma 3.1. This proves (1), and (2) follows from (3.9) taking $\operatorname{div}(f) := \lfloor S \rfloor$. To see (3), use that $\lfloor S \rfloor$ is Cartier in codimension 1 and apply (1) and Lemma 3.1. □

4. Local derivations on a cylinder

In this section, we investigate the condition (3.6). Let D be a $\mathbb{Q}$-divisor on X and write it as in (2.1). Let $U = \operatorname{Spec} B$ be an open affine subset of X on which all the V appearing (2.1) are Cartier. This condition is met, for example, when U is regular.

We first treat the case where D has only one irreducible component. Equivalently, $D = \frac{p}{q} V$. Let g be a defining equation for V on U. The restriction of $\mathscr{A}$ to U is the B-algebra

$$A = \bigoplus_{i \in \mathbb{Z}} B g^{-\lfloor pi/q \rfloor} T^i \subset B_g[T, T^{-1}].$$

We aim for an explicit description of the graded algebra $\operatorname{Der}_k A$. Note that $A_g = B_g[T, T^{-1}]$ and hence

$$\operatorname{Der}_k A \subset \operatorname{Der}_k A_g = A_g \operatorname{Der}_k B_g \oplus A_g T \partial_T. \tag{4.1}$$

Accordingly any derivation $\delta \in \operatorname{Der}_k A$ decomposes uniquely as

$$\delta = \sigma + \alpha T \partial_T, \quad \sigma \in A_g \operatorname{Der}_k B_g, \quad \alpha \in A_g.$$

For a homogeneous $\delta \in (\operatorname{Der}_k A)_d$, by degree reasons,

$$\sigma' := T^{-d} \sigma \in \operatorname{Der}_k B_g, \quad \alpha' := T^{-d} \alpha \in B_g.$$

Correspondingly we consider

$$\delta' := T^{-d} \delta = \sigma' + \alpha' T \partial_T \in \operatorname{Der}_k A_g. \tag{4.2}$$

For notational convenience, we abbreviate

$$s_i := m_{i+d} - m_i, \quad m_i := \lfloor pi/q \rfloor. \tag{4.3}$$

Then

$$\delta' = \sigma' + \alpha' T \partial_T : B g^{-m_i} T^i \to B g^{-m_{i+d}} T^i$$

and hence, cancelling T^i,

$$\sigma' + i\alpha' : Bg^{-m_i} \to Bg^{-m_{i+d}}.$$

For any $b \in B$, this means that

$$\sigma'(b) + b(i\alpha' - m_i\sigma'(g)/g) \in g^{-s_i}B. \tag{4.4}$$

The minimal s_i will be relevant for further investigations and we denote it by

$$s := \min\{s_i \mid i \in \mathbb{Z}\}, \tag{4.5}$$

Setting $b = 1$, we see that (4.4) is equivalent to the following two conditions

$$\sigma' : B \to g^{-s}B, \tag{4.6}$$

$$i\alpha' - m_i\sigma'(g)/g \in g^{-s_i}B \quad \text{for all } i \in \mathbb{Z}. \tag{4.7}$$

Condition (4.6) implies that $\mathrm{ord}_g(g^s\sigma'(g)/g) \geq -1$. Two cases are possible:

(a) $\mathrm{ord}_g(g^s\sigma'(g)/g) \geq 0$
(b) $\mathrm{ord}_g(g^s\sigma'(g)/g) = -1$

We shall find out a numerical criterion, in terms of p, q and

$$w := \mathrm{ch}(k),$$

to tell when (b) and (4.7) can simultaneously hold. Our guiding principle is the following: If (b) holds then for $s_i = s$ the excessive pole of $-m_ig^s\sigma'(g)/g$ in (4.7) must be canceled by $ig^s\alpha'$. Indeed, notice that (4.7) requires in particular that

$$\mathrm{ord}_g(ig^s\alpha' - m_ig^s\sigma'(g)/g) \geq 0 \quad \text{if } s_i = s. \tag{4.8}$$

We first need to see for what i can s_i achieve the minimum s. We use the notation $\{x\} := x - \lfloor x \rfloor$ in the following paragraphs. Then

$$\begin{aligned} s_i &= \lfloor p(i+d)/q \rfloor - \lfloor pi/q \rfloor \\ &= \lfloor pd/q \rfloor + \lfloor pi/q \rfloor + \lfloor \{pd/q\} + \{pi/q\} \rfloor - \lfloor pi/q \rfloor \\ &= \lfloor pd/q \rfloor + \lfloor \{pd/q\} + \{pi/q\} \rfloor. \end{aligned}$$

Thus, we obtain the part (1) of the following lemma. For part (2), pick $i = q$ if $w \nmid q$ and, otherwise, pick $i \in \mathbb{Z}$ with $pi \equiv 1 \mod q$.

Lemma 4.1. *We have $s = \lfloor pd/q \rfloor$. Moreover, $s_i = s$ if and only if $\{pd/q\} + \{pi/q\} < 1$. In particular,*

(1) if $pd \equiv -1 \mod q$ then $s_i = s$ if and only if $i \in q\mathbb{Z}$;
(2) if $pd \not\equiv -1 \mod q$ or $w \nmid q$ then $s_i = s$ and $w \nmid i$ for some $i \in \mathbb{Z}$. □

We can now characterize the case where (b) and (4.7) both hold explicitly as follows.

Lemma 4.2. *Assume that both (b) and (4.7) hold. Then $pd \equiv -1 \mod q$ and $w \nmid q$.*

Proof. We treat both the case $w \neq 0$ and $w = 0$ simultaneously. Note that, in the latter case, $w \nmid i$ is equivalent to $i \neq 0$ for $i \in \mathbb{Z}$.

Let $i = q$, then $m_i = p$, and $s_i = s$ by Lemma 4.1. Then condition (4.8) reads

$$\mathrm{ord}_g(g^s(q\alpha' - p\sigma'(g)/g)) \geq 0 \tag{4.9}$$

This inequality tells us that $w \nmid q$. In fact, if $w \mid q$, we would have $\mathrm{ord}_g(-pg^s\sigma'(g)/g) \geq 0$. By (b), this would force $w \mid p$. However, p and q should have no common factor according to our initial choice.

Suppose $\{pd/q\} \neq (q-1)/q$. Take an integer i such that $pi \equiv 1 \mod q$. Such an integer exists because p and q are coprime. According to Lemma 4.1, $s_{i+qj} = s$ for all $j \in \mathbb{Z}$. Condition (4.8) with i replaced by $i + qj$ reads

$$\operatorname{ord}_g(g^s((i+qj)\alpha' - m_{i+qj}\sigma'(g)/g)) \geq 0.$$

Comparing with (4.9) implies that

$$\frac{m_{i+qj}}{i+qj} = \frac{p}{q} \quad \text{in } k \text{ for all } j \in \mathbb{Z} \text{ such that } w \nmid (i+qj). \tag{4.10}$$

Since $w \nmid q$, there exists a j such that $w \nmid (i+qj)$. Thus the content of condition (4.10) is non vacant.

Using the definition $m_{i+qj} = \lfloor p(i+qj)/q \rfloor$ and simplifying the above equation, we get

$$pi/q - \lfloor pi/q \rfloor = 0 \quad \text{in } F_w \subset k.$$

This equation tells us that $pi - q\lfloor pi/q \rfloor$ is a multiple of w in $\mathbb{Z}$. However $pi - q\lfloor \frac{p}{q} i \rfloor = 1$ by our choice of i. This is a contradiction. ☐

There is also an inverse construction. Suppose $pd \equiv -1 \mod q$ and $w \nmid q$. Given a $\sigma' \in \operatorname{Der}_k B_g$ with the additional properties (4.6) and (b), we can choose any

$$\alpha' := \frac{p}{q}\frac{\sigma'(g)}{g} + g^{-s}a, \quad a \in B, \tag{4.11}$$

and lift σ' to $\delta' = \sigma' + \alpha' T \partial_T$. The condition (4.7) is satisfied because when $i \in q\mathbb{Z}$ the left hand side of (4.7) is $g^{-s}a$ by our choice of α'. If on the other hand $i \notin q\mathbb{Z}$ we have $s_i > s$ according to Lemma 4.1; consequently, using (b),

$$\operatorname{ord}_g(g^{s_i}(i\alpha' - m_i\sigma'(g)/g)) \geq \min\{\operatorname{ord}_g(g^{s_i}\sigma'(g)/g), \operatorname{ord}_g(g^{s_i}\alpha')\} \geq 0.$$

It is time to explain the geometric meaning of this construction. Condition (4.6) tells us that restricting $\pi_*\Psi$ from (3.3) to $(\pi_*\Theta_C)_d|_U$ induces a morphism

$$\psi_U : (\pi_*\Theta_C)_d|_U \to \Theta_U(sV), \quad \delta \mapsto \sigma' = \pi_*\Psi(T^{-d}\delta). \tag{4.12}$$

Lemma 4.2 combined with the above inverse construction shows that ψ_U is surjective if and only if $pd \equiv -1 \mod q$ and $w \nmid q$. If this numerical condition is not satisfied, then condition (a) must hold and we see that the image of ψ_U is contained in $\mathscr{D}\mathrm{er}_U(-\log V)(sV)$. The morphism is in fact surjective since for a given σ' satisfying (a) we can lift it to a section of $(\pi_*\Theta_C)_d$ on U by choosing any

$$\alpha' \in g^{-s}B \tag{4.13}$$

and δ' according to (4.2). This lift apparently satisfies condition (4.7).

Summarizing the above, we have

Proposition 4.3. *Let $s = \lfloor pd/q \rfloor$ and $w = \operatorname{ch}(k)$. Then there is a short exact sequence based on the following numerical conditions. If $pd \equiv -1 \mod q$ and $w \nmid q$, then*

$$0 \longrightarrow \mathscr{O}_U(sV) \xrightarrow{\varphi_U} (\pi_*\Theta_C)_d|_U \xrightarrow{\psi_U} \Theta_U(sV) \longrightarrow 0. \tag{4.14}$$

If $pd \equiv -1 \mod q$ and $w \mid q$, then

$$0 \longrightarrow \mathscr{O}_U(sV+V) \xrightarrow{\varphi_U} (\pi_*\Theta_C)_d|_U \xrightarrow{\psi_U} \mathscr{D}\mathrm{er}_U(-\log V)(sV) \longrightarrow 0.$$

Otherwise,

$$0 \longrightarrow \mathscr{O}_U(sV) \xrightarrow{\varphi_U} (\pi_*\Theta_C)_d|_U \xrightarrow{\psi_U} \mathscr{D}\mathrm{er}_U(-\log V)(sV) \longrightarrow 0.$$

Proof. It remains to determine the kernel of ψ_U. To this end, let $\delta \in \ker \psi_U$. By (4.12), this means that $\sigma' = 0$. By condition (4.7), it follows that

$$i\alpha' \in g^{-s_i}B, \quad \text{for all } i \in \mathbb{Z}. \tag{4.15}$$

If $pd \equiv -1 \mod q$ and $w \mid q$, then (4.15) implies $\alpha' \in g^{-s-1}B$ by Lemma (4.1).(1). In all other cases, (4.15) implies $\alpha' \in g^{-s}B$ by Lemma (4.1).(2). This shows that

$$\varphi_U : \alpha' \mapsto \alpha T \partial_T = \pi_* \Phi(T^d \alpha') \tag{4.16}$$

induced by $\pi_*\Phi$ is the kernel of ψ_U. ∎

Remark 4.4. In the case where D is a Cartier divisor and $w = \mathrm{ch}(k) = 0$, we obtain an exact sequence

$$0 \longrightarrow \mathscr{O}_U(sD) \xrightarrow{\varphi_U} (\pi_*\Theta_C)_d|_U \xrightarrow{\psi_U} \Theta_U(sD) \longrightarrow 0.$$

similar to the sequence (4.14), without having to assume each irreducible component of D is Cartier. To see this, let $g \in Q(B)$ be a defining equation for D and read the preceding arguments in this section with $V = D$ and $p = q = 1$. Note that, in this case, $m_i = i$ and $s_i = s$ for all $i \in \mathbb{Z}$ and hence (4.7) reduces to a single condition giving the lift as in (4.11).

Finally, we deal with the general case where D has multiple components as in (2.1).

Definition 4.5. For each $d \in \mathbb{Z}$ and $w = \mathrm{ch}(k)$, define Weil divisors

$$W_{D,d} := \sum_{\substack{p_V d \equiv -1 \mod q_V \\ \text{and } w | q_V}} V, \quad L_{D,d} := \sum_{\substack{p_V d \not\equiv -1 \mod q_V \\ \text{or } w | q_V}} V.$$

Note $W_{D,d} = 0$ if $w = 0$ and that $L_{D,d}$ is reduced effective as required in Definition 3.5.

Remark 4.6.

(1) In part (A)⇝(B) of Demazure's construction (see §2), one can choose different homogeneous rational $T \in Q(A)$, resulting in different $\mathbb{Q}$-ample $\mathbb{Q}$-Cartier divisors D. However, the divisors $W_{D,d}$ and $L_{D,d}$ in Definition 4.5 are independent of this choice. To see that one uses the fact that $\mathrm{div}(f)$ in (2.8) has integer coefficients which do not contribute to $W_{D,d}$ and $L_{D,d}$.

(2) By definition,

$$\mathscr{O}_X(dD - L_{D,d}) = \mathscr{O}_X(dD - D'), \quad D' := \sum_V (1 - 1/q_V)V.$$

The divisor D' occurs in Watanabe's formula relating the canonical module K_A of A to the canonical divisor K_X of X,

$$K_A = \bigoplus_{i \in \mathbb{Z}} H^0(X, \mathscr{O}_X(K_X + D' + iD)).$$

In case X is a regular curve, we have

$$\mathscr{D}\mathrm{er}_X(dD - \log L_{D,d}) \cong \Theta_X(dD - L_{D,d}) \cong \mathscr{O}_X(-K_X - D' + dD)$$

and a generalized (short exact) Euler sequence was described already by Wahl (see [Wah15, Prop. 3.1, Proof of Thm. 3.3]. However, the first isomorphism above is a special feature of this case.

Proposition 4.7. *There is a short exact sequence*

$$0 \longrightarrow \mathscr{O}_U(dD + W_{D,d}) \xrightarrow{\varphi_U} (\pi_*\Theta_C)_d|_U \xrightarrow{\psi_U} \mathscr{D}\mathrm{er}_U(dD - \log L_{D,d}) \longrightarrow 0.$$

Remark 4.8. In particular, in case $D = \frac{p}{q}V$, $\mathscr{O}_U(dD + W_{D,d}) = \mathscr{O}_U(sV + W_{D,d})$ which reduces to $\mathscr{O}_U(sV)$ or $\mathscr{O}_U(sV+V)$ depending on the numerical conditions in Proposition 4.3. Similarly, $\mathscr{O}_U(dD - L_{D,d}) = \mathscr{O}_U(sV - L_{D,d})$ and hence

$$\mathscr{D}\mathrm{er}_U(dD - \log L_{D,d}) = \mathscr{D}\mathrm{er}_U(sV - \log L_{D,d}) = \mathscr{D}\mathrm{er}_U(-\log L_{D,d})(sV)$$

by Definition 3.5 and Lemma 3.7. Again this reduces to either $\Theta_U(sV)$ or $\mathscr{D}\mathrm{er}_U(-\log V)(sV)$ depending on the numerical conditions in Proposition 4.3. For this reason the sequence in Proposition 4.7 unifies the three sequences in Proposition 4.3.

Proof of Proposition 4.7. We need to first establish the existence of the morphisms φ_U and ψ_U. We denote the irreducible components and coefficients of D by

$$D =: \sum_i \frac{p_i}{q_i} V_i.$$

There exists an affine open covering of $U' = U \setminus \bigcup_{i\neq j} V_i \cap V_j$ by $U_i := U \setminus \bigcup_{i\neq j} V_j$. On each U_i, D has only one irreducible component. Therefore, according to Remark 4.8, we have an exact sequence on each U_i,

$$0 \longrightarrow \mathscr{O}_{U_i}(dD + W_{D,d}) \xrightarrow{\varphi_{U_i}} (\pi_*\Theta_C)_d|_{U_i} \xrightarrow{\psi_{U_i}} \mathscr{D}\mathrm{er}_U(dD - \log L_{D,d})|_{U_i} \longrightarrow 0.$$

These exact sequences glue to an exact sequence on U' because all φ_{U_i} and ψ_{U_i} are induced by Φ and Ψ from (3.3) via (4.12) and (4.16). By Lemmas 3.1, 3.2, 3.7, pushing forward along $i : U' \to U$ yields a left exact sequence

$$0 \longrightarrow \mathscr{O}_U(dD + W_{D,d}) \xrightarrow{i_*\varphi_{U'}} (\pi_*\Theta_C)_d|_U \xrightarrow{i_*\psi_{U'}} \mathscr{D}\mathrm{er}_U(dD - \log L_{D,d})|_U.$$

We define φ_U and ψ_U by $i_*\varphi_{U'}$ and $i_*\psi_{U'}$, respectively.

To check surjectivity of the map ψ_U, it suffices to construct a lift along ψ_U for any

$$\sigma \in H^0(U, \mathscr{D}\mathrm{er}_U(dD - \log L_{D,d})).$$

Let g_i be the defining equation for V_i and denote by

$$s^{(i)} := \min\{\lfloor \frac{(j+d)p_i}{q_i} \rfloor - \lfloor \frac{dp_i}{q_i} \rfloor \mid j \in \mathbb{Z}\} = \lfloor \frac{p_i d}{q_i} \rfloor$$

the "s" for the basic case $D|_{U_i} = \frac{p_i}{q_i} V_i \cap U_i$ (see (4.3) and (4.5)). Restricting to U_i, we find

$$\mathrm{ord}_{g_i}(\sigma(b)) \geq -s^{(i)} \tag{4.17}$$

all $b \in B$ and all i (see (a) and (b)). We consider $\delta := \sigma + \alpha T \partial_T \in \mathrm{Der}_k(K(C))$ where

$$\alpha := \sum_i \alpha_i, \quad \alpha_i := \begin{cases} \frac{p_i}{q_i}\frac{\sigma(g_i)}{g_i}, & \text{if } p_i d \equiv -1 \mod q_i \text{ and } w \nmid q_i, \\ 0, & \text{otherwise.} \end{cases} \tag{4.18}$$

Regrouping the terms of $\alpha' = T^{-d}\alpha$ as

$$\alpha' = \alpha'_i + \sum_{i\neq j} \alpha'_j$$

and using (4.17), we see that $\alpha'|_{U_i}$ has the shape required in (4.11) and (4.13) and hence $\delta \in H^0(U_i, (\pi_*\Theta_C)_d)$ by (the proof of) Proposition 4.3. Thus, $\delta \in H^0(U', (\pi_*\Theta_C)_d) = H^0(U, (\pi_*\Theta_C)_d)$ by gluing and Lemma 3.2. By construction and (4.12), δ lifts the given σ proving surjectivity of ψ_U. ∎

Remark 4.9. In the special case $q=1$, it is clear from our construction that

$$(\pi_*\Theta_C)_{d_2}|_U = (\pi_*\Theta_C)_{d_1}|_U \otimes_{\mathscr{O}_U} \mathscr{O}_U((d_2-d_1)D).$$

5. Generalized Euler sequence

We can now globalize our local construction from §4.

Proposition 5.1. *Denote by $Z := X \setminus X^{\mathrm{reg}}$ the complement of the regular part of X. Then, for each $d \in \mathbb{Z}$, there is an exact sequence*

(5.1)
$$0 \to \mathscr{O}_X(dD + W_{D,d}) \to (\pi_*\Theta_C)_d \to \mathscr{D}\mathrm{er}_X(dD - \log L_{D,d}) \to \mathscr{H}^2_Z(\mathscr{O}_X(dD+W_{D,d}))$$

whose first three terms are coherent reflexive $\mathscr{O}_X$-modules.

Proof. Denote by $i : X^{\mathrm{reg}} \hookrightarrow X$ the inclusion. By Proposition 4.7,

$$0 \to i^*\mathscr{O}_X(dD+W_{D,d}) \to i^*(\pi_*\Theta_C)_d \to i^*\,\mathscr{D}\mathrm{er}_X(dD - \log L_{D,d}) \to 0 \tag{5.2}$$

is exact. Applying i_* yields

$$\begin{aligned} 0 \to i_*i^*\mathscr{O}_X(dD+W_{D,d}) \to i_*i^*(\pi_*\Theta_C)_d \to i_*i^*\,\mathscr{D}\mathrm{er}_X(dD - \log L_{D,d}) \to \cdots \\ \cdots \to R^1 i_*i^*\mathscr{O}_X(dD+W_{D,d}). \end{aligned} \tag{5.3}$$

By Lemmas 3.2 and 3.7, $(\pi_*\Theta_C)_d$ and $\mathscr{D}\mathrm{er}_X(dD - \log L_{D,d})$ are coherent reflexive $\mathscr{O}_X$-modules. By Lemma 3.1, $\mathscr{F} = i_*i^*\mathscr{F}$ for any coherent reflexive $\mathscr{O}_X$-module and, by [Har67, Cor. 1.9], $R^1 i_* i^* = \mathscr{H}^2_Z$. This turns (5.3) into (5.1). □

Motivated by Lemma 3.3, we would like to derive a short exact sequence of global sections from (5.1). This is possible in case $\mathscr{O}_X(dD+W_{D,d})$ has sufficient depth. Restricting ourselves to the case where $W_{D,d} = 0$, we are concerned with the depth of $\mathscr{O}_X(dD) \cong \mathscr{O}_X(d)$. Watanabe [Wat81, Rem. (2.11)] showed that X is Cohen–Macaulay if A is so. More generally, Flenner [Fle81b, (2.3) Kor.] showed that Serre's condition (S_ℓ) passes from Γ to X. We extend his result to $\mathscr{O}_X(i)$ as follows.

Lemma 5.2. *Let A be a finitely generated positively graded algebra over a field k of characteristic $\operatorname{ch} k = 0$. Let $f \in A_d$ be a non zero divisor. If A satisfies Serre's condition (S_ℓ) then the same holds true for the $(A_f)_0$-modules $(A_f)_i = (A(i)_f)_0$ where $i \in \mathbb{Z}$.*

Proof. Clearly $(A_f)_i \otimes_k \bar{k} \cong ((A \otimes_k \bar{k})_{f\otimes 1})_i$ and A satisfies (S_ℓ) if and only if $A \otimes_k \bar{k}$ does. We may therefore assume that $k = \bar{k}$. Let $\mathbb{Z}_d$ be the cyclic group of order d. For any $k[\mathbb{Z}_d]$-module M we denote by $M^{\bar{i}}$ the eigenspace for the character $\bar{1} \mapsto \zeta_d^i$ where $\zeta_d \in k$ is a primitive d-th root of unity. Setting $A_i \subset A^{\bar{i}}$, A becomes a $k[\mathbb{Z}_d]$-module. Since $f \in A_d$ is $\mathbb{Z}_d$-invariant, there is an induced action on $B := A/\langle f-1\rangle$. One easily shows (see [Dol03, Thm. 3.1]) that B, and hence each $B^{\bar{i}}$, is a finitely generated module over the finitely generated k-algebra $B^{\bar{0}}$. By [Fle81b, §2], B is (S_ℓ) and $B^{\bar{0}} = (A_f)_0$; analogously, $B^{\bar{i}} = (A_f)_i$. For $\mathfrak{p} \in \operatorname{Spec} B^{\bar{0}}$, $(B^{\bar{i}})_{\mathfrak{p}} \cong (B_{\mathfrak{p}})^{\bar{i}}$. We may therefore assume that $B^{\bar{0}}$ is local with maximal ideal $\mathfrak{m}$. The primes over $\mathfrak{m}$ are conjugates under $\mathbb{Z}_d$. In fact, assume that there are two invariant non-empty sets $\{\mathfrak{p}_i\}$ and $\{\mathfrak{q}_j\}$ of primes over $\mathfrak{m}$. Then prime avoidance yields an element $b \in \bigcap_i \mathfrak{p}_i \setminus \bigcup_j \mathfrak{q}_j$. Its norm $b_0 = \prod_{\bar{i}\in\mathbb{Z}_d} \bar{i}.b$ has the same property and $b_0 \in \mathfrak{p}_i \cap B^{\bar{0}} = \mathfrak{m} \subset \mathfrak{q}_j$, contradicting to $b_0 \notin \mathfrak{q}_j$. It follows the $\dim B_{\mathfrak{p}} = \dim B^{\bar{0}}$ for all $\mathfrak{m} \subset \mathfrak{p} \in \operatorname{Spec} B$. By [BH93, Prop. 1.2.10.(a)] and since B is (S_ℓ),

$$\operatorname{grade}(\mathfrak{m}B, B) = \min\{\operatorname{depth} B_{\mathfrak{p}} \mid \mathfrak{m} \subset \mathfrak{p} \in \operatorname{Spec} B\} \geq \min\{\ell, \dim B^{\bar{0}}\}$$

Now we apply the argument from the proof of [Fog81, Prop. 1]: If $b \in \mathfrak{m}B$ is B-regular, then its norm $b_0 = \prod_{\bar{i}\in\mathbb{Z}_d} \bar{i}.b$ is B-regular and $b_0 \in \mathfrak{m}B \cap B^{\bar{0}} = \mathfrak{m}$. Moreover, $(B/Bb_0)^{\bar{0}} = B^{\bar{0}}/B^{\bar{0}}b_0$ using the Reynolds operator (which is defined due to the hypothesis $\operatorname{ch} k = 0$). By [BH93, Prop. 1.2.10.(d)] and induction, it follows that

$$\operatorname{grade}(\mathfrak{m}, B) = \operatorname{grade}(\mathfrak{m}B, B) \geq \min\{\ell, \dim B^{\bar{0}}\}.$$

Since $B \cong \bigoplus_{\bar{i}\in\mathbb{Z}_d} B^{\bar{i}}$ as $B^{\bar{0}}$-module, any B-regular sequence in $\mathfrak{m}$ is also $B^{\bar{i}}$-regular. Thus,

$$\operatorname{grade}(\mathfrak{m}, B^{\bar{i}}) \geq \operatorname{grade}(\mathfrak{m}, B) \geq \min\{\ell, \dim B^{\bar{0}}\} \geq \min\{\ell, \dim B^{\bar{i}}\}. \qquad \square$$

Proposition 5.3. *If* $\operatorname{ch} k = 0$ *then global sections of* (5.1) *form a short exact sequence*

$$0 \longrightarrow H^0(X, \mathscr{O}_X(d)) \longrightarrow (\operatorname{Der}_k A)_d \longrightarrow H^0(X, \mathscr{D}\mathrm{er}_X(dD - \log L_{D,d})) \dashrightarrow 0,$$

with the dashed arrow if A is (S_3) and X is (R_2) of dimension $\dim X \geq 2$.

Proof. The first claim follows from Proposition 5.1 using Lemma 3.3. Now assume the additional hypotheses. Since $\dim X \geq 2$ and A is (S_3), we have $\operatorname{depth} A \geq 3$ and hence $H^1(X, \mathscr{O}_X(d)) = 0$ for all $d \in \mathbb{Z}$ by [Wat81, Cor. 2.3]. Since X is (R_2), we have $\operatorname{codim}_X Z \geq 3$. We may therefore assume that $\dim X \geq 3$ as otherwise $Z = \emptyset$. By Lemma 5.2, $\mathscr{O}_X(d)$ is (S_3) and, in particular, $\operatorname{depth}_Z \mathscr{O}_X(d) \geq 3$. By [Har67, Thm. 3.8], this gives $\mathscr{H}^i_Z(\mathscr{O}_X(d)) = 0$ and hence the second claim. $\square$

For the remainder of the section, we assume that X is regular, that $W_{D,d} = 0$, and that $L_{D,d}$ is a free divisor. In this case, (5.1) is a short exact sequence whose extension class can be described explicitly. Due to the first two hypotheses, the sequence reads

$$0 \longrightarrow \mathscr{O}_X(dD) \xrightarrow{\varphi} (\pi_*\Theta_C)_d \xrightarrow{\psi} \mathscr{D}\mathrm{er}_X(dD - \log L_{D,d}) \longrightarrow 0. \tag{5.4}$$

By definition, the extension class of a short exact sequence of $\mathscr{O}_X$-modules

$$0 \to \mathscr{F} \to \mathscr{E} \to \mathscr{G} \to 0 \tag{5.5}$$

is the image of the identity morphism under the connecting homomorphism

$$\operatorname{Hom}_{\mathscr{O}_X}(\mathscr{F}, \mathscr{F}) \xrightarrow{\partial} \operatorname{Ext}^1_{\mathscr{O}_X}(\mathscr{G}, \mathscr{F}); \tag{5.6}$$

it parametrizes extensions (5.5) of $\mathscr{F}$ by $\mathscr{G}$ up to isomorphism. As one derives from the spectral sequence associated to the composition of functors $\operatorname{Hom}_{\mathscr{O}_X}(-,-) = H^0 \circ \mathscr{H}\mathrm{om}_{\mathscr{O}_X}(-,-)$, the target of (5.6) fits into an exact sequence

$$\begin{aligned} 0 \to H^1(X, \mathscr{H}\mathrm{om}(\mathscr{G}, \mathscr{F})) \to \operatorname{Ext}^1_{\mathscr{O}_X}(\mathscr{G}, \mathscr{F}) \to H^0(X, \mathscr{E}\mathrm{xt}^1_{\mathscr{O}_X}(\mathscr{G}, \mathscr{F})) \to \cdots \\ \cdots \to H^2(X, \mathscr{H}\mathrm{om}(\mathscr{G}, \mathscr{F})) \to 0. \end{aligned} \tag{5.7}$$

Due to the hypotheses $\mathscr{D}\mathrm{er}_X(-\log L_{D,d})$ is locally free and, by Lemma 3.7.(2) and Remark 3.6,

$$\begin{aligned} \mathscr{H}\mathrm{om}_{\mathscr{O}_X}(\mathscr{D}\mathrm{er}_X(dD - \log L_{D,d}), \mathscr{O}_X(dD)) &\cong \mathscr{H}\mathrm{om}_{\mathscr{O}_X}(\mathscr{D}\mathrm{er}_X(-\log L_{D,d})(dD), \mathscr{O}_X(dD)) \\ &\cong \mathscr{H}\mathrm{om}_{\mathscr{O}_X}(\mathscr{D}\mathrm{er}_X(-\log L_{D,d}), \mathscr{O}_X) \\ &\cong \Omega^1_X(\log L_{D,d}). \end{aligned}$$

It follows from (5.7) that the extension class of the sequence (5.4) is the image of $1 \in H^0(X, \mathscr{O}_X)$ in $H^1(X, \Omega^1_X(\log L_{D,d}))$ under the connecting homomorphism in the long exact cohomology sequence associated to the dual sequence

$$0 \leftarrow \mathscr{H}\mathrm{om}_{\mathscr{O}_X}(\mathscr{O}_X(dD), \mathscr{O}_X(dD)) \leftarrow \mathscr{H}\mathrm{om}_{\mathscr{O}_X}((\pi_*\Theta_C)_d, \mathscr{O}_X(dD)) \leftarrow \Omega^1(\log L_{D,d}) \leftarrow 0. \tag{5.8}$$

This image can be described explicitly as a Čech cocycle. To this end, take an affine open covering

$$X = \bigcup_\alpha X_\alpha$$

and local defining equations $g_i^{(\alpha)}$ for V_i on X_α. First, restrict the sequence (5.8) to each X_α and lift $\mathrm{id}_{\mathscr{O}_{X_\alpha}(dD)}$ to the middle term with the help of the splitting of the sequence (5.4) on X_α defined by (4.18). Rewriting (4.18), this lift is given explicitly by

$$\delta \mapsto \omega_\alpha(\psi(\delta)), \quad \omega_\alpha := \sum_{p_i d \equiv -1 \mod q_i} \frac{p_i}{q_i} \frac{dg_i^{(\alpha)}}{g_i^{(\alpha)}}.$$

Applying the Čech differential to the 0-cocycle $(\omega_\alpha \circ \psi)_\alpha$, we obtain a 1-cocycle $\varepsilon_D \in H^1(X, \Omega_X^1)$ given by

$$(\varepsilon_D)_{\alpha,\beta} := \sum_{p_i d \equiv -1 \mod q_i} \frac{p_i}{q_i} \left(\frac{dg_i^{(\beta)}}{g_i^{(\beta)}} - \frac{dg_i^{(\alpha)}}{g_i^{(\alpha)}} \right) \tag{5.9}$$

$$\text{“} = d\log \prod_{p_i d \equiv -1 \mod q_i} \left(\frac{g_i^{(\beta)}}{g_i^{(\alpha)}} \right)^{\frac{p_i}{q_i}} \text{”}.$$

The preceding arguments now prove the following generalization of a result of Wahl (see [Wah76, Prop. 3.3]).

Proposition 5.4. *Assume that X is regular, that $W_{D,d} = 0$, and that $L_{D,d}$ is a free divisor. Then the extension class of the short exact sequence (5.4) is the image of ε_D under the morphism induced by the natural inclusion $\Omega_X^1 \to \Omega_X^1(\log L_{D,d})$.* □

6. Examples

6.1. **A normal ICIS with negative derivation.** We apply our result to a graded isolated complete intersection singularity with negative derivation found in [GS14]. Let $P = k[x_1, \dots, x_6]$ be the graded polynomial algebra with weights $w = (8, 8, 5, 2, 2, 2)$ on the variables $x_1, \dots, x_6$. The weighted homogeneous polynomials

$$g_1 := x_1 x_4 + x_2 x_5 + x_3^2 - x_4^5,$$
$$g_2 := x_1 x_5 + x_2 x_6 + x_3^2 + x_6^5$$

of degree 10 define a 4-dimensional graded isolated complete intersection singularity $A := P/\langle g_1, g_2 \rangle$ with a derivation

$$\eta := \begin{vmatrix} \partial_1 & \partial_2 & \partial_3 \\ x_4 & x_5 & 2x_3 \\ x_5 & x_6 & 2x_3 \end{vmatrix} = 2x_3(x_5 - x_6)\partial_1 - 2x_3(x_4 - x_5)\partial_2 + (x_4 x_6 - x_5^2)\partial_3 \tag{6.1}$$

of degree -1. Consider the corresponding 3-dimensional projective k-variety $X := \operatorname{Proj} A \subset \mathbb{P}_k^w$. By Delorme's reduction [Del75, 1.3. Prop.] (see also [Dol82, 1.3.1. Prop.]), the second Veronese subring of P is the polynomial subring $P^{(2)} = k[x_1, x_2, y, x_4, x_5, x_6]$ where $y = x_3^2$ is a new variable of weight 10. Because g_1 and g_2 are contained in $P^{(2)}$, the second Veronese subring of A is

$$A^{(2)} = P^{(2)}/\langle g_1, g_2 \rangle$$

where

$$g_1 = x_1x_4 + x_2x_5 + y - x_4^5,$$
$$g_2 = x_1x_5 + x_2x_6 + y + x_6^5.$$

Since y occurs as a linear term of g_1 and g_2, there is an isomorphism

$$A^{(2)} = P^{(2)}/\langle g_1 - g_2, g_1\rangle \cong P^{(2)}/\langle g, y\rangle = k[x_1, x_2, x_4, x_5, x_6]/\langle g\rangle \cong P'/\langle g\rangle =: A'$$

where

$$g := g_1 - g_2 = x_1x_4 - x_1x_5 + x_2x_5 - x_2x_6 - x_4^5 - x_6^5$$

and $P' := k[x_1, x_2, x_4, x_5, x_6]$ is the polynomial ring with weights renormalized to $w' = 4, 4, 1, 1, 1$. Then we have

$$X = \operatorname{Proj} A \cong \operatorname{Proj} A^{(2)} \cong \operatorname{Proj} A'.$$

Thus, A' is a graded isolated hypersurface singularity and we can consider $X \subset \mathbb{P}_k^{w'}$. In particular,

$$X^{\text{sing}} \subseteq V(x_4, x_5, x_6). \tag{6.2}$$

In order to pass to the chart $x_1 = 1$ we introduce a 4th root of x_1 and divide g by its 5th power. As a result we obtain

$$g = (1 - x_4^4)x_4 - (1 - x_2)x_5 - (x_2 + x_6^4)x_6.$$

Then $X|_{x_1=1}$ is the quotient of the affine k-variety defined by g by the cyclic group μ_4 acting diagonally on x_4, x_5, x_6. Unless $x_2 = 1$, g serves to eliminate x_5 and hence, using the notation from [Kol07, §2.4],

$$X|_{x_1=1\neq x_2} \cong (\mathbb{A}_k^1 \setminus \{0\}) \times (\mathbb{A}_k^2/\tfrac{1}{4}(1,1))$$

corresponding to variables x_2, x_4, x_6. An analogous argument applies to the chart $x_2 = 1$. Therefore, (6.2) is an equality

$$X^{\text{sing}} = V(x_4, x_5, x_6).$$

In particular, $\operatorname{codim}_X X^{\text{sing}} = 2$ and X is not (R_2).

Following Demazure's construction (see [Dem88, (2.6) Lem. (iii), (2.10)]), one can choose $T = x_3/x_4^2$ and hence $D = \frac{1}{2}V(x_3) - 2V(x_4)$. In particular, $L_{D,-1} = 0$ (see Definition 4.5). Since $H^0(X, \mathscr{O}_X(-1)) = 0$, Proposition 5.3 serves to interpret

$$\eta \in H^0(X, \mathscr{D}\text{er}_X(-D)).$$

6.2. **A nonsingular plane conic.** In this section, we compute explicitly the sheaves $\mathscr{M}_{D,d} = (\pi_*\Theta_C)_d$ on the toy model (X, D) where $X = \mathbb{P}_k^1$ with $\mathscr{O}_X(D) = \mathscr{O}_X(2)$. The section ring is isomorphic to the subring

$$A := k[x^2, xy, y^2] \subset k[x, y],$$

where the generators x^2, xy, y^2 all have degree 1. We see that $y^{-1}\partial_x$ is a derivation of degree -1 if $\operatorname{ch}(k) = 2$. This can also be seen by passing to the isomorphic ring

$$A \cong k[x, z, y]/\langle z^2 - xy\rangle$$

where the degrees of x, y, z are all 1. There is no derivation of degree -1 if $\operatorname{ch}(k) \neq 2$, but ∂_z is such a derivation if $\operatorname{ch}(k) = 2$.

We follow closely the idea used in our construction of the generalized Euler sequence. In Demazure's construction (see §2), one has to choose a homogeneous fraction of degree

1 in $K(C)$, and uses it to define an ample $\mathbb{Q}$-Cartier $\mathbb{Q}$-divisor. We choose $T = xy$ for our purpose and pass to the isomorphic ring

$$k(t,T) \supset k[tT, T, \frac{T}{t}] \cong A, \quad T \mapsto xy, \quad t \mapsto \frac{x}{y}.$$

In this particular model for the section ring A, $\deg t = 0, \deg T = 1$, and t is the parameter for $\mathbb{P}^1_k$. The ample $\mathbb{Q}$-Cartier $\mathbb{Q}$-divisor D on $\mathbb{P}^1_k$ is defined by $\operatorname{div}(T) = \pi^{-1}(D)$. It is clear that

$$D = \{0\} + \{\infty\}$$

because $xy = 0$ implies either $x = 0$ or $y = 0$.

We use the standard affine cover of $\mathbb{P}^1_k$ by $U_1 = \operatorname{Spec} k[t]$ and $U_2 = \operatorname{Spec} k[s]$ with gluing given by $s = \frac{1}{t}$. Correspondingly, the cylinder C is covered by $C_i = \pi^{-1}(U_i)$, $i = 1, 2$. We have $C_1 = \operatorname{Spec} k[t][\frac{T}{t}, \frac{t}{T}]$ and $C_2 = \operatorname{Spec} k[s][\frac{T}{s}, \frac{s}{T}]$. Notice in particular that

$$k[tT, T, \frac{T}{t}] = k[t][\frac{T}{t}, \frac{t}{T}] \cap k[s][\frac{T}{s}, \frac{s}{T}],$$

so a degree--1 derivation on A is simultaneously a degree--1 derivation on both C_1 and C_2.

Let us find the specific form of degree--1 derivations on C_1. Given such a derivation δ, we write $T\delta = \sigma' + \alpha' T\partial_T$ as in §4. Since δ is of degree -1, it must take the degree n part of $k[t][\frac{T}{t}, \frac{t}{T}]$ into its degree $n-1$ part. In other words,

$$\delta(t^\ell(\frac{T}{t})^n) \in k[t] \cdot (\frac{T}{t})^{n-1}, \quad \text{for all } n \in \mathbb{Z}.$$

Using the decomposition of δ, we get

$$\sigma'(t)t^{-1} \in k[t], \quad n \cdot (\frac{\alpha'}{t} - \frac{\sigma'(t)}{t^2}) \in k[t], \quad \text{for all } n \in \mathbb{Z}.$$

The set of pairs $(\sigma'(t), \alpha')$ satisfying this condition is a free $k[t]$-module of rank 2 with basis $(t, 1)$ and $(t^2, 0)$. Therefore, the $\mathscr{O}_{U_1}$-module $(\pi_*(\Theta_C))_{-1}|_{U_1}$ is free of rank 2 with basis $\frac{t}{T}\partial_t + \partial_T$ and $\frac{t^2}{T}\partial_t$. Symmetrically, one can find that $(\pi_*(\Theta_C))_{-1}|_{U_2}$ is free with basis $\frac{s}{T}\partial_s + \partial_T$ and $\frac{s^2}{T}\partial_s$. How do these bases glue on $U_1 \cap U_2$? If $\operatorname{ch}(k) = 2$, we see that

$$\frac{t}{T}\partial_t + \partial_T = (\frac{t}{T}\partial_t s)\partial_s + \partial_T = -\frac{s}{T}\partial_s + \partial_T = \frac{s}{T}\partial_s + \partial_T,$$
$$\frac{t^2}{T}\partial_t = \frac{t^2}{T}(\partial_t s)\partial_s = -\frac{1}{T}\partial_s = -s^{-2}(\frac{s^2}{T}\partial_s) = s^{-2}(\frac{s^2}{T}\partial_s).$$

So the two pieces glue like $\mathscr{O}_X \oplus \mathscr{O}_X(-2)$. If $\operatorname{ch}(k) \neq 2$, we can use the modified basis

$$\frac{t}{T}\partial_t + \partial_T, \quad t(\frac{t}{T}\partial_t + \partial_T) - 2\frac{t^2}{T}\partial_t = -\frac{t^2}{T}\partial_t + t\partial_T$$

on U_1, and correspondingly the basis

$$\frac{s}{T}\partial_s + \partial_T, \quad -\frac{s^2}{T}\partial_s + s\partial_T$$

on U_2. Changing coordinate on $\mathbb{P}^1_k$ from t to s, we see that

$$\frac{t}{T}\partial_t + \partial_T = -\frac{s}{T}\partial_s + \partial_T = s^{-1}(-\frac{s^2}{T}\partial_s + s\partial_T),$$
$$-\frac{t^2}{T}\partial_t + t\partial_T = \frac{1}{T}\partial_s + \frac{1}{s}\partial_T = s^{-1}(\frac{s}{T}\partial_s + \partial_T).$$

So this time the two pieces glue like $\mathscr{O}_X(-1) \oplus \mathscr{O}_X(-1)$.

We summarize the computation as follows

Proposition 6.1. *For* $X = \mathbb{P}^1_k$, $\mathscr{L} \cong \mathscr{O}_X(2)$, *we have*

$$\mathscr{M}_{D,0} \cong \begin{cases} \mathscr{O}_X(2) \oplus \mathscr{O}_X, & \textit{if } \operatorname{ch}(k) = 2, \\ \mathscr{O}_X(1) \oplus \mathscr{O}_X(1), & \textit{if } \operatorname{ch}(k) \neq 2. \end{cases}$$

The generalized Euler sequence

$$0 \to \mathscr{L}^d \to \mathscr{M}_{D,d} \to \Theta_{\mathbb{P}^1} \otimes_{\mathscr{O}_X} \mathscr{L}^d \to 0$$

is obtained by twisting the 0*th sequence by the* d*th power of* $\mathscr{L}$*.*

Proof. We showed above that $\mathscr{M}_{D,-1} \cong \mathscr{O}_X \oplus \mathscr{O}_X(-2)$ and $\mathscr{M}_{D,-1} \cong \mathscr{O}_X(-1) \oplus \mathscr{O}_X(-1)$ in the case that $\operatorname{ch}(k) = 2$ or otherwise. The remaining argument follows from Remark 4.9. □

Remark 6.2. The splitting of the short exact sequence in Proposition 6.1 in case $\operatorname{ch}(k) = 2$ was observed already by Wahl (see [Wah76, Rem. (3.4)]). It also follows from Proposition 5.4 since $\frac{ds}{s} - \frac{dt}{t} = 2\frac{ds}{s} = 0$.

References

[AH06] Klaus Altmann and Jürgen Hausen, *Polyhedral divisors and algebraic torus actions*, Math. Ann. **334** (2006), no. 3, 557–607. MR 2207875 (2006m:14062) 5

[AHS08] Klaus Altmann, Jürgen Hausen, and Hendrik Süss, *Gluing affine torus actions via divisorial fans*, Transform. Groups **13** (2008), no. 2, 215–242. MR 2426131 (2009f:14107) 5

[Ale85] A. G. Aleksandrov, *Cohomology of a quasihomogeneous complete intersection*, Izv. Akad. Nauk SSSR Ser. Mat. **49** (1985), no. 3, 467–510, 672. MR 794953 (87h:32014) 1

[BH93] Winfried Bruns and Jürgen Herzog, *Cohen-Macaulay rings*, Cambridge Studies in Advanced Mathematics, vol. 39, Cambridge University Press, Cambridge, 1993. MR MR1251956 (95h:13020) 5

[Bou98] Nicolas Bourbaki, *Commutative algebra. Chapters 1–7*, Elements of Mathematics (Berlin), Springer-Verlag, Berlin, 1998, Translated from the French, Reprint of the 1989 English translation. MR 1727221 (2001g:13001) 2

[CLS11] David A. Cox, John B. Little, and Henry K. Schenck, *Toric varieties*, Graduate Studies in Mathematics, vol. 124, American Mathematical Society, Providence, RI, 2011. MR 2810322 (2012g:14094) 4

[Del75] Charles Delorme, *Espaces projectifs anisotropes*, Bull. Soc. Math. France **103** (1975), no. 2, 203–223. MR 0404277 (53 #8080a) 6.1

[Dem88] Michel Demazure, *Anneaux gradués normaux*, Introduction à la théorie des singularités, II, Travaux en Cours, vol. 37, Hermann, Paris, 1988, pp. 35–68. MR 1074589 (91k:14004) 2, 2, 2.1, 2, 2, 2, 3, 6.1

[Dol82] Igor Dolgachev, *Weighted projective varieties*, Group actions and vector fields (Vancouver, B.C., 1981), Lecture Notes in Math., vol. 956, Springer, Berlin, 1982, pp. 34–71. MR 704986 (85g:14060) 6.1

[Dol03] ———, *Lectures on invariant theory*, London Mathematical Society Lecture Note Series, vol. 296, Cambridge University Press, Cambridge, 2003. MR 2004511 (2004g:14051) 5

[Fle81a] Hubert Flenner, *Divisorenklassengruppen quasihomogener Singularitäten*, J. Reine Angew. Math. **328** (1981), 128–160. MR 636200 (83a:13009) 4

[Fle81b] ———, *Rationale quasihomogene Singularitäten*, Arch. Math. (Basel) **36** (1981), no. 1, 35–44. MR 612234 (82g:14036) 5, 5

[Fog81] John Fogarty, *On the depth of local rings of invariants of cyclic groups*, Proc. Amer. Math. Soc. **83** (1981), no. 3, 448–452. MR 627666 (83c:13004) 5

[GS14] Michel Granger and Mathias Schulze, *Derivations of negative degree on quasihomogeneous isolated complete intersection singularities*, arXiv.org **1403.3844** (2014). 1, 6.1

[Har67] Robin Hartshorne, *Local cohomology*, A seminar given by A. Grothendieck, Harvard University, Fall, vol. 1961, Springer-Verlag, Berlin-New York, 1967. MR 0224620 (37 #219) 5, 5

[Har80] ———, *Stable reflexive sheaves*, Math. Ann. **254** (1980), no. 2, 121–176. MR 597077 (82b:14011) 3, 3.1

[Kni73] Carol M. Knighten, *Differentials on quotients of algebraic varieties*, Trans. Amer. Math. Soc. **177** (1973), 65–89. MR 0323788 (48 #2144) 3

[Kol07] János Kollár, *Lectures on resolution of singularities*, Annals of Mathematics Studies, vol. 166, Princeton University Press, Princeton, NJ, 2007. MR 2289519 (2008f:14026) 6.1

[Mat89] Hideyuki Matsumura, *Commutative ring theory*, second ed., Cambridge Studies in Advanced Mathematics, vol. 8, Cambridge University Press, Cambridge, 1989, Translated from the Japanese by M. Reid. MR 1011461 (90i:13001) 3

[Sai80] Kyoji Saito, *Theory of logarithmic differential forms and logarithmic vector fields*, J. Fac. Sci. Univ. Tokyo Sect. IA Math. **27** (1980), no. 2, 265–291. MR MR586450 (83h:32023) 1.4, 3.6

[Wah76] Jonathan M. Wahl, *Equisingular deformations of normal surface singularities. I*, Ann. of Math. (2) **104** (1976), no. 2, 325–356. MR 0422270 (54 #10261) 5, 6.2

[Wah83a] J. M. Wahl, *A cohomological characterization of* $\mathbf{P}^n$, Invent. Math. **72** (1983), no. 2, 315–322. MR 700774 (84h:14024) 1, 3, 2.1

[Wah83b] Jonathan M. Wahl, *Derivations, automorphisms and deformations of quasihomogeneous singularities*, Singularities, Part 2 (Arcata, Calif., 1981), Proc. Sympos. Pure Math., vol. 40, Amer. Math. Soc., Providence, RI, 1983, pp. 613–624. MR 713285 (85g:14008) 1

[Wah88] Jonathan Wahl, *Deformations of quasihomogeneous surface singularities*, Math. Ann. **280** (1988), no. 1, 105–128. MR 928300 (89j:14002) 2

[Wah15] ———, *Milnor and Tjurina numbers for smoothings of surface singularities*, Algebr. Geom. **2** (2015), no. 3, 315–331. MR 3370125 2

[Wat81] Keiichi Watanabe, *Some remarks concerning Demazure's construction of normal graded rings*, Nagoya Math. J. **83** (1981), 203–211. MR 632654 (83g:13016) 1, 2, 5, 5

[Zie89] Günter M. Ziegler, *Multiarrangements of hyperplanes and their freeness*, Singularities (Iowa City, IA, 1986), Contemp. Math., vol. 90, Amer. Math. Soc., Providence, RI, 1989, pp. 345–359. MR MR1000610 (90e:32015) 3.6

X. Liao, Department of Mathematics, University of Kaiserslautern, 67663 Kaiserslautern, Germany
E-mail address: liao@mathematik.uni-kl.de

M. Schulze, Department of Mathematics, University of Kaiserslautern, 67663 Kaiserslautern, Germany
E-mail address: mschulze@mathematik.uni-kl.de