

ADJOINT DIVISORS AND FREE DIVISORS

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ABSTRACT. We describe two situations where adding the adjoint divisor to a divisor D with smooth normalization yields a free divisor. Both also involve stability or versality. In the first, D is the image of a corank 1 stable map-germ $(\mathbb{C}^n, 0) \rightarrow (\mathbb{C}^{n+1}, 0)$, and is not free. In the second, D is the discriminant of a versal deformation of a weighted homogeneous function with isolated critical point (subject to certain numerical conditions on the weights). Here D itself is already free.

We also prove an elementary result, inspired by these first two, from which we obtain a plethora of new examples of free divisors. The presented results seem to scratch the surface of a more general phenomenon that is still to be revealed.

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1. INTRODUCTION

Let M be an n -dimensional complex analytic manifold and D a hypersurface in M . The \mathcal{O}_M -module $\text{Der}(-\log D)$ of *logarithmic vector fields* along D consists of all vector fields on M tangent to D at all smooth points of D . If this module is locally free, D is called a *free divisor*. This terminology was introduced by Kyoji Saito in [Sai80b]. As freeness is evidently a local condition, so we may pass to germs of analytic spaces $D \subset X := (\mathbb{C}^n, 0)$, and pick coordinates x_1, \dots, x_n on X and a defining equation $h \in \mathcal{O}_X = \mathbb{C}\{x_1, \dots, x_n\}$ for D .

The module $\text{Der}(-\log D)$ is an infinite-dimensional Lie sub-algebra of Der_X , to be more precise, a Lie algebroid. Thus free divisors bring together commutative algebra, Lie theory and the theory of \mathcal{D} -modules, see [CMNM05]. Freeness has been

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used by Jim Damon and the first author to give an algebraic method for computing the vanishing homology of sections of discriminants and other free divisors, see [DM91] and [Dam96]. More recently the idea of adding a divisor to another in order to make the union free has been used by Damon and Brian Pike as a means of extending this technique to deal with sections of non-free divisors, see [DP11a] and [DP11b].

Saito formulated the following elementary freeness test, called *Saito's criterion* (see [Sai80b, Thm. 1.8.(ii)]): If the determinant of the so-called *Saito matrix* $(\delta_i(x_j))$ generates the defining ideal $\langle h \rangle$ for some $\delta_1, \dots, \delta_n \in \text{Der}(-\log D)$, then D is free and $\delta_1, \dots, \delta_n$ is a basis of $\text{Der}(-\log D)$. While any smooth hypersurface is free, singular free divisors are in fact highly singular: Let $\text{Sing } D$ be the singular locus of D with structure defined by the Jacobian ideal of D . By the theorem of Aleksandrov–Terao (see [Ale88, §1 Thm.] or [Ter80, Prop. 2.4]), freeness of D is equivalent to $\text{Sing } D$ being a Cohen–Macaulay space of (pure) codimension 1 in D .

The simplest example of a free divisor, whose importance in algebraic geometry is well-known, is the normal crossing divisor D defined by $h := x_1 \cdots x_n$; here, due to Saito's criterion, $\text{Der}(-\log D)$ is freely generated by the vector fields $x_1 \frac{\partial}{\partial x_1}, \dots, x_n \frac{\partial}{\partial x_n}$. In general free divisors are rather uncommon: given n vector fields $\delta_1, \dots, \delta_n \in \text{Der}_X$, let h be the determinant of their Saito matrix, and suppose that h is reduced. Then h defines a free divisor if and only if the $\mathcal{O}_{\mathbb{C}^n}$ -submodule of Der_M generated by the δ_j is a Lie algebra, see [Sai80b, Lem. 1.9]. Thus to generate examples, special techniques are called for. Non-trivial examples of free divisors first appeared as discriminants and bifurcation sets in the base of versal deformations of isolated hypersurface singularities, see [Sai80a], [Ter83], [Loo84], [Bru85], [vS95], [Dam98], [BEGvB09]. Here freeness follows essentially from the fact that $\text{Der}(-\log D)$ is the kernel of the *Kodaira–Spencer map* from the module of vector fields on the base to the relative T^1 of the deformation.

In this paper, we construct new examples of free divisors by a quite different procedure. Recall that we denote by X the germ $(\mathbb{C}^n, 0)$. Let now $f: X \rightarrow (\mathbb{C}^{n+1}, 0) =: T$ be a finite and generically 1-to-1 holomorphic map germ. In particular, X is a normalization of the reduced image D of f . Denote by \mathcal{F}_i the i th Fitting ideal of \mathcal{O}_X considered as \mathcal{O}_T -module. Mond and Pellikaan [MP89, Props. 3.1, 3.4, 3.5] showed that D is defined by \mathcal{F}_0 , \mathcal{F}_1 is perfect ideal of height 2 restricting to the *conductor ideal* $\mathcal{C}_D := \text{Ann}_{\mathcal{O}_D}(\mathcal{O}_X/\mathcal{O}_D)$ which in turn is a principal ideal of \mathcal{O}_X . In particular, the reduced singular locus Σ of D is the closure of the set of double points of f and \mathcal{C}_D is radical provided D is normal crossing in codimension 1. We call any member of \mathcal{F}_1 whose pull-back under f generates \mathcal{C}_D an *adjoint equation*, and its zero locus A an *adjoint divisor*. Set-theoretically, this implies that $\Sigma = A \cap D$. Ragni Piene [Pie79, §3] showed that an adjoint equation is given by the quotient

$$\frac{\partial h / \partial x_j}{\partial(f_1, \dots, \widehat{f_j}, \dots, f_{n+1}) / \partial(x_1, \dots, x_n)}.$$

For example, if $f(x_1, x_2) = (x_1, x_2^2, x_1 x_2)$ is the parameterisation of the Whitney umbrella, with image $D = \{t_3^2 - t_1^2 t_2 = 0\}$, then this recipe gives t_1 as adjoint equation. Figure 1 below shows $D + A$ in this example.

We show

Theorem 1.1. *Let D be the image of a stable map germ $(\mathbb{C}^n, 0) \rightarrow (\mathbb{C}^{n+1}, 0)$ of corank 1, and let A be an adjoint divisor for D . Then $D + A$ is a free divisor.*

However, we show by an example that $D + A$ is not free when D is the image of a stable germs of corank ≥ 2 (see Example 2.4.(3)). We recall the standard normal forms for stable map germs of corank 1 in §2.

In §3, we prove the following analogous result for the discriminants of certain weighted homogeneous isolated function singularities. Recall that the discriminant has smooth normalization, so that the preceding definitions apply.

Theorem 1.2. *Let $f: (\mathbb{C}^n, 0) \rightarrow (\mathbb{C}, 0)$ be a weighted homogeneous polynomial of degree d with isolated critical point and Milnor number μ . Let $d_1 \geq d_2 \geq \dots \geq d_\mu$ denote the degrees of the members of a weighted homogeneous \mathbb{C} -basis of the Jacobian algebra $\mathcal{O}_{\mathbb{C}^n, 0} / \left\langle \frac{\partial f}{\partial x_1}, \dots, \frac{\partial f}{\partial x_n} \right\rangle$. Assume that $d - d_1 + 2d_i \neq 0 \neq d - d_i$ for $i = 1, \dots, \mu$.*

Let D be the discriminant in the base space of an \mathcal{R}_e -versal deformation of f and let A be an adjoint divisor for D . Then $D + A$ is a free divisor.

Theorem 1.2 evidently applies to the simple singularities, since for these $d_1 < d$. It also applies in many other cases. For example, it is easily checked that the hypotheses on the weights hold for plane curve singularities of the form $x^p + y^q$ with p and q coprime. We do not know whether the conclusion continues to hold without the hypotheses on the weights.

The adjoint A of the discriminant is closely related to the bifurcation set; this is discussed at the end of Section 3.

We remark that an adjoint can be defined verbatim in case X is merely Gorenstein rather than smooth. In [GMS12, §6] the techniques used here are applied to construct new free divisors from certain Gorenstein varieties lying canonically over the discriminants of Coxeter groups.

In Theorem 1.2, D is already a free divisor as remarked above. In contrast, in Theorem 1.1, D itself is not free: the argument with the Kodaira Spencer map referred to above shows that $\text{Der}(-\log D)$ has depth n rather than $n + 1$. So by adding A we are making a non-free divisor free. Something similar was already done by Jim Damon, for the same divisor D , in [Dam98, Ex. 8.4]; Damon showed that after the addition of a certain divisor E (not an adjoint divisor for D) with two irreducible components, $D + E$ is the discriminant of a \mathcal{K}_V -versal deformation of a non-linear section of another free divisor V . Freeness of $D + E$ followed by his general theorem on \mathcal{K}_V -versal discriminants. It seems that our divisor $D + A$ does not arise as a discriminant using Damon's procedure.

A crucial step in the proofs of both of Theorems 1.1 and 1.2 is the following fact (see Propositions 2.8 and 3.4).

Proposition 1.3. *In the situations of Theorems 1.1 and 1.2, \mathcal{F}_1 is cyclic as module over the Lie algebra $\text{Der}(-\log D)$, and generated by an adjoint equation $a \in \mathcal{O}_{\mathbb{C}^{n+1}, 0}$ for D .*

Indeed, Theorem 1.2 follows almost trivially from this (see Proposition 3.9 below). We cannot see how to deduce Theorem 1.1 in an equally transparent way. Unfortunately our proof of Proposition 1.3 is combinatorial and not very revealing, something we hope to remedy in future work.

From Proposition 1.3 it follows that the adjoint is unique up to isomorphism preserving D (see Corollaries 2.9 and 3.8).

In the process of proving Theorem 1.2, we note an easy argument which shows that the preimage of the adjoint divisor in the normalization of the discriminant is itself a free divisor. This is our Theorem 3.2.

Motivated by Theorems 1.1 and 1.2, we describe in §4 a general procedure which constructs, from a triple consisting of a free divisor D with k irreducible components and a free divisor in $(\mathbb{C}^k, 0)$ containing the coordinate hyperplanes, a new free divisor containing D . By this means we are able to construct a surprisingly large number of new examples of free divisors.

Notation. We shall denote by $\mathcal{F}_\ell^R(M)$ the ℓ th *Fitting ideal* of the R -module M . For any presentation

$$R^m \xrightarrow{A} R^k \longrightarrow M \longrightarrow 0$$

it is generated by all $(k - \ell)$ -minors of A and defines the locus where M requires more than ℓ generators.

For any analytic space germ X , we denote by $\Theta_X := \text{Hom}_{\mathcal{O}_X}(\Omega_X^1, \mathcal{O}_X)$ the \mathcal{O}_X -module of vector fields on X . The \mathcal{O}_X -module of vector fields along an analytic map germ $f: X \rightarrow Y$ is defined by $\Theta(f) := f^*\Theta_Y$. For X and Y smooth, we shall use the standard operators $tf: \Theta_X \rightarrow \Theta(f)$ and $\omega f: \Theta_Y \rightarrow \Theta(f)$ of singularity theory defined by $tf(\xi) := Tf \circ \xi$ and $\omega f(\eta) = \eta \circ f$. For background in singularity theory we suggest the survey paper [Wal81] of C.T.C. Wall.

Throughout the paper, all the hypersurfaces we consider will be assumed reduced, without further mention.

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2. IMAGES OF STABLE MAPS

Let $f: X := (\mathbb{C}^n, 0) \rightarrow (\mathbb{C}^{n+1}, 0) =: T$ be a finite and generically one-to-one map-germ with image D . Note that $X = \bar{D}$ is a normalization of D . By [MP89, Prop. 2.5], the \mathcal{O}_T -module \mathcal{O}_X has a free resolution of the form

$$(2.1) \quad 0 \longrightarrow \mathcal{O}_T^k \xrightarrow{\lambda} \mathcal{O}_T^k \xrightarrow{\alpha} \mathcal{O}_X \longrightarrow 0,$$

in which the matrix λ can be chosen symmetric (we shall recall the proof below). For $1 \leq i, j \leq k$, we denote by m_j^i the minor obtained from λ by deleting the i th row and the j th column. The map α sends the i th basis vector e_i to $g_i \in \mathcal{O}_X$, where $g = g_1, \dots, g_k$ generates \mathcal{O}_X over \mathcal{O}_T . It will be convenient to assume, after reordering the g_i , that $g_k = 1$. This leads to a free presentation

$$\mathcal{O}_T^k \xrightarrow{\lambda^k} \mathcal{O}_T^{k-1} \xrightarrow{\alpha} \mathcal{O}_X / \mathcal{O}_D \longrightarrow 0,$$

where λ^k is obtained from λ by deleting the k th row (corresponding to the generator $g_k = 1$ of \mathcal{O}_X). By a theorem of Buchsbaum and Eisenbud [BE77], this shows that

$$(2.2) \quad \mathcal{F}_1' := \mathcal{F}_0^{\mathcal{O}_T}(\mathcal{O}_X / \mathcal{O}_D) = \text{Ann}_{\mathcal{O}_T}(\mathcal{O}_X / \mathcal{O}_D) = \mathcal{O}_D \mathcal{O}_T.$$

As $\mathcal{F}_\ell^{\mathcal{O}_T}(\mathcal{O}_X)$ defines the locus where \mathcal{O}_X requires more than ℓ \mathcal{O}_T -generators, $\det(\lambda)$ is an equation for D . By the hypothesis that f is generically one-to-one, it is a reduced equation for D (see [MP89, Prop. 3.1]).

By Cramer's rule one finds that in \mathcal{O}_X , $g_i m_s^j = \pm g_j m_s^i$ for $1 \leq i, j, s \leq k$ (see [MP89, Lem. 3.3]); invoking the symmetry of λ , this gives

$$(2.3) \quad g_i m_j^k = \pm g_j m_i^k, \quad 1 \leq i, j \leq k.$$

Combining this with the structure equations $g_i g_j = \sum_{\ell=1}^k \alpha_{i,j}^\ell g_\ell$ for \mathcal{O}_X as \mathcal{O}_T -algebra, one shows that all of the m_j^i lie in $\text{Ann}_{\mathcal{O}_T}(\mathcal{O}_X/\mathcal{O}_D)$; then from (2.2), one deduces that (see [MP89, Thm. 3.4])

Lemma 2.1. $\mathcal{F}_1 = \mathcal{F}'_1 = \langle m_j^\mu \mid j = 1, \dots, \mu \rangle$ is a determinantal ideal.

Since \mathcal{F}_1 therefore gives $\text{Sing } D$ a Cohen-Macaulay structure, this structure is reduced if generically reduced. If f is stable, then at most points of $\text{Sing } D$, D consists of two smooth irreducible components meeting transversely, from which generic reducedness follows.

As $g_k = 1$, it follows from (2.3) that $m_i^k = \pm g_i m_k^k$ and hence

Lemma 2.2. $\mathcal{F}_1 \mathcal{O}_X = \langle m_k^k \rangle_{\mathcal{O}_X}$ is a principal ideal.

From this, we deduce

Lemma 2.3. Any adjoint divisor A for D is of the form

$$A = V(m_k^k + \sum_{j=1}^{k-1} c_j m_j^k), \quad c_j \in \mathcal{O}_T.$$

Mather [Mat69] showed how to construct normal forms for stable map germs: one begins with a germ $f: (\mathbb{C}^k, 0) \rightarrow (\mathbb{C}^\ell, 0)$ whose components lie in $\mathfrak{m}_{\mathbb{C}^k, 0}^2$, and unfolds it by adding to f terms of the form $u_i g_i$, where the u_i are unfolding parameters and the g_i form a basis for $\mathfrak{m}_{\mathbb{C}^k, 0} \Theta(f) / (tf(\Theta_{\mathbb{C}^k, 0}) + \mathfrak{m}_{\mathbb{C}^\ell, 0} \Theta(f))$. Applying this construction to $f(x) = (x^k, 0)$, one obtains the stable corank-1 map germ $f_k: (\mathbb{C}^{2k-2}, 0) \rightarrow (\mathbb{C}^{2k-1}, 0)$ given by

$$(2.4) \quad f_k(u, v, x) := (u, v, x^k + u_1 x^{k-2} + \dots + u_{k-2} x, v_1 x^{k-1} + \dots + v_{k-1} x) = (u, v, w),$$

where we abbreviate $u := u_1, \dots, u_{k-2}$, $v := v_1, \dots, v_{k-1}$, $w := w_1, w_2$.

Example 2.4.

(1) When $k = 2$ in (2.4), $f_2(v, x) = (v, x^2, vx)$ parameterizes the Whitney umbrella in $(\mathbb{C}^3, 0)$, whose equation is $w_2^2 - v^2 w_1 = 0$. With respect to the basis $g = x, 1$ of \mathcal{O}_S over \mathcal{O}_T , one has the symmetric presentation matrix

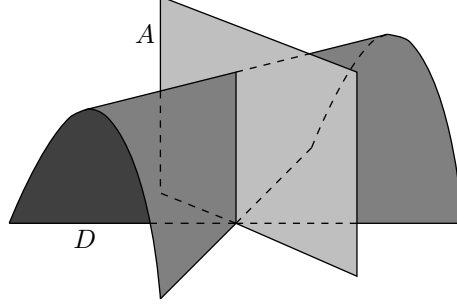
$$\lambda = \begin{pmatrix} v & -w_2 \\ -w_2 & v w_1 \end{pmatrix},$$

and this gives $a = v$ as equation of an adjoint A (see Figure 1).

One calculates that $\text{Der}(-\log D)$ and $\text{Der}(-\log(D + A))$ are generated, respectively, by the vector fields whose coefficients are displayed as the columns of the matrices

$$\begin{pmatrix} v & v & 0 & w_2 \\ 2w_1 & 0 & 2w_2 & 0 \\ 0 & w_2 & v^2 & v w_1 \end{pmatrix} \quad \text{and} \quad \begin{pmatrix} v & v & 0 \\ 2w_1 & 0 & 2w_2 \\ 2w_2 & w_2 & w^2 \end{pmatrix}.$$

FIGURE 1. Whitney umbrella with adjoint



Note that here the basis of the free module $\text{Der}(-\log(D + A))$ is included in a (minimal) list of generators of $\text{Der}(-\log D)$. As we will see, this is always the case for the germs f_k described above.

(2) Let D_0 be the image of the stable corank-2 map germ

$$(u_1, u_2, u_3, u_4, x, y) \mapsto (u_1, u_2, u_3, u_4, x^2 + u_1y, xy + u_2x + u_3y, y^2 + u_4x).$$

One calculates that $D_0 + A$ is *not* free.

(3) Every stable map germ of corank $k > 2$ is adjacent to the germ considered in the preceding example. That is, there are points on the image D where D is isomorphic to the product of the divisor D_0 we have just considered in (2) and a smooth factor. It follows that that $D + A$ also is not free.

(4) If D is the image of an unstable corank 1 germ then in general $D + A$ is not free.

(5) A multi-germ of immersions $(\mathbb{C}^n, \{p_1, \dots, p_k\}) \rightarrow (\mathbb{C}^{n+1}, 0)$ is stable if and only if it is a normal crossing. There are strata of such normal crossing points, of different multiplicities, on the image D of a stable map germ such as (2.4). It is easy to show that adding an adjoint divisor to D gives a free divisor at such normal crossing points. The normal crossing divisor $D = V(y_1 \cdots y_k) \subset (\mathbb{C}^k, 0)$ is normalized by separating its irreducible components. The equation $a := \sum_{j=1}^k y_1 \cdots \hat{y}_j \cdots y_k$ restricts to $y_1 \cdots \hat{y}_j \cdots y_k$ on the component $V(y_j)$, and so generates the conductor there, and thus $A := V(a)$ is an adjoint for D . The Euler vector field $\chi = \sum_{j=1}^k y_j \partial_{y_j}$, together with the $n - 1$ vector fields $\delta_j := y_j^2 \partial_{y_j} - y_{j+1}^2 \partial_{y_{j+1}}$, $j = 1, \dots, k - 1$, all lie in $\text{Der}(-\log(D + A))$. An application of Saito's criterion yields freeness of $D + A$ and shows that $\chi, \delta_1, \dots, \delta_{k-1}$ form a basis for $\text{Der}(-\log(D + A))$.

Our proof of Theorem 1.1 is based on Saito's criterion. By Mather's construction, we are concerned with the map $f := f_k$ of (2.4) where now $n = 2k - 2$. Using an explicit list of generators of $\text{Der}(-\log D)$ constructed by Houston and Littlestone in [HL09], and testing them on the equation m_k^k of A , we find a collection of vector fields ξ_1, \dots, ξ_{2k-1} in $\text{Der}(-\log D)$ which are in $\text{Der}(-\log A)$ "to first order", in the sense that for $j = 1, \dots, 2k - 1$, we have $\xi_j \cdot m_k^k \in \langle m_k^k \rangle + \mathfrak{m}_T \mathcal{F}_1$. Note that \mathcal{F}_1 is intrinsic to D , and therefore invariant under any infinitesimal automorphism of D , so that necessarily $\xi_j \cdot m_k^k \in \mathcal{F}_1$. In the process of testing, we show that the map $\text{Der}(-\log D) \rightarrow \mathcal{F}_1$ sending ξ to $\xi \cdot m_k^k$ is surjective. Using this, we can then adjust the ξ_j , without altering their linear part, so that now $\xi_j \cdot m_k^k \in \langle m_k^k \rangle$ for $j = 1, \dots, 2k - 1$. As a consequence, the determinant of their Saito matrix must

be divisible by the equation of $D + A$. This determinant contains a distinguished monomial also present in the equation of $D + A$, so the quotient of the determinant by the equation of $D + A$ is a unit, the determinant is a reduced equation for $D + A$, and $D + A$ is a free divisor, by Saito's criterion.

To begin this process, we need more detailed information about the matrix λ of (2.1). We use a trick from [MP89, §2]: embed X as $X \times \{0\}$ into $X \times (\mathbb{C}, 0) := S$, and let the additional variable in S be denoted by t . Extend $f: X \rightarrow T$ to a map $F: S \rightarrow T$ by adding t to the last component of f . Applying this procedure to the map $f = f_k$ of (2.4), gives

$$(2.5) \quad F(u, v, x, t) := (u, v, x^k + u_1 x^{k-2} + \cdots + u_{k-2} x, v_1 x^{k-1} + \cdots + v_{k-1} x + t) = (u, v, w).$$

It is clear that $\mathcal{O}_S/\mathfrak{m}_T \mathcal{O}_S$ is generated over \mathbb{C} by the classes of $1, x, \dots, x^{k-1}$, from which it follows by [Gun74, Cor. 2, p. 137] that \mathcal{O}_S is a free \mathcal{O}_T -module on the basis

$$(2.6) \quad g_i := x^{k-i}, \quad i = 1, \dots, k.$$

Consider the diagram

$$(2.7) \quad \begin{array}{ccccccc} 0 & \longrightarrow & \mathcal{O}_S & \xrightarrow{t} & \mathcal{O}_S & \longrightarrow & \mathcal{O}_X \longrightarrow 0 \\ & & \uparrow \varphi_g & & \uparrow \varphi_g & & \parallel \\ 0 & \longrightarrow & \mathcal{O}_T^k & \xrightarrow{[t]_g^g} & \mathcal{O}_T^k & \longrightarrow & \mathcal{O}_X \longrightarrow 0 \end{array}$$

in which φ_g is the \mathcal{O}_T -isomorphism sending $(c_1, \dots, c_k) \in \mathcal{O}_T^k$ to $\sum_{j=1}^k c_j g_j \in \mathcal{O}_S$, and where now $[t]_g^g$ denotes the matrix of multiplication by t with respect to the basis g of \mathcal{O}_S as \mathcal{O}_T -module. The lower row is thus a presentation of \mathcal{O}_X as \mathcal{O}_T -module. This can be improved by a change of basis on the source of $[t]_g^g$, as follows.

Since \mathcal{O}_S is Gorenstein, $\mathrm{Hom}_{\mathcal{O}_T}(\mathcal{O}_S, \mathcal{O}_T) \cong \mathcal{O}_S$ as \mathcal{O}_S -module. Let Φ be an \mathcal{O}_S -generator of $\mathrm{Hom}_{\mathcal{O}_T}(\mathcal{O}_S, \mathcal{O}_T)$. It induces a symmetric perfect pairing

$$\langle \cdot, \cdot \rangle : \mathcal{O}_S \times \mathcal{O}_S \rightarrow \mathcal{O}_T, \quad \langle a, b \rangle = \Phi(ab)$$

with respect to which multiplication by t is self-adjoint. We refer to this pairing as the Gorenstein pairing. Now choose a basis $\check{g} = \check{g}_1, \dots, \check{g}_k$ for \mathcal{O}_S over \mathcal{O}_T dual to g with respect to $\langle \cdot, \cdot \rangle$; that is, such that $\langle g_i, \check{g}_j \rangle = \delta_{i,j}$. Then the (i, j) th entry of $[t]_g^{\check{g}}$ equals $\langle t\check{g}_j, \check{g}_i \rangle$, and so redefining

$$(2.8) \quad \lambda = (\lambda_j^i) = (\lambda_1, \dots, \lambda_k) := [t]_g^{\check{g}}$$

yields a symmetric presentation matrix.

Lemma 2.5. *With an appropriate choice of generator Φ of $\mathrm{Hom}_{\mathcal{O}_T}(\mathcal{O}_S, \mathcal{O}_T)$, we have*

$$(2.9) \quad \lambda \equiv \begin{pmatrix} -v_1 & -v_2 & -v_3 & \cdots & -v_{k-1} & w_2 \\ -v_2 & -v_3 & & \ddots & w_2 & 0 \\ -v_3 & & & \ddots & \ddots & \vdots \\ \vdots & \ddots & \ddots & & & \vdots \\ -v_{k-1} & w_2 & \ddots & & & \vdots \\ w_2 & 0 & \cdots & \cdots & \cdots & 0 \end{pmatrix} \pmod{\langle u, w_1 \rangle}.$$

Proof. Let $b \in \mathcal{O}_S$ map to the socle of the 0-dimensional Gorenstein ring $\mathcal{O}_S/\mathfrak{m}_T\mathcal{O}_S$. This means that

$$\mathfrak{m}_S b \subset \mathfrak{m}_T \mathcal{O}_S.$$

Thus, for any $a \in \mathfrak{m}_S$ and $\Psi \in \text{Hom}_{\mathcal{O}_T}(\mathcal{O}_S, \mathcal{O}_T)$, we have

$$(a\Psi)(b) = \Psi(ab) = \Psi\left(\sum_i c_i b_i\right) = \sum_i c_i \Psi(b_i) \in \mathfrak{m}_T$$

where $c_i \in \mathfrak{m}_T$ and $b_i \in \mathcal{O}_S$. In other words, any $\Phi \in \mathfrak{m}_S \text{Hom}_{\mathcal{O}_T}(\mathcal{O}_S, \mathcal{O}_T)$ maps b to \mathfrak{m}_T . Conversely, for any $\Phi \in \text{Hom}_{\mathcal{O}_T}(\mathcal{O}_S, \mathcal{O}_T)$ with $\Phi(b) \in \mathcal{O}_T^*$, its class $\bar{\Phi}$ in $\text{Hom}_{\mathcal{O}_T}(\mathcal{O}_S, \mathcal{O}_T)/\mathfrak{m}_S \text{Hom}_{\mathcal{O}_T}(\mathcal{O}_S, \mathcal{O}_T)$ is non-zero. As $\text{Hom}_{\mathcal{O}_T}(\mathcal{O}_S, \mathcal{O}_T) \cong \mathcal{O}_S$, this latter space is 1-dimensional over \mathbb{C} which shows that $\bar{\Phi}$ is a generator. By Nakayama's lemma, Φ is then an \mathcal{O}_S -basis of $\text{Hom}_{\mathcal{O}_T}(\mathcal{O}_S, \mathcal{O}_T)$. In particular, we may take as $\Phi(h)$, for $h \in \mathcal{O}_S$, the coefficient of $b = x^{k-1}$ in the representation of h in the basis (2.6); then $\Phi(b) = 1$.

In the following, we implicitly compute modulo $\langle u, w_1 \rangle$. Using the relation

$$(2.10) \quad w_1 = x^k + \sum_{i=1}^{k-2} u_i x^{k-1-i}$$

from (2.5) we compute

$$\check{g}_1 = 1, \quad \check{g}_j = \left(w_1 - \sum_{i=j-1}^{k-2} u_i x^{k-1-i} \right) / x^{k+1-j} = x^{j-1} + \sum_{i=1}^{j-2} u_i x^{j-i-2}, \quad j = 2, \dots, k.$$

Note that

$$(2.11) \quad \check{g}_2 = x\check{g}_1, \quad \check{g}_j = x\check{g}_{j-1} + u_{j-2}\check{g}_1, \quad j = 3, \dots, k.$$

Now let us calculate the columns $\lambda_1, \dots, \lambda_k$ of the matrix (2.8). Using $\check{g}_1 = 1$ and the relation $t + \sum_{i=1}^{k-1} v_i g_i = w_2 = w_2 g_k$ from (2.5), we first compute

$$\lambda_1 = [t\check{g}_1]_g = (\langle t, \check{g}_j \rangle)_j = \left(\left\langle w_2 g_k - \sum_{i=1}^{k-1} v_i g_i, \check{g}_j \right\rangle \right)_j = (-v_1, \dots, -v_{k-1}, w_2)^t.$$

By (2.11), each of the remaining columns λ_j is obtained by multiplying by x the vector represented by its predecessor, and, for $j \geq 3$, adding $u_{j-2}\lambda_1$. Thus,

$$(2.12) \quad \lambda_2 = [x]_g^g \lambda_1, \quad \lambda_j = [x]_g^g \lambda_{j-1} + u_{j-2} \lambda_1, \quad j = 3, \dots, k.$$

Using (2.10) again, observe that

$$(2.13) \quad [x]_g^g = \begin{pmatrix} w_1 & 1 & 0 & \cdots & 0 \\ 0 & 0 & 1 & \ddots & \vdots \\ -u_1 & \vdots & \ddots & \ddots & 0 \\ \vdots & \vdots & & \ddots & 1 \\ -u_{k-2} & 0 & \cdots & \cdots & 0 \end{pmatrix} \equiv \begin{pmatrix} 0 & 1 & 0 & \cdots & 0 \\ 0 & 0 & 1 & \ddots & \vdots \\ \vdots & \vdots & \ddots & \ddots & 0 \\ \vdots & \vdots & & \ddots & 1 \\ 0 & 0 & \cdots & \cdots & 0 \end{pmatrix} \pmod{\langle u, w_1 \rangle}.$$

The result follows. \square

Corollary 2.6. *The reduced equation h of the image D of the map f_k of (2.4) contains the monomial w_2^k with coefficient ± 1 . The minor m_k^k contains the monomial $w_2^{k-2} v_1$ with coefficient ± 1 .*

Proof. The determinant of the matrix λ of (2.9) is a reduced equation for the image of f (see [MP89, Prop. 3.1]). Both statements then follow from Lemma 2.5. \square

Example 2.7. For the stable map-germs

$$f_3(u_1, v_1, v_2, x) = (u_1, v_1, v_2, x^3 + u_1x, v_1x^2 + v_2x)$$

and

$$f_4(u_1, u_2, v_1, v_2, v_3, x) = (u_1, u_2, v_1, v_2, v_3, x^4 + u_1x^2 + u_2x, v_1x^3 + v_2x^2 + v_3x)$$

of (2.4), the matrix λ is equal to

$$\begin{pmatrix} -v_1 & -v_2 & w_2 \\ -v_2 & w_2 + u_1v_1 & -v_1w_1 \\ w_2 & -v_1w_1 & v_2w_1 - u_1w_2 \end{pmatrix}$$

and to

$$\begin{pmatrix} -v_1 & -v_2 & -v_3 & w_2 \\ -v_2 & u_1v_1 - v_3 & w_2 + u_2v_1 & -v_1w_1 \\ -v_3 & w_2 + u_2v_1 & u_2v_2 - u_1v_3 - v_1w_2 & -v_2w_1 + u_1w_2 \\ w_2 & -v_1w_2 & -v_2w_1 + u_1w_2 & -v_3w_1 + u_2w_2 \end{pmatrix}$$

respectively.

Proof of Theorem 1.1. In [HL09, Thms. 3.1-3.3], Houston and Littlestone give an explicit list of generators for $\text{Der}(-\log D)$. Their proof that these generators lie in $\text{Der}(-\log D)$ simply exhibits, for each member ξ of the list, a lift $\eta \in \Theta_X$ in the sense that $tf(\eta) = \omega f(\xi)$. The Houston-Littlestone list consists of the Euler field ξ_e and three families ξ_j^i , $1 \leq i \leq 3$, $1 \leq j \leq k-1$. Denote by $\bar{\xi}_j^i$ the linear part of ξ_j^i . After dividing by 1, k , k , k and k^2 respectively, these linear parts are

$$\begin{aligned} \bar{\xi}_e &= \sum_{i=1}^{k-2} (i+1)u_i\partial_{u_i} + \sum_{i=1}^{k-1} iv_i\partial_{v_i} + kw_1\partial_{w_1} + kw_2\partial_{w_2}, \\ \bar{\xi}_j^1 &= -w_2\partial_{v_j} + \sum_{i<j} v_{i-j+k}\partial_{v_i}, \quad 1 \leq j \leq k-1, \\ \bar{\chi} &= -\bar{\xi}_1^2 = -\sum_{i=1}^{k-2} (i+1)u_i\partial_{u_i} + \sum_{i=1}^{k-1} (k-i)v_i\partial_{v_i} + kw_1\partial_{w_1}, \\ \bar{\xi}_j^2 &= \sum_{i<k-j} (i+j)u_{i+j-1}\partial_{u_i} - \sum_{i<k-j+1} (k-i-j+1)v_{i+j-1}\partial_{v_i}, \quad 1 < j \leq k-1, \\ \bar{\xi}_j^3 &= -(1-\delta_{j,1})w_2\partial_{u_{k-j}} + \sum_{i<k-j} v_{i+j}\partial_{u_i} + \delta_{j,1}w_2\partial_{w_1}, \quad 1 \leq j \leq k-1. \end{aligned}$$

Also let

$$\bar{\sigma} = (\bar{\xi}_e + \bar{\chi})/k = \sum_{i=1}^{k-1} v_i\partial_{v_i} + w_2\partial_{w_2}.$$

We now test the above vector fields for tangency to $A = V(m_k^k)$. Vector fields in $\text{Der}(-\log D)$ preserve \mathcal{F}_1 , so for each $\xi \in \text{Der}(-\log D)$ there exist $c_j \in \mathcal{O}_T$, unique modulo \mathfrak{m}_T , such that

$$\xi(m_k^k) = \sum_j c_j m_j^k.$$

We determine their value modulo \mathfrak{m}_T with the help of distinguished monomials. Let ι be the sign of the order-reversing permutation of $1, \dots, k-1$. Then, by Lemma 2.5, for $1 < j \leq k$, the monomial $w_2^{k-2}v_{k-j+1}$ appears in the polynomial expansion of m_j^k with coefficient $(-1)^{j-1}\iota$ but does not appear in the polynomial expansion of m_ℓ^k for $\ell \neq j$. Similarly, the monomial w_2^{k-1} has coefficient ι in the polynomial expansion of m_1^k but does not appear in that of m_j^k for $j \geq 2$.

Let $\lambda'_1, \dots, \lambda'_k$ denote the columns of the matrix λ of (2.9) with its last row deleted. For any $\delta \in \Theta_T$, we have

$$(2.14) \quad \delta(m_k^k) = \sum_{r=1}^{k-1} \det(\lambda'_1, \dots, \delta(\lambda'_r), \dots, \lambda'_{k-1}).$$

For $\delta = \xi_j^2$, the only distinguished monomial to appear in any of the summands in (2.14) is $v_j w_2^{k-2}$, which appears in the summand for $r = 1$, with coefficient $(k-j)(-1)^k \iota$. Thus

$$(2.15) \quad \xi_j^2(m_k^k) = (-1)^j(k-j)m_{k-j+1}^k \pmod{\mathfrak{m}_T \mathcal{F}_1}, \quad \text{for } 1 \leq j \leq k-1.$$

Similarly we find

$$\xi_j^1(\lambda'_r) = \begin{cases} \lambda'_{k-j+r} & \text{if } r \leq j, \\ 0 & \text{otherwise.} \end{cases}$$

Using (2.14) with $\delta = \xi_j^1$, it follows that

$$(2.16) \quad \xi_j^1(m_i^k) = (-1)^{k-j+1} m_{i+j-k}^k \pmod{\mathfrak{m}_T \mathcal{F}_1}, \quad \text{for } 1 \leq i \leq k, 1 \leq j \leq k-1.$$

Note that (2.15), and (2.16) with $i = k$, imply

Proposition 2.8. *The map*

$$(2.17) \quad dm_k^k : \text{Der}(-\log D) \rightarrow \mathcal{F}_1$$

sending $\xi \in \text{Der}(-\log D)$ to $\xi(m_k^k)$ is an \mathcal{O}_T -linear surjection. \square

Combining (2.15) and (2.16) with $i = k$, we construct vector fields

$$\eta_j = (j-1)\xi_j^1 - \xi_{k-j+1}^2, \quad 2 \leq j \leq k-1$$

with linear part

$$\tilde{\eta}_j \equiv (1-j)w_2 \partial_{v_j} + \sum_{i < j} (2j-i-1)v_{k+i-j} \partial_{v_i} \pmod{\langle \partial_u, \partial_{w_1} \rangle},$$

which lie in $\text{Der}(-\log(D+A))$ to first order, since $\eta_j(m_k^k) \in \mathfrak{m}_T \mathcal{F}_1$.

Both $\tilde{\chi}$ and $\tilde{\sigma}$ are semi-simple, and so by consideration of the distinguished monomials, χ and σ must therefore lie in $\text{Der}(-\log A)$ to first order. The vector fields ξ_j^3 lie in $\text{Der}(-\log A)$ to first order, since it is clear by consideration of the distinguished monomials that $\xi_j^3(m_k^k) \in \mathfrak{m}_T \mathcal{F}_1$.

Thus we have $2k-1$ vector fields $\eta_2, \dots, \eta_{k-1}, \chi, \sigma, \xi_1^3, \dots, \xi_{k-1}^3$ in $\text{Der}(-\log D)$ which are also in $\text{Der}(-\log A)$ to first order. By Proposition 2.8, we can modify these by the addition of suitable linear combinations, with coefficients in \mathfrak{m}_T , of the Houston–Littlestone generators of $\text{Der}(-\log D)$, so that they are indeed in $\text{Der}(-\log A)$ and therefore in $\text{Der}(-\log(D+A))$. The determinant of the Saito matrix of the modified vector fields $\tilde{\eta}_2, \dots, \tilde{\eta}_{k-1}, \tilde{\chi}, \tilde{\sigma}, \tilde{\xi}_1^3, \dots, \tilde{\xi}_{k-1}^3$ must be a multiple αhm_k^k of the equation of $D+A$. We now show that α is a unit, from which it follows, by Saito's criterion, that $D+A$ is a free divisor.

The modification of the vector fields does not affect the lowest order terms in the determinant of their Saito matrix, and these are the same as the lowest order terms in the determinant of the Saito matrix of their linear parts. With the rows representing the coefficients of $\partial_{u_1}, \dots, \partial_{u_{k-2}}, \partial_{w_1}, \partial_{v_1}, \dots, \partial_{v_{k-1}}, \partial_{w_2}$ in this order, this matrix is of the form

$$\begin{pmatrix} * & \bar{B}_1 \\ \bar{B}_2 & 0 \end{pmatrix},$$

with

$$\bar{B}_1 = \begin{pmatrix} v_2 & v_3 & v_4 & \cdots & v_{k-1} & -w_2 \\ v_3 & v_4 & & \ddots & -w_2 & 0 \\ v_4 & & & \ddots & \ddots & \vdots \\ \vdots & \ddots & \ddots & & & \\ v_{k-1} & -w_2 & \ddots & & & \vdots \\ w_2 & 0 & \cdots & \cdots & \cdots & 0 \end{pmatrix}$$

and

$$\bar{B}_2 = \begin{pmatrix} 2v_{k-1} & 4v_{k-2} & 6v_{k-3} & \cdots & \cdots & (2k-4)v_2 & (k-1)v_1 & v_1 \\ -w_2 & 3v_{k-1} & 5v_{k-2} & \cdots & \cdots & (2k-5)v_3 & (k-2)v_2 & v_2 \\ 0 & -2w_2 & 4v_{k-1} & \ddots & & \vdots & \vdots & \vdots \\ \vdots & 0 & -3w_2 & \ddots & \ddots & \vdots & \vdots & \vdots \\ \vdots & \vdots & 0 & \ddots & \ddots & kv_{k-2} & 3v_{k-3} & v_{k-3} \\ \vdots & \vdots & \vdots & \ddots & \ddots & (k-1)v_{k-1} & 2v_{k-2} & v_{k-2} \\ 0 & 0 & 0 & \cdots & 0 & -(k-2)w_2 & v_{k-1} & v_{k-1} \\ 0 & 0 & 0 & \cdots & \cdots & 0 & 0 & w_2 \end{pmatrix}$$

In its determinant we find the monomial $w_2^{2k-2}v_1$ with coefficient $\pm(k-1)!$. By Corollary 2.6, this monomial is present in the equation of $D + A$. This proves that α is a unit and completes the proof that $D + V(m_k^k)$ is a free divisor.

The following consequence of Proposition 2.8 is needed to prove that Theorem 1.1 holds for *any* adjoint divisor A of D , and not just for $A = V(m_k^k)$.

Corollary 2.9. *The adjoint divisor A is unique up to isomorphism preserving D .*

Proof. Let $A_0 := V(m_k^k)$. By Lemma 2.3, any adjoint divisor A_1 must have an equation of the form $m_1 := m_k^k + \sum_{i=1}^{k-1} c_i m_i^k$. Consider the family of divisors

$$(2.18) \quad A := V(m) \subset T \times (\mathbb{C}, 1), \quad m := m_k^k + s \sum_{j=1}^{k-1} c_j m_j^k,$$

where s is a coordinate on $(\mathbb{C}, 1)$. We claim that there exists a vector field

$$\Xi \in \text{Der}_{T \times (\mathbb{C}, 1) / (\mathbb{C}, 1)}(-\log(D \times (\mathbb{C}, 1)))$$

such that

$$(2.19) \quad \Xi(m) = \partial_s(m).$$

Then the vector field $\partial_s - \Xi$ is tangent to $D \times (\mathbb{C}, 1)$, and its integral flow trivializes the family (2.18). Passing to representatives of germs and scaling the c_j , the latter holds true in an open neighborhood of any point of the interval $\{0\} \times [0, 1] \subset$

$\mathbb{C}^{n+1} \times \mathbb{C}$. A finite number of such neighborhoods cover this interval and it follows that A_0 and A_1 are isomorphic by an isomorphism preserving D .

To construct Ξ we first show, using Proposition 2.8, that

$$(2.20) \quad dm_1 : \text{Der}(-\log D) \rightarrow \mathcal{F}_1$$

is surjective. By (2.17), there are vector fields $\delta_1, \dots, \delta_k \in \text{Der}(-\log D)$, homogeneous with respect to the grading determined by the vector field χ , such that $\delta_j(m_k^k) = m_j^k$ for $j = 1, \dots, k$. Pick $\alpha_{i,j}^\ell \in \mathcal{O}_T$ such that

$$\delta_i(m_j^k) = \sum_{\ell=1}^k \alpha_{i,j}^\ell m_\ell^k, \quad 1 \leq i, j \leq k,$$

and set $L_j := (\alpha_{i,j}^\ell)_{1 \leq \ell, i \leq k}$. The constant parts $L_j(0)$ are uniquely defined and L_k is the identity matrix by choice of the δ_j . For $j < k$, with respect to the grading determined by χ , we have

$$\deg(m_k^k) < \deg(m_{k-1}^k) \leq \dots \leq \deg(m_1^k)^1.$$

This gives

$$\deg(\delta_i(m_j^k)) > \deg(\delta_i(m_k^k)) = \deg(m_i^k) \geq \deg(m_{i+1}^k) \geq \dots \geq \deg(m_k^k),$$

and hence

$$\delta_i(m_j^k) \in \langle m_1^k, \dots, m_{i-1}^k \rangle + \mathfrak{m}_T \mathcal{F}_1.$$

The constant matrices $L_j(0)$, $j < k$, are therefore strictly upper triangular, and the constant part $L_k + \sum_{j=1}^{k-1} c_j(0)L_j(0)$ of the matrix of dm_1 is invertible. Thus,

$$dm_1(\text{Der}(-\log D)) + \mathfrak{m}_T \mathcal{F}_1 = \mathcal{F}_1,$$

and (2.20) follows by Nakayama's Lemma. As $m \equiv m_1 \pmod{\mathfrak{m}_{T \times (\mathbb{C}, 1)}}$, (2.20) gives

$$dm(\text{Der}_{T \times (\mathbb{C}, 1)/(\mathbb{C}, 1)}(-\log(D \times (\mathbb{C}, 1))) + \mathfrak{m}_{T \times (\mathbb{C}, 1)} \mathcal{O}_{T \times (\mathbb{C}, 1)} \mathcal{F}_1 \supset \mathcal{O}_{T \times (\mathbb{C}, 1)} \mathcal{F}_1$$

and Nakayama's Lemma yields

$$(2.21) \quad dm(\text{Der}_{T \times (\mathbb{C}, 1)/(\mathbb{C}, 1)}(-\log(D \times (\mathbb{C}, 1))) \supset \mathcal{O}_{T \times (\mathbb{C}, 1)} \mathcal{F}_1.$$

Then any preimage Ξ of $\partial_s(m) \in \mathcal{F}_1$ under dm solves (2.19). \square

The proof of Theorem 1.1 is now complete. \square

3. DISCRIMINANTS OF HYPERSURFACE SINGULARITIES

Let $f: X := (\mathbb{C}^n, 0) \rightarrow (\mathbb{C}, 0) =: T$ be weighted homogeneous of degree d (with respect to positive weights) and have an isolated critical point at 0. Let χ_0 be an Euler vector field for f , with $\chi_0(f) = d \cdot f$. Denote by $J_f := \langle \partial_{x_1}(f), \dots, \partial_{x_n}(f) \rangle$ the Jacobian ideal of f . Pick a weighted homogeneous $g = g_1, \dots, g_\mu \in \mathcal{O}_X$ with decreasing degrees $d_i := \deg(g_i)$ inducing a \mathbb{C} -basis of the Jacobian algebra

$$M_f := \mathcal{O}_X / J_f.$$

We may take $g_\mu := 1$ and $g_1 := H_0$ to be the Hessian determinant H_0 of f , which generates the socle of M_f . Then

$$(3.1) \quad F(x, u) := f(x) + g_1(x)u_1 + \dots + g_\mu(x)u_\mu$$

¹All the inequalities here are in fact strict, but we want to use the argument again later in a context where we assume only what is written here (see Corollary 3.8).

defines an \mathcal{R}_e -versal unfolding

$$F \times \pi_S: Y := X \times S \rightarrow T \times S$$

of f , with base space $S := (\mathbb{C}^\mu, 0)$, where

$$\pi = \pi_S: Y = X \times S \rightarrow S$$

is the natural projection. Setting $\deg(u_i) = w_i := d - d_i$ makes F weighted homogeneous of degree $\deg(F) = d = \deg(f)$. We denote by χ the Euler vector field $\chi_0 + \delta_1$ where $\delta_1 = \sum_{i=1}^\mu w_i u_i \partial_{u_i}$.

Let $\Sigma \subset Y$ be the relative critical locus of F , defined by the relative Jacobian ideal $J_F^{\text{rel}} = \langle \partial_{x_1}(F), \dots, \partial_{x_\mu}(F) \rangle$, and set $\Sigma^0 = \Sigma \cap V(F)$. Then \mathcal{O}_Σ is a finite free \mathcal{O}_S -module with basis g . As Σ is smooth and hence Gorenstein, $\text{Hom}_{\mathcal{O}_S}(\mathcal{O}_\Sigma, \mathcal{O}_S) \cong \mathcal{O}_\Sigma$ as \mathcal{O}_Σ -modules, and a basis element Φ defines a symmetric perfect pairing

$$\langle \cdot, \cdot \rangle: \mathcal{O}_\Sigma \otimes_{\mathcal{O}_S} \mathcal{O}_\Sigma \rightarrow \mathcal{O}_S, \quad \langle g, h \rangle := \Phi(gh),$$

which we refer to as the Gorenstein pairing. As in §2 (see the proof of Lemma 2.5), a generator Φ may be defined by projection to the socle of the special fiber: We let $\Phi(h)$ be the coefficient of the Hessian g_1 in the expression of h in the basis g . By

$$\langle \cdot, \cdot \rangle_0: M_f \otimes_{\mathbb{C}} M_f \rightarrow \mathbb{C}$$

we denote the induced Gorenstein pairing on $\mathcal{O}_\Sigma/\mathfrak{m}_S \mathcal{O}_\Sigma = \mathcal{O}_X/J_f = M_f$.

Let $\check{g} = \check{g}_1, \dots, \check{g}_\mu$ denote the dual basis of g with respect to the Gorenstein pairing, and denote by \check{d}_i the degree of \check{g}_i . We have $d_i + \check{d}_i = d_1$, so $\check{d}_i = d_{\mu+i-1}$ (recall that we have ordered the g_i by descending degree).

The discriminant $D = \pi_S(\Sigma^0) \subset S$ was shown by Kyoji Saito (see [Sai80a]) to be a free divisor. The following argument proves this, and shows also that it is possible to choose a basis for $\text{Der}(-\log D)$ whose Saito matrix is symmetric.

Theorem 3.1. *There is a free resolution of \mathcal{O}_{Σ^0} as \mathcal{O}_S -module*

$$0 \longrightarrow \mathcal{O}_S^\mu \xrightarrow{\Lambda} \Theta_S \xrightarrow{dF} \mathcal{O}_{\Sigma^0} \longrightarrow 0$$

in which Λ is symmetric, and is the Saito matrix of a basis of $\text{Der}(-\log D)$.

Proof. As in (2.7), there is a commutative diagram with exact rows

$$\begin{array}{ccccccc} 0 \rightarrow & \mathcal{O}_\Sigma & \xrightarrow{F} & \mathcal{O}_\Sigma & \longrightarrow & \mathcal{O}_{\Sigma^0} & \rightarrow 0 \\ & \uparrow \varphi_{\check{g}} \cong & & \uparrow \varphi_g \cong & & \nearrow dF & \\ 0 \rightarrow & \mathcal{O}_S^\mu & \xrightarrow{\Lambda} & \Theta_S & & & \\ & \searrow \cong & & \nearrow & & & \\ & & & \text{Der}(-\log D) & & & \end{array}$$

where $\Lambda = (\lambda_j^i)_{1 \leq i, j \leq \mu}$ is the matrix of multiplication by F with respect to bases \check{g} in the source and g in the target. As in (2.8), symmetry of Λ follows from self-adjointness of multiplication by F with respect to the Gorenstein pairing. Because of the form of F , the map $\varphi_g: \Theta_S \rightarrow \mathcal{O}_\Sigma$ sending $\eta = \sum_j \alpha_j \partial_{u_j}$ to $\sum_j \alpha_j g_j$ coincides with evaluation of dF on a(ny) lift $\tilde{\eta} \in \Theta_Y$ of η ; different lifts of the same vector field differ by a sum $\sum_j \alpha_j \partial_{x_j} \in \Theta_{Y/S}$, and the evaluation of dF on such a sum vanishes on Σ . The kernel of the composite $\Theta_S \rightarrow \mathcal{O}_{\Sigma^0}$ consists of vector fields

on S which lift to vector fields on Y which are tangent to $V(F)$, since $dF(\tilde{\eta})$ is divisible by F if and only if $\tilde{\eta} \in \text{Der}(-\log V(F))$. It is well known (see e.g. [Loo84, Lem. 6.14]) that the set of vector fields on S which lift to vector fields tangent to $V(F)$ is equal to $\text{Der}(-\log D)$. \square

Denote by m_j^i the $(\mu - 1)$ -minor of Λ obtained by deleting the i th row and the j th column. Then Lemmas 2.1, 2.2 and 2.3 of §2 remain valid in this new context. That is,

$$(3.2) \quad \mathcal{F}_1 := \mathcal{F}_1^{\mathcal{O}_S}(\mathcal{O}_{\Sigma^0}) = \langle m_j^\mu \mid j = 1, \dots, \mu \rangle_{\mathcal{O}_S}, \quad \mathcal{F}_1 \mathcal{O}_{\Sigma^0} = \langle m_\mu^\mu \rangle_{\mathcal{O}_{\Sigma^0}},$$

and the adjoints of D are divisors of the form

$$A = V(m_\mu^\mu + \sum_{j=1}^{\mu-1} c_j m_j^\mu).$$

Although it is not part of the main thrust of our paper, the following result seems to be new, and is easily proved. It assumes that D is the discriminant of an \mathcal{R}_e -versal deformation, but does not require any assumption of weighted homogeneity. We denote by H the relative Hessian determinant of the deformation (3.1).

Theorem 3.2. *Let A be any adjoint divisor for D . Then*

$$\tilde{A} := \pi^{-1}(A) \cap \Sigma^0$$

is a free divisor in Σ^0 containing $V(H) \cap \Sigma^0$, with reduced defining equation $(m_\mu^\mu \circ \pi)/H$.

Proof. By Corollary 3.8 below, we may assume that $A = V(m_\mu^\mu)$, and hence $\tilde{A} = V(m_\mu^\mu \circ \pi) \cap \Sigma^0$. First, it is necessary to show that H^2 divides $m_\mu^\mu \circ \pi$ and that m_μ^μ/H is reduced. Since Σ^0 is smooth, it is enough to check this at generic points of $V(H)$. This reduces to checking that it holds at an A_2 -point. The miniversal deformation of an A_2 -singularity is given by $G(x, v_1, v_2) = x^3 + v_1 x + v_2$. In this case, m_μ^μ is, up to multiplication by a unit, simply the coefficient of ∂_{v_1} in the Euler vector field, namely v_1 , and $v_1 = -3x^2$ on Σ^0 . The Hessian H is equal to $6x$, so H^2 does divide $m_\mu^\mu \circ \pi$, and moreover the quotient $(m_\mu^\mu \circ \pi)/H$ is reduced.

As Σ^0 is the normalization of D (see [Loo84, Thm. 4.7]), vector fields tangent to D lift to vector fields tangent to Σ^0 (see [Sei66]). Let $\tilde{\delta}_1, \dots, \tilde{\delta}_\mu$ be the lifts to Σ^0 of the symmetric basis $\delta_1, \dots, \delta_\mu$ of $\text{Der}(-\log D)$ constructed in Theorem 3.1. We will show that $\tilde{\delta}_1, \dots, \tilde{\delta}_{\mu-1}$ form a basis for $\text{Der}(-\log \tilde{A})$.

To see this, pick coordinates for Σ^0 , and denote by $(\tilde{\delta}_1, \dots, \tilde{\delta}_\mu)$ the matrix whose j th column consists of the coefficients of the vector field $\tilde{\delta}_j$ with respect to these coordinates. Similarly, denote by $(\delta_1, \dots, \delta_\mu)$ the matrix whose j th column consists of the coefficients of δ_j with respect to the coordinates u_1, \dots, u_μ . We abbreviate $\pi_{\Sigma^0} := \pi|_{\Sigma^0}$ and $\pi_\Sigma := \pi|_\Sigma$. There is a matrix equality

$$(3.3) \quad [T\pi_{\Sigma^0}] \cdot (\tilde{\delta}_1, \dots, \tilde{\delta}_\mu) = (\delta_1, \dots, \delta_\mu) \circ \pi,$$

where $[T\pi_{\Sigma^0}]$ is the Jacobian matrix of π with respect to the chosen coordinates. Let π^μ be obtained from π by omitting the μ th component, and let $(\delta_1, \dots, \delta_\mu)_\mu^\mu$ denote the submatrix of $(\delta_1, \dots, \delta_\mu)$ obtained by omitting its μ th row and column. Then (3.3) gives

$$[T\pi_{\Sigma^0}^\mu] \cdot (\tilde{\delta}_1, \dots, \tilde{\delta}_{\mu-1}) = (\delta_1, \dots, \delta_\mu)_\mu^\mu \circ \pi,$$

so that

$$(3.4) \quad \det[T\pi_{\Sigma^0}^\mu] \det(\tilde{\delta}_1, \dots, \tilde{\delta}_{\mu-1}) = m_\mu^\mu \circ \pi.$$

We will now compute $\det[T\pi_{\Sigma^0}^\mu]$ in terms of F . Because $g_\mu = 1$, u_μ does not appear in the equations of Σ , so $\partial_{u_\mu} \in T_{(x,u)}\Sigma$ for all $(x, u) \in \Sigma$. It follows that at any point of Σ^0 , $T_{(x,u)}\Sigma$ has a basis consisting of a basis of $T_{(x,u)}\Sigma^0$ followed by the vector ∂_{u_μ} . With respect to such a basis, the matrix of $[T\pi_\Sigma]$ takes the form

$$[T\pi_\Sigma] = \begin{bmatrix} T\pi_{\Sigma^0}^\mu & * \\ 0 & 1 \end{bmatrix}$$

from which it follows that

$$(3.5) \quad \det[T\pi_{\Sigma^0}^\mu] = \det[T\pi_\Sigma].$$

In order to express the latter in terms of F , we compare two representations of the zero-dimensional Gorenstein ring

$$(3.6) \quad \mathcal{O}_\Sigma / \langle u_1, \dots, u_\mu \rangle = \mathcal{O}_\Sigma / \pi^* \mathfrak{m}_S = \mathcal{O}_Y / \langle \partial_{x_1}(F), \dots, \partial_{x_n}(F), u_1, \dots, u_\mu \rangle$$

as a quotient of a regular \mathbb{C} -algebra. In both cases, by [SS75, (4.7) Bsp.], the socle is generated by the Jacobian determinant of the generators of the respective defining ideal. The first representation then shows that the socle is generated by $\det[T\pi_\Sigma]$, the second one, that it is also generated by the (relative) Hessian $H = \det(\partial^2 F / \partial x^2)$ of F . Hence, up to multiplication by a unit, we obtain

$$(3.7) \quad \det[T\pi_\Sigma] = H$$

in $\mathcal{O}_\Sigma / \langle u_1, \dots, u_\mu \rangle$. It is easy to see that the non-immersive locus of π_Σ is the vanishing set of H . This, together with (3.7), shows that $\det[T\pi_{\Sigma^0}] = H$. Now combining (3.4), (3.5) and (3.7),

$$\det(\tilde{\delta}_1, \dots, \tilde{\delta}_{\mu-1}) = (m_\mu^\mu \circ \pi) / H,$$

and so is a reduced defining equation for $\tilde{A} = V(m_\mu^\mu \circ \pi)$. Finally, each $\tilde{\delta}_j$ is tangent to $\tilde{A} = (\pi_{\Sigma^0})^{-1}(A)$ at its smooth points, since δ_j is necessarily tangent to the non-normal locus $A \cap D$ of D . The theorem now follows by Saito's criterion. \square

Remark 3.3. Computation with examples appears to show that closure

$$C_v := \overline{\tilde{A} \setminus V(H)}$$

is also a free divisor.

We now go on to show first that the divisor $D + V(m_\mu^\mu)$ is free and then (see Corollary 3.8) that all adjoints are isomorphic. Just as in §2, our proof makes use of the representation of $\text{Der}(-\log D)$ on \mathcal{F}_1 , and relies on the surjectivity of $dm_\mu^\mu: \text{Der}(-\log D) \rightarrow \mathcal{F}_1$.

Proposition 3.4. *Assume that $d - d_1 + 2d_i \neq 0 \neq d - d_i$ for $i = 1, \dots, \mu$. Then*

$$dm_\mu^\mu(\text{Der}(-\log D)) = \mathcal{F}_1.$$

Inclusion of the left hand side in the right is a consequence of the $\text{Der}(-\log D)$ -invariance of \mathcal{F}_1 . To show equality, it is enough to show that it holds modulo $\mathfrak{m}_S \mathcal{F}_1$. This will cover most of the remainder of this section.

Denote by $\bar{\Lambda} = (\bar{\lambda}_j^i)_{1 \leq i, j \leq \mu}$ the linear part of Λ , and let $\bar{\delta}_i = \sum_j \bar{\lambda}_j^i u_i \partial_{u_j}$ be the linear part of δ_i .

Theorem 3.5. *The entries of Λ are given by $\lambda_j^i = \sum_{k=1}^{\mu} \langle \check{g}_i \check{g}_j, g_k \rangle w_k u_k$. In particular, $\bar{\lambda}_j^i = \sum_{k=1}^{\mu} \langle \check{g}_i \check{g}_j, g_k \rangle_0 w_k u_k$.*

Proof. Since $\chi_0(F) \in J_F^{\text{rel}}$, we have

$$F = \chi(F) \equiv \delta_1(F) = \sum_k w_k u_k g_k \pmod{J_F^{\text{rel}}},$$

and hence

$$\lambda_j^i = \langle \check{g}_i, F \check{g}_j \rangle = \langle \check{g}_i \check{g}_j, F \rangle = \left\langle \check{g}_i \check{g}_j, \sum_k w_k u_k g_k \right\rangle = \sum_k \langle \check{g}_i \check{g}_j, g_k \rangle w_k u_k.$$

□

We call a homogeneous basis g of M_f self-dual if

$$(3.8) \quad \check{g}_i = g_{\mu+1-i}.$$

Lemma 3.6. *M_f admits self-dual bases.*

Proof. Denote by $W_j \subset M_f$ the subspace of degree- d_j elements. The space W_1 is 1-dimensional generated by the Hessian of f . Therefore W_j and W_k are orthogonal unless $d_j + d_k = d_1$, in which case $\langle \cdot, \cdot \rangle_0$ induces a non-degenerate pairing $W_j \otimes_{\mathbb{C}} W_k \rightarrow \mathbb{C}$. If $j \neq k$, one can choose the basis of W_j to be the reverse dual basis of a basis of W_k . Otherwise, $W_j = W_k$ and (since quadratic forms are diagonalizable) there is a basis of W_j for which the matrix of $\langle \cdot, \cdot \rangle_0$ is diagonal. Self-duality on W_j is then achieved by a coordinate change with matrix

$$\begin{pmatrix} 1 & 0 & \cdots & \cdots & 0 & 1 \\ 0 & \ddots & & & \ddots & 0 \\ \vdots & & 1 & 1 & & \vdots \\ \vdots & & i & -i & & \vdots \\ 0 & \ddots & & & \ddots & 0 \\ i & 0 & \cdots & \cdots & 0 & -i \end{pmatrix} \quad \text{or} \quad \begin{pmatrix} 1 & 0 & \cdots & \cdots & 0 & 1 \\ 0 & \ddots & & & \ddots & 0 \\ \vdots & & 1 & 1 & & \vdots \\ & & & 1 & & \\ \vdots & & i & -i & & \vdots \\ 0 & \ddots & & & \ddots & 0 \\ i & 0 & \cdots & \cdots & 0 & -i \end{pmatrix}$$

where $i = \sqrt{-1}$, for $\dim_{\mathbb{C}}(W_j)$ even or odd, respectively. A self-dual basis of M_f is then obtained by joining the bases of the W_j constructed above. □

Let \bar{m}_j^i be the $(\mu - 1)$ -jet of m_j^i , that is, the corresponding minor of $\bar{\Lambda}$.

Lemma 3.7. *Suppose g is an \mathcal{O}_S -basis for \mathcal{O}_{Σ} whose restriction to M_f is self-dual, and that $d \neq d_i \neq 0$. Then the following equalities hold true:*

- (a) $\mathfrak{m}_S \mathcal{F}_1 = \mathcal{F}_1 \cap \mathfrak{m}_S^{\mu} \mathcal{O}_S$ and \mathcal{F}_1 is minimally generated by $m_1^{\mu}, \dots, m_{\mu}^{\mu}$. In particular, $\bar{m}_i^{\mu} \equiv m_i^{\mu} \pmod{\mathfrak{m}_S \mathcal{F}_1}$ for $i = 1, \dots, \mu$.
- (b) $\bar{\delta}_i(\bar{m}_{\mu}^{\mu}) = \pm(d - d_1 + 2d_i) \bar{m}_{\mu+1-i}^{\mu}$ for $i = 2, \dots, \mu$.
- (c) $\delta_1(m_{\mu}^{\mu}) \equiv m_{\mu}^{\mu} \pmod{\mathbb{C}^*}$.

Proof. As $w_i = d - d_i \neq 0$ by hypothesis, we may introduce new variables $v_i = w_i u_i$ for $i = 1, \dots, \mu$. Under the self-duality hypothesis, Theorem 3.5 implies that the

matrix $\bar{\Lambda}$ has the form

$$(3.9) \quad \bar{\Lambda} = \begin{pmatrix} v_1 & v_2 & \cdots & \cdots & v_{\mu-1} & v_\mu \\ v_2 & \star & \cdots & \star & v_\mu & 0 \\ \vdots & \vdots & \ddots & \ddots & \ddots & \vdots \\ \vdots & \star & \ddots & \ddots & & \vdots \\ v_{\mu-1} & v_\mu & \ddots & & & \vdots \\ v_\mu & 0 & \cdots & \cdots & \cdots & 0 \end{pmatrix}$$

where \star entries do not involve the variable v_μ . For the first row and column, this is clear. For the remaining entries, we note that, by Theorem 3.5, v_μ appears in $\bar{\lambda}_j^i$ if and only if $0 \neq \langle \check{g}_i \check{g}_j, g_\mu \rangle_0 = \langle \check{g}_i, \check{g}_j \rangle_0$. By the self-duality assumption (3.8), this is equivalent to $i + j = \mu + 1$, in which case $\langle \check{g}_i \check{g}_j, g_\mu \rangle = 1$ and $\bar{\lambda}_{\mu-i+1}^i = v_\mu$.

As in the proof of Theorem 1.1, it is convenient to use ι to denote the sign of the order-reversing permutation of $1, \dots, \mu - 1$. From (3.9) it follows that $\bar{m}_{\mu+1-i}^\mu$ involves a distinguished monomial $v_i v_\mu^{\mu-2}$, with coefficient $(-1)^{\mu-i-1} \iota$ for $i = 1, \dots, \mu - 1$ and ι for $i = \mu$; this monomial does not appear in any other minor $\bar{m}_{\mu+1-j}^\mu$ for $i \neq j$. This implies (a).

In order to prove (b), assume, for simplicity of notation, that Λ and δ are linear, and fix $i \in \{2, \dots, \mu - 1\}$; the case $i = \mu$ is similar. We know that $\delta_i(m_\mu^\mu)$ is a linear combination of $m_1^\mu, \dots, m_\mu^\mu$. We will show that

$$(3.10) \quad \delta_i(m_\mu^\mu) = (-1)^{i-1} (w_1 - 2w_{\mu-i+1}) m_{\mu-i+1}^\mu$$

by computing that the coefficient $c_{i,j}$ of the distinguished monomial $v_j v_\mu^{\mu-2}$ in $\delta_i(m_\mu^\mu)$ satisfies

$$(3.11) \quad c_{i,j} = (-1)^{\mu-2} \iota (w_1 - 2w_{\mu-i+1}) \delta_{i,j}.$$

The self-duality assumption (3.8) implies that $d_1 - d_\ell = \check{d}_\ell = d_{\mu-\ell+1}$. Using $w_\ell = d - d_\ell$, this gives

$$w_1 - 2w_{\mu-i+1} = d - d_1 + 2d_{\mu-i+1} - 2d = d - d_1 + 2d_1 - 2d_i - 2d = -(d - d_1 + 2d_i).$$

So (b) will follow from (3.10).

By linearity of δ_i , the only monomials in the expansion of m_μ^μ that could conceivably contribute to a non-zero $c_{i,j}$ are of the following three forms:

$$(3.12) \quad v_\mu^{\mu-1}, \quad v_j v_\mu^{\mu-2}, \quad v_j v_k v_\mu^{\mu-3}.$$

The first monomial does not figure in the expansion of m_μ^μ . Monomials of the other two types do appear. The second type of monomial in (3.12) must satisfy $j = 1$ and arises as the product

$$(3.13) \quad (-1)^{\mu-2} \iota v_1 v_\mu^{\mu-2} = (-1)^{\mu-2} \iota \lambda_1^1 \lambda_{\mu-1}^2 \lambda_{\mu-2}^3 \cdots \lambda_2^{\mu-1}.$$

Monomials of the third type in (3.12) must satisfy $k = \mu - j + 1$. Each such monomial arises in the expansion of m_μ^μ in two ways, which coincide when $j = \mu - j + 1$:

$$(3.14) \quad (-1)^{\mu-3} \iota v_j v_k v_\mu^{\mu-3} = (-1)^{\mu-3} \iota \lambda_j^1 \lambda_{\mu-1}^2 \cdots \lambda_{j+1}^{\mu-j} \lambda_1^{\mu-j+1} \lambda_{j-1}^{\mu-j+2} \cdots \lambda_2^{\mu-1}$$

$$(3.15) \quad (-1)^{\mu-3} \iota v_j v_k v_\mu^{\mu-3} = (-1)^{\mu-3} \iota \lambda_{\mu-j+1}^1 \lambda_{\mu-1}^2 \cdots \lambda_{\mu-j+2}^{j-1} \lambda_1^j \lambda_{\mu-j}^{j+1} \cdots \lambda_2^{\mu-1}.$$

In terms of the coordinates v_1, \dots, v_μ , δ_i contains monomials

$$(3.16) \quad w_i u_i \partial_{u_1} = w_1 v_i \partial_{v_1},$$

$$(3.17) \quad w_\mu u_\mu \partial_{u_{\mu-i+1}} = w_{\mu-i+1} v_\mu \partial_{v_{\mu-i+1}}.$$

Now (3.16) applied to (3.13) contributes $w_1(-1)^{\mu-2} \iota$ to $c_{i,i}$, (3.17) applied to one or two copies of (3.14) for $i = j$ contributes $2(-1)^{\mu-3} \iota w_{\mu-i+1}$ to $c_{i,i}$ in both cases. There are no contributions to the coefficient of any other distinguished monomial.

We have proved (3.10), from which (b) follows; (c) is clear, since δ_1 is the Euler vector field. \square

By Nakayama's lemma, Proposition 3.4 is now an immediate consequence of (3.2) and Lemma 3.7.

The next result, closely analogous to Corollary 2.9, follows from Proposition 3.4 by the same argument by which Corollary 2.9 is deduced from Proposition 2.8.

Corollary 3.8. *Assume the hypothesis of Proposition 3.4. Then any two adjoint divisors of D are isomorphic by an isomorphism preserving D .* \square

Proposition 3.9. *Let $D = V(h)$ and $A = V(m)$ be divisors in S , and suppose that D is a free divisor. Let \mathcal{F} be the \mathcal{O}_S -ideal $dm(\text{Der}(-\log D))$, and suppose that $m \in \mathcal{F}$. Then the following two statements are equivalent:*

- (1) $\text{depth}_{\mathcal{O}_S} \mathcal{F} = \mu - 1$.
- (2) $D + A$ is a free divisor.

Proof. Apply the depth lemma (see [BH93, Prop. 1.2.9]) to the two short exact sequences:

$$0 \longrightarrow \langle m \rangle \longrightarrow \mathcal{F} \longrightarrow \mathcal{F}/\langle m \rangle \longrightarrow 0$$

$$0 \longrightarrow \text{Der}(-\log(D + A)) \longrightarrow \text{Der}(-\log D) \xrightarrow{dm} \mathcal{F}/\langle m \rangle \longrightarrow 0 \quad \square$$

Proof of Theorem 1.2. By Corollary 3.8, we may assume that $A = V(m_\mu^\mu)$. Recall that $dm_\mu^\mu(\text{Der}(-\log D)) = \mathcal{F}_1$ by Proposition 3.4 and hence $\text{depth}_{\mathcal{O}_S} \mathcal{F}_1 = \mu - 1$ by the Hilbert–Burch Theorem (see [BH93, Thm. 1.4.17]). So Proposition 3.9 with $m = m_\mu^\mu$ and $\mathcal{F} = \mathcal{F}_1$ yields the claim. \square

Remark 3.10. We remark that the very simple deduction of Theorem 1.2 from Propositions 3.4 and 3.9 does not have a straightforward analogue by which Theorem 1.1 can be deduced from Propositions 2.8 and 3.9. Firstly, the image D of a stable map-germ $f : (\mathbb{C}^n, 0) \rightarrow (\mathbb{C}^{n+1}, 0)$ is not free: $\text{Der}(-\log D)$ has depth n and not $n + 1$. Secondly, there can be no way of adapting the argument to deal with this difference without some other input, since when f is the stable germ of corank 2 of Example 2, the corresponding map $\text{Der}(-\log D) \rightarrow \mathcal{F}_1$ is surjective, and \mathcal{F}_1 has depth n , but even so $\text{Der}(-\log(D + A))$ is not free.

We conclude this section with a description of the relation between the adjoint divisor of D and the bifurcation set of the deformation. For $u \in S$, we set $X_u := \pi_S^{-1}(u)$ and define $f_u : X_u \rightarrow T$ by $f_u(x) := F(x, u)$. We consider $S' := (\mathbb{C}^{\mu-1}, 0)$ with coordinates $u' = u_1, \dots, u_{\mu-1}$, and we denote by

$$(3.18) \quad \rho : S \rightarrow S' \quad u \mapsto u',$$

the natural projection forgetting the last coordinate. Recall that the *bifurcation set* is the set $B \subset S'$ of parameter values u' such that $f_{u'} := f_{(u', 0)}$ has fewer than

μ distinct critical values. The coefficient u_μ of $g_\mu = 1$ is set to 0 since it has no bearing on the number of critical values. The bifurcation set consists of two parts: the level bifurcation set B_v consisting of parameter values u' for which $f_{u'}$ has distinct critical points with the same critical value, and the local bifurcation set B_ℓ where $f_{u'}$ has a degenerate critical point. H. Terao, in [Ter83], and J.W. Bruce in [Bru85] proved that B is a free divisor and gave algorithms for constructing a basis for $\text{Der}(-\log B)$. The free divisor B is of course singular in codimension 1. The topological double points (points at which B is reducible) are of four generic types:

- Type 1: $f_{u'}$ has two distinct degenerate critical points, x_1 and x_2 .
- Type 2: $f_{u'}$ has two distinct pairs of critical points, x_1, x_2 and x_3, x_4 , such that $f_{u'}(x_1) = f_{u'}(x_2)$ and $f_{u'}(x_3) = f_{u'}(x_4)$.
- Type 3: $f_{u'}$ has a pair of critical points x_1 and x_2 with the same critical value, and also a degenerate critical point x_3 .
- Type 4: $f_{u'}$ has three critical points x_1, x_2 and x_3 with the same critical value.

In the neighborhood of a double point of type 1, 2 or 3, B is a normal crossing of two smooth sheets. In the neighborhood of a double point of type 4, B is isomorphic to a product $B_0 \times (\mathbb{C}^{\mu-2}, 0)$ where $B_0 = V(uv(u-v)) \subset (\mathbb{C}^2, 0)$.

Proposition 3.11. *For any adjoint divisor A for D , (3.18) induces a surjection*

$$(3.19) \quad \rho: D \cap A \rightarrow B.$$

Proof. A point $u \in S$ lies in $D \cap A$ if the sum of the lengths of the Jacobian algebras of f_u at points $x \in f_u^{-1}(0)$ is greater than 1. The sum may be greater than 1 because for some x the dimension of the Jacobian algebra is greater than 1 – in which case f_u has a degenerate critical point at x – or because f_u has two or more critical points with critical value 0. In either case, it is clear that $\rho(u) \in B$. If $u' \in B$, then $f_{u'}$ has either a degenerate critical point or a repeated critical value (or both). In both, cases let v be the corresponding critical value. Then $(u', -v) \in D \cap A$, proving surjectivity. \square

Remark 3.12. The projection (3.19) is a partial normalization, in the sense that generically, topological double points of $u' \in B$ of types 1, 2 and 3 are separated. Indeed, in each such case $f_{u'}$ has two critical points with different critical values, and hence with different preimages under ρ . However, a general point u' of type 4 has only one preimage, $(u', -f_{(u',0)}(x_i))$, in $D \cap A$. Generically, at such a point D is a normal crossing of three smooth divisors, and $D \cap A$ is the union of their pairwise intersections. Thus B_0 is improved to a curve isomorphic to the union of three coordinate axes in 3-space.

Finally, our free divisors $D + A$ and \tilde{A} of Theorems 1.2 and 3.2, and the conjecturally free divisor C_v of Remark 3.3, fit into the following commutative diagram, in which A is any adjoint divisor for D , and the simple and double underlinings

indicate conjecturally free and free divisors.

$$\begin{array}{ccccc}
 V(H) \cap \Sigma^0 & \xrightarrow{C_v} & \tilde{A} & \xrightarrow{\quad} & \Sigma^0 \\
 \downarrow \rho \circ \pi & & \downarrow \pi & & \downarrow \pi \\
 B_\ell & \xrightarrow{B_v} & \underline{B} & \xrightarrow{\quad} & \underline{D+A} \\
 & & \downarrow \rho & & \\
 & & & & \underline{D}
 \end{array}$$

$\xrightarrow{\quad} \underline{D} \xrightarrow{\quad} \underline{D+A}$

4. PULL-BACK OF FREE DIVISORS

In this section, we describe a procedure for constructing new free divisors from old by a pull-back construction. It is motivated by Example 2.4.(5).

Theorem 4.1. *Suppose that $D = \bigcup_{i=1}^k D_i \subset (\mathbb{C}^n, 0) =: X$ is a germ of a free divisor. Let $f: X \rightarrow Y := (\mathbb{C}^k, 0)$ be the map whose i th component $f_i \in \mathcal{O}_X$, for $i = 1, \dots, k$, is a reduced equation for D_i . Suppose that, for $j = 1, \dots, k$, there exist vector fields $\varepsilon_j \in \Theta_X$ such that*

$$(4.1) \quad df_i(\varepsilon_j) = \delta_{i,j} \cdot f_i.$$

Let $N := V(y_1 \cdots y_k) \subset Y$ be the normal crossing divisor, so that $D = f^{-1}(N)$. Let $E \subset Y$ be a divisor such that $N + E$ is free. Then provided that no component of $f^{-1}(E)$ lies in D , $f^{-1}(N + E) = D + f^{-1}(E)$ is a free divisor.

Proof. The vector fields $\varepsilon_1, \dots, \varepsilon_k$ can be incorporated into a basis $\varepsilon_1, \dots, \varepsilon_n$ for $\text{Der}(-\log D)$ such that (4.1) holds for $j = 1, \dots, n$ and hence

$$(4.2) \quad tf(\varepsilon_j) = \sum_{i=1}^k df_i(\varepsilon_j) \partial_{y_i} = \begin{cases} \omega f(y_j \partial_{y_j}), & \text{if } j \leq k, \\ 0, & \text{otherwise.} \end{cases}$$

Any Saito matrix of $N + E$ can be written in the form $S_{N+E} = S_N \cdot A$, where $S_N = \text{diag}(y_1, \dots, y_k)$ is the standard Saito matrix of N and $A = (a_{i,j}) \in \mathcal{O}_Y^{k \times k}$. Then, by Saito's criterion, $h := \det A$ and $g := y_1 \cdots y_n h$ are reduced equations for E and $N + E$ respectively. For $j = 1, \dots, k$, consider the vector fields $v_j := \sum_{i=1}^k a_{i,j} y_i \partial_{y_i} \in \text{Der}(-\log(N + E))$ whose coefficients are the columns of S_{N+E} and let

$$\tilde{v}_j := \sum_{i=1}^k (a_{i,j} \circ f) \varepsilon_i \in \Theta_X.$$

By (4.2), we have $tf(\tilde{v}_j) = \omega f(v_j)$; by construction of the v_j , it follows that

$$d(g \circ f)(\tilde{v}_j) = (dg(v_j)) \circ f \in (g \circ f) \mathcal{O}_X,$$

so that $\tilde{v}_j \in \text{Der}(-\log f^{-1}(N + E))$, and moreover $\varepsilon_j \in \text{Der}(-\log f^{-1}(N + E))$ for $j > k$.

Let S_D be the Saito matrix of D , whose columns are the coefficients of the vector fields $\varepsilon_1, \dots, \varepsilon_n$. The matrix of coefficients of the vector fields $\tilde{v}_1, \dots, \tilde{v}_k, \varepsilon_{k+1}, \dots, \varepsilon_n$ is equal to

$$S_D \cdot \begin{pmatrix} A \circ f & 0 \\ 0 & I_{n-k} \end{pmatrix}$$

and thus its determinant $\det(S_D) \cdot (h \circ f)$ defines $D + f^{-1}(E) = f^{-1}(N + E)$. By Saito's criterion, this shows that the latter is a free divisor, provided

- (1) $h \circ f$ is reduced and
- (2) $h \circ f$ has no irreducible factor in common with $f_1 \cdots f_k$.

Now $h \circ f$ is reduced where f is a submersion. So provided no component of $f^{-1}(E)$ is contained in the critical set Σ_f of f , $h \circ f$ is reduced. In fact $\Sigma_f = D_{\text{Sing}}$. To see this, consider the “logarithmic Jacobian matrix” $(\varepsilon_i(f_j))_{1 \leq i \leq n, 1 \leq j \leq k}$ of f . The determinant of the first k columns is equal to $f_1 \cdots f_k$; thus $f_1 \cdots f_k$ is in the Jacobian ideal of f , so $\Sigma_f \subset D$. Thus $D_{\text{Sing}} = D \cap \Sigma_f = \Sigma_f$. This shows that condition (2) above implies condition (1). \square

Example 4.2. Let $D := V(x_1 \cdots x_n) \subset (\mathbb{C}^n, 0)$ be the normal crossing divisor, define $f: (\mathbb{C}^n, 0) \rightarrow (\mathbb{C}^2, 0)$ by $f(x_1, \dots, x_n) = (x_1 \cdots x_k, x_{k+1} \cdots x_n)$, and let $g \in \mathcal{O}_{\mathbb{C}^2, 0}$ be any germ not divisible by either of the coordinates. Take $N := V(y_1 y_2)$ and $E := V(g(y_1, y_2))$. By Theorem 4.1 it follows that $V(x_1 \cdots x_n g(x_1 \cdots x_k, x_{k+1} \cdots x_n))$ is a free divisor. The condition that no component of $f^{-1}(E)$ should lie in D is guaranteed by the requirement that neither y_1 nor y_2 should divide $g(y_1, y_2)$.

The existence hypothesis (4.1) in the theorem is not fulfilled for every reducible free divisor. In the graded case, the vector fields ε_j must have degree zero. If D is the discriminant of a versal deformation of a weighted homogeneous isolated hypersurface singularity meeting the hypotheses of Theorem 1.2, and A is an adjoint divisor, then $D + A$ is free but the only vector fields of weight zero in $\text{Der}(-\log D)$ are multiples of the Euler field. Thus, hypothesis (4.1) cannot hold.

However there is an interesting class, namely linear free divisors, for which this requirement always holds. We recall from [GMNRS09] that a free divisor D in the n -dimensional vector space V is said to be linear if $\text{Der}(-\log D)$ has a basis consisting of vector fields of weight 0. The linear span $\text{Der}(-\log D)_0$ of the basic fields is an n -dimensional Lie algebra. It is naturally identified with the Lie algebra of the algebraic subgroup $\iota: G_D \hookrightarrow \text{GL}(V)$ consisting of the identity component of the set of automorphisms preserving D . It follows that (V, G_D, ι) is a prehomogeneous vector space (see [SK77]) with discriminant D .

Let $D \subset V$ be a linear free divisor and $D = \bigcup_{i=1}^k D_i$ a decomposition into irreducible components. The corresponding defining equations f_1, \dots, f_k are polynomial relative invariants of (V, G_D, ι) with associated characters χ_1, \dots, χ_k ; that is, for $g \in G_D$ and $x \in V$, $f_j(gx) = \chi_j(g)f_j(x)$. These characters are multiplicatively independent, by [SK77, §4, proof of Prop. 5]. Let \mathfrak{g}_D denote the Lie algebra of G_D . By differentiating the character map $\chi = (\chi_1, \dots, \chi_k): G \rightarrow (\mathbb{C}^*)^k$ we obtain an epimorphism of Lie algebras $d\chi: \mathfrak{g}_D \rightarrow \mathbb{C}^k$. This yields a decomposition

$$\mathfrak{g}_D = \ker d\chi \oplus \bigoplus_{i=1}^k \mathbb{C}\varepsilon_i, \quad d\chi_i(\varepsilon_j) = \delta_{i,j}.$$

For $\delta \in \mathfrak{g}_D$, the equality $f_i(gx) = \chi_i(g)f_i(x)$ differentiates to $\delta(f_i) = d\chi_i(\delta) \cdot f_i$, which implies (4.1). We have proved

Proposition 4.3. *Any germ of a linear free divisor D satisfies the existence hypothesis (4.1) of Theorem 4.1.* \square

Example 4.4. Let $\sigma_i(y)$ be the i th symmetric function of $y = y_1, \dots, y_k$ and set $N := V(\sigma_k(y))$ and $E := V(\sigma_{k-1}(y))$. As seen in Example 2.4.(5), the divisor

$N + E$ is free. So by Theorem 4.1 and Proposition 4.3, for any germ D of a linear free divisor with distinct irreducible components $D_i = V(f_i)$, the divisor germ $V(\sigma_k(f_1, \dots, f_k)\sigma_{k-1}(f_1, \dots, f_k))$ is also free. No component of $V(\sigma_{k-1}(f_1, \dots, f_k))$ can lie in D , since were this the case, some f_i would divide $\sigma_{k-1}(f_1, \dots, f_k)$ and therefore would divide $f_1 \cdots \widehat{f_i} \cdots f_k$.

If each of the D_i is normal, then in fact $f^{-1}(E)$ is an adjoint divisor of the normalization of D . As the singular locus of any free divisor has pure codimension 1, the singular locus of D is equal to its non-normal locus. The ring of functions on the normalization $\bar{D} = \coprod_{i=1}^k D_i$ has presentation matrix $\text{diag}(f_1, \dots, f_k)$. Thus $V(\sigma_{k-1} \circ f)$ is an adjoint divisor of D .

There is another class of divisors that fits naturally into the setup of Theorem 4.1, namely that of hyperplane arrangements.

Proposition 4.5. *Given the hypothesis (4.1), any germ $N + E$ of a free hyperplane arrangement automatically satisfies the hypothesis on $f^{-1}(E)$ in Theorem 4.1.*

Proof. By assumption, $N = V(y_1, \dots, y_k)$ and $E = \bigcup_{i=k+1}^m H_i$ where $H_i = V(\ell_i)$ for some linear equation $\ell_i(y) = \sum_j \alpha_{i,j} y_j$ for $i = k+1, \dots, m$. We need to show that no component of any of the $\ell_i \circ f$ is divisible by any f_j . Suppose to the contrary that $\ell_i \circ f = g \cdot f_t$. For $s \neq t$, ε_s applied to this equation gives $\alpha_{i,s} f_s = \varepsilon_s(g) \cdot f_t$. Since f_t cannot divide f_s as D is reduced, it follows that $\alpha_{i,s} = 0$ for any $s \neq t$, and thus $\ell_i = \alpha_{i,t} y_t$. This is absurd since A is supposed reduced. \square

Combining Propositions 4.3 and 4.5 proves

Corollary 4.6. *Let $A = \bigcup_{i=1}^m H_i \subset (\mathbb{C}^k, 0)$ be the germ of a free hyperplane arrangement containing the normal crossing divisor $\{y_1 \cdots y_k = 0\}$, and let $D \subset (\mathbb{C}^n, 0)$ be the germ of a linear free divisor whose irreducible components have equations f_1, \dots, f_k . If $f: (\mathbb{C}^n, 0) \rightarrow (\mathbb{C}^k, 0)$ is defined by $f(x) = (f_1(x), \dots, f_k(x))$, then $f^{-1}(A)$ is a free divisor.*

Note that the corollary applies to any essential free arrangement, once suitable coordinates are chosen.

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