Good bases for tame polynomials

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Abstract

An algorithm to compute a good basis of the Brieskorn lattice of a cohomologically tame polynomial is described. This algorithm is based on the results of C. Sabbah and generalizes the algorithm by A. Douai for convenient Newton non–degenerate polynomials.

Key words: tame polynomial, Gauss–Manin system, Brieskorn lattice, V–filtration, mixed Hodge structure, monodromy, good basis
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Introduction

Let $f:\mathbb{C}^{n+1}\longrightarrow\mathbb{C}$ with $n\geq 1$ be a cohomologically tame polynomial function [1]. This means that no modification of the topology of the fibres of f comes from infinity. In particular, the set of critical points $\mathrm{C}(f)$ of f is finite. Then the reduced cohomology of the fibre $f^{-1}(t)$ for $t\notin\mathrm{C}(f)$ is concentrated in dimension n and equals $\mathbb{C}^\mu$ where μ is the Milnor number of f. Moreover, the n–th cohomology of the fibres of f forms a local system H^n on $\mathbb{C}\setminus\mathrm{D}(f)$ where $\mathrm{D}(f)=f(\mathrm{C}(f))$ is the discriminant of f. Hence, there is a monodromy action of the fundamental group $\Pi_1(\mathbb{C}\setminus\mathrm{D}(f),t)$ on H^n_t.

The Gauss–Manin system M of f is a regular holonomic module over the Weyl algebra $\mathbb{C}[t]\langle\partial_t\rangle$ with associated local system H^n on $\mathbb{C}\setminus\mathrm{D}(f)$. The Fourier transform $G:=\widehat{M}$ of M is the $\mathbb{C}[\tau]\langle\partial_\tau\rangle$–module defined by $\tau:=\partial_t$ and

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$\partial_\tau = -t$. The monodromy T_∞ of M around $\mathrm{D}(f)$ can be identified with the inverse of the monodromy $\widehat{T}_0$ of G at 0. It turns out that ∂_t is invertible on M and hence G is a $\mathbb{C}[\tau,\theta]$–module where $\theta := \tau^{-1}$. A finite $\mathbb{C}[\tau]$– resp. $\mathbb{C}[\theta]$–submodule $L \subset G$ such that $L[\theta] = G$ resp. $L[\tau] = G$ is called a $\mathbb{C}[\tau]$– resp. $\mathbb{C}[\theta]$–lattice. The regularity of M at ∞ implies that G is singular at most in $\{0,\infty\}$ and where $0 := \{\tau = 0\}$ is regular and $\infty := \{\theta = 0\}$ of type 1. In particular, the V–filtration $V_\bullet$ on G at 0 consists of $\mathbb{C}[\tau]$–lattices.

The Brieskorn lattice $G_0 \subset G$ is a t–invariant $\mathbb{C}[\theta]$–submodule of G such that $G = G_0[\tau]$. C. Sabbah [1] proved that G_0 is a free $\mathbb{C}[t]$– and $\mathbb{C}[\theta]$–module of rank μ. In particular, G is a free $\mathbb{C}[\tau,\theta]$–module of rank μ. By definition, the spectrum of a $\mathbb{C}[\theta]$–lattice $L \subset G$ is the spectrum of the induced V–filtration $V_\bullet(L/\theta L)$ and the spectrum of f is the spectrum of G_0.

C. Sabbah [1] showed that there is a natural mixed Hodge structure on the moderate nearby cycles of G with Hodge filtration induced by G_0. This leads to the existence of good bases of the Brieskorn lattice. For a basis $\underline{\phi} = \phi_1,\dots,\phi_\mu$ of a t–invariant $\mathbb{C}[\theta]$–lattice,

$$t \circ \underline{\phi} = \underline{\phi} \circ (A^{\underline{\phi}} + \theta^2\partial_\theta)$$

where $A^{\underline{\phi}} \in \mathbb{C}[\theta]^{\mu\times\mu}$. A $\mathbb{C}[\theta]$–basis $\underline{\phi}$ of G_0 is called good if $A^{\underline{\phi}} = A_0^{\underline{\phi}} + \theta A_1^{\underline{\phi}}$ where $A_0^{\underline{\phi}}, A_1^{\underline{\phi}} \in \mathbb{C}^{\mu\times\mu}$,

$$A_1^{\underline{\phi}} = \begin{pmatrix} \alpha_1 & & \\ & \ddots & \\ & & \alpha_\mu \end{pmatrix}$$

and $\phi_i \in V_{\alpha_i}G_0$ for all $i \in [1,\mu]$. One can read off the monodromy $T_\infty = \widehat{T}_0^{-1}$ from $A^{\underline{\phi}}$ immediately. The diagonal $\underline{\alpha} = \alpha_1,\dots,\alpha_\mu$ is the spectrum of f and determines with $\mathrm{gr}_1^V A_0$ the spectral pairs of f. The latter correspond to the Hodge numbers of the above mixed Hodge structure.

Analogous results to those above were first obtained in a local situation where $f : (\mathbb{C}^n,\underline{0}) \longrightarrow (\mathbb{C},0)$ is a holomorphic function germ with an isolated critical point [2–7]. In this situation, the role of the Fourier transform is played by microlocalization and the algorithms in [8,9] compute A_0 and A_1 for a good $\mathbb{C}\{\{\theta\}\}$–basis of the (local) Brieskorn lattice. But [9] and [8, 7.4–5] do not apply to the global situation.

A. Douai [10] explained how to compute a good basis of G_0 if f is convenient and Newton non–degenerate using the equality of the V– and Newton filtration [11,1] and a division algorithm with respect to the Newton filtration [12,13].

The intention of this article is to describe an explicit algorithm to compute a good basis of G_0 for an arbitrary cohomologically tame polynomial f. This algorithm is based on the following idea:

Let $\underline{x} = x_0, \ldots, x_n$ be a coordinate system on $\mathbb{C}^{n+1}$. Then the Brieskorn lattice G_0 can be identified with the quotient

$$\mathbb{C}[\underline{x}, \theta] \Big/ \sum_{i=0}^{n} (\partial_{x_i}(f) - \theta\partial_{x_i})(\mathbb{C}[\underline{x}, \theta])$$

of non–finite $\mathbb{C}[\theta]$–modules. The degree with respect to $\underline{x}$ defines an increasing filtration $\mathbb{C}[\underline{x}, \theta]_\bullet$ by finite $\mathbb{C}[\theta]$–modules on $\mathbb{C}[\underline{x}, \theta]$ and hence

$$G_0^{k,l} := \mathbb{C}[\underline{x}, \theta]_k \Big/ \Big(\mathbb{C}[\underline{x}, \theta]_k \cap \sum_{i=0}^{n} (\partial_{x_i}(f) - \theta\partial_{x_i})(\mathbb{C}[\underline{x}, \theta]_l) \Big)$$

are finite $\mathbb{C}[\theta]$–modules. For $k \gg 0$ and $l \gg 0$, $G_0^{k,l} = G_0$ by the finiteness of G_0. But, a priori, there is no bound for these indices.

By Gröbner basis methods, one can compute cyclic generators $\underline{\phi}$ of a t–invariant $\mathbb{C}[\theta]$–sublattice $G_0^{k,l} \subset G_0$. By an argument of A. Khovanskii and A. Varchenko [11], $G_0^{k,l} = G_0$ if and only if the mean values of the spectra coincide. By the t–invariance of $G_0^{k,l}$, one can compute the spectrum of $G_0^{k,l}$ like that of G_0 below. The mean value of the spectrum of G_0 is known to be $\frac{n+1}{2}$. So if the mean value of the spectrum of $G_0^{k,l}$ is not $\frac{n+1}{2}$ then one has to increase k. This process terminates with $G_0^{k,l} = G_0$.

Then one can compute $A^{\underline{\phi}}$ for the $\mathbb{C}[\theta]$–basis $\underline{\phi}$ of G_0. By a saturation process, one can compute the V–filtration and, by a Gröbner basis computation, the spectrum of G_0 and the Hodge filtration. Then one can compute a $\mathbb{C}[\tau, \theta]$–basis of G which is compatible with the V–filtration refined by an opposite Hodge filtration. In terms of this basis, one can compute a good basis of G_0 by a simultaneous normal form computation and basis transformation.

We denote rows vectors $\underline{v}$ by a lower bar and column vectors $\overline{v}$ by an upper bar. In general, lower indices are column indices and upper indices are row indices. We denote by $\{M\}$ the set and by $\langle M \rangle R$ the R–linear span of the of columns of a matrix M. We denote by lead the leading term and by lexp the leading exponent with respect to a monomial ordering. We denote by E the unit matrix and by $\overline{e}_i$ the ith unit vector.

1 Gauss–Manin system

Let $f : \mathbb{C}^{n+1} \longrightarrow \mathbb{C}$ with $n \geq 1$ be a polynomial function. Let $\mathcal{O}$ be the sheaf of regular functions and $(\Omega^\bullet, \mathrm{d})$ the complex of polynomial differential forms on

$\mathbb{C}^{n+1}$. Then the Gauss–Manin System $f_+\mathcal{O}$ of f is represented by the complex of left $\mathbb{C}[t]\langle\partial_t\rangle$–modules

$$(\Omega^{\bullet+n+1}[\partial_t], \mathrm{d} - \partial_t \mathrm{d}f)$$

[5, 15] and has regular holonomic cohomology [14, VII.12.2]. The coefficients of the differentials are the differentials of the complexes $(\Omega^\bullet, \mathrm{d})$ and $(\Omega^\bullet, \mathrm{d}f)$.

Lemma 1 (Poincaré Lemma) *The complex of* $\mathbb{C}$*–vector spaces*

$$0 \longrightarrow \mathbb{C} \longrightarrow (\Omega^\bullet, \mathrm{d}) \longrightarrow 0$$

is exact [15, Ex. 16.15].

From now on, we assume that set of critical points $\mathrm{C}(f)$ of f is finite. Then the following lemma holds.

Lemma 2 (De Rham Lemma)

(1) $\mathrm{H}^k(\Omega^\bullet, \mathrm{d}f) = 0$ *for* $k \neq n+1$.
(2) $\dim_\mathbb{C} \mathrm{H}^{n+1}(\Omega^\bullet, \mathrm{d}f) < \infty$.

Proof. If $\mathrm{C}(f)$ is finite then

$$\mathbb{C}[\underline{x}]/\langle\underline{\partial}(f)\rangle \cong \Omega^{n+1}/\mathrm{d}f \wedge \Omega^{n-1} = \mathrm{H}^{n+1}(\Omega^\bullet, \mathrm{d}f)$$

is a finite $\mathbb{C}$-vector space and hence $\underline{\partial}(f)$ is a regular sequence in $\mathbb{C}[\underline{x}]$. Then the cohomology of the Koszul complex $(\Omega^\bullet, \mathrm{d}f)$ is concentrated in dimension $n+1$ [15, Cor. 17.5].

The image M_0 of Ω^{n+1} in $M := \mathrm{H}^0(f_+\mathcal{O})$ is the key for an algorithmic approach to the Gauss–Manin system. It determines the differential structure of M and it can be identified with a quotient of $\mathbb{C}[\underline{x}]$.

Proposition 3

(1) $\mathrm{H}^k(f_+\mathcal{O}) = 0$ *for* $k \notin \{-n, 0\}$.
(2) ∂_t *is invertible on* M.
(3) $M_0 = \Omega^{n+1}/\mathrm{d}f \wedge \mathrm{d}\Omega^{n-1}$.

Proof. This follows from Lemma 1 and 2 and [5, 15.2.2].

The Fourier transform $\widehat{M}$ of M is the left $\mathbb{C}[\tau]\langle\partial_\tau\rangle$–module defined by the isomorphism

$$\tau := \partial_t, \quad \partial_\tau := -t$$

of $\mathbb{C}[\tau]\langle\partial_\tau\rangle$ and $\mathbb{C}[t]\langle\partial_t\rangle$ [16, 2.1]. By Proposition 3, M is a $\mathbb{C}[\theta]\langle\partial_\theta\rangle$–module with

$$\theta := \tau^{-1}, \quad \partial_\theta := -\tau^2\partial_\tau$$

and M_0 is a $\mathbb{C}[\theta]$-submodule. Note that $t = \theta^2\partial_\theta$.

Definition 4 *Let G be the $\mathbb{C}[\theta]\langle\partial_\theta\rangle$-module $\widehat{M}$. Then the Brieskorn lattice G_0 of f is the $\mathbb{C}[\theta]\langle t\rangle$-submodule M_0 of G.*

Since M is regular at ∞, G is singular at most in $\{0, \infty\}$ where $0 := \{\tau = 0\}$ is regular and $\infty := \{\theta = 0\}$ of type 1 [17, V.2.a].

From now on, we assume that f is cohomologically tame. By definition [1, 8], this means that there is a compactification

$$\begin{array}{ccc} \mathbb{C}^{n+1} & \overset{j}{\hookrightarrow} & \overline{\mathbb{C}^{n+1}} \\ & \searrow^{f} & \downarrow{\scriptstyle \overline{f}} \\ & & \mathbb{C} \end{array}$$

where $\overline{\mathbb{C}^{n+1}}$ is quasi–projective and $\overline{f}$ is proper such that, for all $t \in \mathbb{C}$, the support of the vanishing cycle complex $\phi_{\overline{f}-t}\,\mathrm{R}\,j_*\mathbb{Q}$ is a finite subset of $\mathbb{C}^{n+1}$. In particular, $\mathrm{C}(f)$ is finite and hence, by Lemma 2, the Milnor number

$$\mu := \dim_{\mathbb{C}} \mathrm{H}^{n+1}(\Omega^\bullet, \mathrm{d}f) = \dim_{\mathbb{C}}(\Omega^{n+1}/\mathrm{d}f \wedge \Omega^n)$$

of f is finite. Then the following theorem holds.

Theorem 5 (C. Sabbah [1, 10.1–3]) *G_0 is a free $\mathbb{C}[t]$– and $\mathbb{C}[\theta]$–module of rank μ.*

In particular, G is a free $\mathbb{C}[\tau, \theta]$–module of rank μ.

2 Brieskorn lattice

A finite $\mathbb{C}[\tau]$– resp. $\mathbb{C}[\theta]$–submodule $L \subset G$ such that $L[\theta] = G$ resp. $L[\tau] = G$ is called a $\mathbb{C}[\tau]$– resp. $\mathbb{C}[\theta]$–lattice. By Theorem 5, a lattice is free of rank μ. In terms of a $\mathbb{C}[\theta]$–basis of G_0, the $\mathbb{C}[\theta]\langle\partial_\theta\rangle$–module structure of G is determined by the basis representation of t on G_0. The following lemma shows that the latter is determined by a matrix with coefficients in $\mathbb{C}[\theta]$.

Definition 6 *Let $\underline{\phi}$ be a basis of a t–invariant $\mathbb{C}[\theta]$–sublattice $L \subset G$. Then the matrix $A^{\underline{\phi}} \in \mathbb{C}[\theta]^{\mu\times\mu}$ of t with respect to $\underline{\phi}$ is defined by*

$$\underline{\phi}A^{\underline{\phi}} := t\underline{\phi}.$$

Lemma 7 *Let $\underline{\phi}$ be a basis of a t–invariant $\mathbb{C}[\theta]$–sublattice $L \subset G$. Then*

$$t \circ \underline{\phi} = \underline{\phi} \circ (A^{\underline{\phi}} + \theta^2\partial_\theta).$$

Proof. Since $t = \theta^2\partial_\theta$,

$$\begin{aligned} t \circ \underline{\phi}\Big(\sum_k \overline{p}_k\theta^k\Big) &= t\sum_k \underline{\phi}\overline{p}_k\theta^k \\ &= \sum_k t(\underline{\phi}\overline{p}_k)\theta^k + \underline{\phi}\overline{p}_k\theta^2\partial_\theta\theta^k \\ &= \underline{\phi} \circ \Big(A^{\underline{\phi}} + \theta^2\partial_\theta\Big)\Big(\sum_k \overline{p}_k\theta^k\Big) \end{aligned}$$

and hence $t \circ \underline{\phi} = \underline{\phi} \circ (A^{\underline{\phi}} + \theta^2\partial_\theta)$.

The following lemma gives a presentation of the $\mathbb{C}[\theta]\langle t\rangle$–module G_0. This presentation shall be used to compute t on G_0.

Lemma 8 *There is an isomorphism of $\mathbb{C}[\theta]\langle t\rangle$–modules*

$$G_0 = \Omega^{n+1}[\theta]/(\mathrm{d}f - \theta\mathrm{d})(\Omega^n[\theta]).$$

Proof. By Theorem 5,

$$G = \Omega^{n+1}[\tau,\theta]/(\mathrm{d}f - \theta\mathrm{d})(\Omega^n[\tau,\theta]).$$

By definition, G_0 is the image of $\Omega^{n+1}[\theta]$ in G and hence

$$G_0 = \Omega^{n+1}[\theta]\Big/\Big((\mathrm{d}f - \theta\mathrm{d})(\Omega^n[\tau,\theta]) \cap \Omega^{n+1}[\theta]\Big).$$

By Lemma 2, $\mathrm{d}\ker(\mathrm{d}f) \subset \mathrm{d}f \wedge \mathrm{d}\Omega^{n-1} \subset \ker(\mathrm{d}f)$ and hence

$$(\mathrm{d}f - \theta\mathrm{d})(\Omega^n[\tau,\theta]) \cap \Omega^{n+1}[\theta] = (\mathrm{d}f - \theta\mathrm{d})(\Omega^n[\theta]).$$

Let $\underline{x} = x_0, \ldots, x_n$ be coordinates on $\mathbb{C}^{n+1}$ with corresponding partial derivatives $\underline{\partial} = \partial_{x_0}, \ldots, \partial_{x_n}$. Let

$$t := f + \theta^2\partial_\theta \in \mathbb{C}[\underline{x},\theta]\langle\partial_\theta\rangle.$$

Then, by Lemma 8, we can identify

$$G = \mathbb{C}[\underline{x}, \tau, \theta]/(\underline{\partial}(f) - \theta\underline{\partial})(\mathbb{C}[\underline{x}, \tau, \theta]^{n+1})$$

as $\mathbb{C}[\tau, \theta]\langle t\rangle$–modules and

$$G_0 = \mathbb{C}[\underline{x}, \theta]/(\underline{\partial}(f) - \theta\underline{\partial})(\mathbb{C}[\underline{x}, \theta]^{n+1})$$

as $\mathbb{C}[\theta]\langle t\rangle$–modules. These modules are quotients of non–finite $\mathbb{C}[\theta]$–modules. On the numerator and denominator, the degree with respect to $\underline{x}$ defines an increasing filtration by finite $\mathbb{C}[\theta]$–modules. The following algorithm computes t on G_0 by an approximation process with respect to these filtrations.

Definition 9 *The degree* $\deg_{\underline{x}}$ *with respect to* $\underline{x}$ *defines an increasing filtration* $\mathbb{C}[\underline{x}, \theta]_\bullet$ *on* $\mathbb{C}[\underline{x}, \theta]$ *by finite* $\mathbb{C}[\theta]$*–modules*

$$\mathbb{C}[\underline{x}, \theta]_k := \{p \in \mathbb{C}[\underline{x}, \theta] \mid \deg_{\underline{x}}(p) \leq k\}$$

such that $t\mathbb{C}[\underline{x}, \theta]_\bullet \subset \mathbb{C}[\underline{x}, \theta]_{\bullet+\deg(f)}$. *We define the finite* $\mathbb{C}[\theta]$*–modules*

$$G_0^k := \mathbb{C}[\underline{x}, \theta]_k \Big/ \Big((\underline{\partial}(f) - \underline{\partial}\theta)(\mathbb{C}[\underline{x}, \theta]^{n+1}) \cap \mathbb{C}[\underline{x}, \theta]_k\Big),$$
$$G_0^{k,l} := \mathbb{C}[\underline{x}, \theta]_k \Big/ \Big((\underline{\partial}(f) - \underline{\partial}\theta)(\mathbb{C}[\underline{x}, \theta]_l^{n+1}) \cap \mathbb{C}[\underline{x}, \theta]_k\Big).$$

Algorithm 1

Input: *(a) A cohomologically tame polynomial* $f \in \mathbb{C}[\underline{x}]$.
(b) An integer $k \geq 0$.

Output: *(a) A vector* $\underline{\phi} \in \mathbb{C}[\underline{x}, \theta]^\mu$ *such that* $[\underline{\phi}]$ *is a basis of a* t*–invariant* $\mathbb{C}[\theta]$*–lattice* $L_k \subset G_0$ *and* $L_k = G_0$ *for* $k \gg 0$.
(b) The matrix $A = A^{[\underline{\phi}]} \in \mathbb{C}[\theta]^{\mu\times\mu}$.

(1) Set $l := k$.

(2) Set $l := l + 1$.

(3) Compute a reduced Gröbner basis

$$\underline{g} := \mathrm{GB}\Big((\underline{\partial}(f) - \underline{\partial}\theta)(\underline{x}^{\underline{\alpha}}\overline{e}_i) \mid (\underline{\alpha}, i) \in \mathbb{N}^{n+1} \times [0, n], |\underline{\alpha}| \leq l\Big)$$

of $(\underline{\partial}(f) - \underline{\partial}\theta)\Big(\mathbb{C}[\underline{x}, \theta]_l^{n+1}\Big)$ *with respect to a monomial ordering* $>$ *on* $\{\underline{x}^{\underline{\alpha}}\theta^i \mid (\underline{\alpha}, i) \in \mathbb{N}^{n+1} \times \mathbb{N}\}$ *such that*

$$|\underline{\alpha}| > |\underline{\beta}| \Rightarrow \underline{\alpha} > \underline{\beta},$$
$$(\underline{\alpha}, i) > (\underline{\beta}, j) \Leftrightarrow \underline{\alpha} > \underline{\beta} \vee (\underline{\alpha} = \underline{\beta} \wedge i > j)$$

for all $(\underline{\alpha}, i), (\underline{\beta}, j) \in \mathbb{N}^{n+1} \times \mathbb{N}$.

(4) Find the minimal k_0 *with*

$$k_0 < |\alpha| \leq k \Rightarrow \underline{x}^{\underline{\alpha}} \in \langle \mathrm{lead}(\underline{g})\rangle\mathbb{C}[\theta].$$

(5) Compute $\underline{\phi} \in \mathbb{C}[\underline{x},\theta]_{k_0}^{\gamma}$ *such that* $[\underline{\phi}]$ *are cyclic generators of*

$$G_0^{k_0,l} = \mathbb{C}[\underline{x},\theta]_{k_0}/\langle g_i \mid \deg_{\underline{x}}(g_i) \leq k_0\rangle\mathbb{C}[\theta] \cong G_0^{k,l}$$

and $\rho := \mathrm{rk}(G_0^{k_0,l})$ *using [18, 2.6.3].*

(6) If $\rho > \mu$ *or* $\gamma > \rho = \mu$ *then go to (2).*

(7) If $\rho < \mu$ *then set* $k := k+1$ *and go to (2).*

(8) If $k_0 + \deg(f) > k$ *then set* $k := k+1$ *and go to (2).*

(9) If $[\underline{\phi}]$ *is not a* $\mathbb{C}$*–basis of* $\mathbb{C}[\underline{x}]/\langle\underline{\partial}(f)\rangle\mathbb{C}[\underline{x}]$ *then set* $k := k+1$ *and go to (2).*

(10) Compute a normal form $\mathrm{NF}(t\underline{\phi},\underline{g})$ *of* $t\underline{\phi}$ *with respect to* $\underline{g}$*.*

(11) Compute the basis representation $A \in \mathbb{C}[\theta]^{\mu\times\mu}$ *of* $[\mathrm{NF}(t\underline{\phi},\underline{g})]$ *with respect to the* $\mathbb{C}[\theta]$*–basis* $[\underline{\phi}]$ *of* $L_k := G_0^{k_0,l}$*.*

(12) Return $\underline{\phi}$ *and* A*.*

Lemma 10 *Algorithm 1 terminates and is correct.*

Proof. Since $\underline{\partial}(f) - \underline{\partial}\theta$ is $\mathbb{C}[\theta]$–linear,

$$(\underline{\partial}(f) - \underline{\partial}\theta)(\mathbb{C}[\underline{x},\theta]_l^{n+1}) = \langle(\underline{\partial}(f) - \underline{\partial}\theta)(\underline{x}^{\underline{\alpha}}\overline{e}_i) \mid (\underline{\alpha}, i) \in \mathbb{N}^{n+1} \times [0,n], |\underline{\alpha}| \leq l\rangle\mathbb{C}[\theta].$$

By definition of the monomial ordering,

$$(\underline{\partial}(f) - \underline{\partial}\theta)(\mathbb{C}[\underline{x},\theta]_l^{n+1}) \cap \mathbb{C}[\underline{x},\theta]_k = \langle g_i \mid \deg_{\underline{x}}(g_i) \leq k\rangle\mathbb{C}[\theta]$$

and hence, by definition of k_0,

$$\begin{aligned} G_0^{k,l} &= \mathbb{C}[\underline{x},\theta]_k/\langle g_i \mid \deg_{\underline{x}}(g_i) \leq k\rangle\mathbb{C}[\theta] \\ &\cong \mathbb{C}[\underline{x},\theta]_{k_0}/\langle g_i \mid \deg_{\underline{x}}(g_i) \leq k_0\rangle\mathbb{C}[\theta] = G_0^{k_0,l}. \end{aligned}$$

Because of step (2), l is strictly increasing for fixed k. There are $\mathbb{C}[\theta]$–linear maps

$$G_0^{k,l} \xrightarrow{\pi_{k,l}} G_0^{k} \xhookrightarrow{\iota_k} G_0$$

where ι_k is an isomorphism for $k \gg 0$ and $\pi_{k,l}$ is an isomorphism for fixed k and $l \gg 0$. By Theorem 5, G_0^k is a free $\mathbb{C}[\theta]$–module of rank at most μ. Hence, if condition (6) holds then $\pi^{k,l}$ is not an isomorphism and if condition (7) holds then ι^k is not an isomorphism.

By Theorem 5, there is a $\underline{\psi} \in \mathbb{C}[\underline{x},\theta]^{\mu}$ such that $[\underline{\psi}]$ is a $\mathbb{C}[\theta]$–basis of G_0. In particular, ι^k is an isomorphism for $k \geq \deg_{\underline{x}}(\underline{\psi})$ and hence

$$\iota^k \circ \pi^{k,l} : G_0^{k,l} \hookrightarrow\!\!\!\!\rightarrow G_0$$

is an isomorphism and conditions (6) and (7) do not hold for $k \geq \deg_{\underline{x}}(\underline{\psi})$ and $l \gg 0$. By Lemma 8, for each $\underline{\alpha} \in \mathbb{N}^{n+1}$, there is a matrix $M^{\underline{\alpha}} \in \mathbb{C}[\theta]^{\mu\times\mu}$ such that

$$\underline{x}^{\underline{\alpha}} - \underline{\psi} M^{\underline{\alpha}} \in (\underline{\partial}(f) - \underline{\partial}\theta)(\mathbb{C}[\underline{x},\theta]^{n+1}).$$

If $|\underline{\alpha}| > \deg_{\underline{x}}(\underline{\psi})$ then $\underline{x}^{\underline{\alpha}} - \underline{\psi} M^{\underline{\alpha}} \in (\underline{\partial}(f) - \underline{\partial}\theta)(\mathbb{C}[\underline{x},\theta]^n_l) \cap \mathbb{C}[\underline{x},\theta]_{|\underline{\alpha}|}$ and hence

$$\underline{x}^{\underline{\alpha}} \in \mathrm{lead}((\underline{\partial}(f) - \underline{\partial}\theta)(\mathbb{C}[\underline{x},\theta]^{n+1}_l)) = \langle \mathrm{lead}(\underline{g})\rangle \mathbb{C}[\theta]$$

for $l \gg 0$. Hence, by definition of k_0, $k_0 \leq \deg_{\underline{x}}(\underline{\psi})$ for $k > \deg_{\underline{x}}(\underline{\psi})$ and $l \gg 0$ and, in particular, condition (8) does not hold for $k \geq \deg_{\underline{x}}(\underline{\psi}) + \deg(f)$ and $l \gg 0$. Since $[\underline{\psi}]$ is a $\mathbb{C}[\theta]$–basis of G_0, $[\underline{\psi}]$ is a $\mathbb{C}$–basis of

$$G_0/\theta G_0 = \mathbb{C}[\underline{x}]/\langle \underline{\partial}(f)\rangle \mathbb{C}[\underline{x}]$$

and hence condition (9) does not hold for $k \geq \deg_{\underline{x}}(\underline{\psi})$ and $l \gg 0$. This proves that the algorithm terminates.

Since $[\underline{\phi}]$ is a $\mathbb{C}$–basis of $\mathbb{C}[\underline{x}]/\langle \underline{\partial}(f)\rangle \mathbb{C}[\underline{x}] = G_0/\theta G_0$, $\iota^k \circ \pi^{k,l}$ is injective and $[\underline{\phi}]$ is a basis of the $\mathbb{C}[\theta]$–lattice $L_k = G_0^{k_0,l} \subset G_0$. Since $\underline{\phi} \in \mathbb{C}[\underline{x},\theta]^\mu_{k_0}$ and $k_0 + \deg(f) \leq k$, $t\underline{\phi} \in \mathbb{C}[\underline{x},\theta]^\mu_k$ and hence, by definition of k_0, $\mathrm{NF}(t\underline{\phi}, \underline{g}) \in \mathbb{C}[\underline{x},\theta]^\mu_{k_0}$. By Lemma 8,

$$t[\underline{\phi}] = [t\underline{\phi}] = [\mathrm{NF}(t\underline{\phi},\underline{g})] = [\underline{\phi}A] = [\underline{\phi}]A$$

and hence L_k is t–invariant and $A = A^{[\underline{\phi}]}$. This proves that the algorithm is correct.

A priori, we do not know a k_0 such that $L_k = G_0$ for all $k \geq k_0$. We shall solve this problem by a criterion on the spectrum with respect to the V–filtration.

3 V–filtration

Definition 11 *The V–filtration $V_\bullet$ on $\mathbb{C}[\tau]\langle\partial_\tau\rangle$ is the increasing filtration by $V_0\mathbb{C}[\tau]\langle\partial_\tau\rangle$–modules*

$$\begin{aligned} V_{-k}\mathbb{C}[\tau]\langle\partial_\tau\rangle &:= \tau^k \mathbb{C}[\tau]\langle\tau\partial_\tau\rangle, \\ V_{k+1}\mathbb{C}[\tau]\langle\partial_\tau\rangle &:= V_k\mathbb{C}[\tau]\langle\partial_\tau\rangle + \partial_\tau V_k\mathbb{C}[\tau]\langle\partial_\tau\rangle \end{aligned}$$

for all $k \geq 0$.

Proposition 12 *There is a unique $V_\bullet\mathbb{C}[\tau]\langle\partial_\tau\rangle$–good filtration $V_\bullet$ on G by $\mathbb{C}[\tau]$–lattices such that $\tau\partial_\tau + \alpha$ is nilpotent on $\mathrm{gr}^V_\alpha G$ for all α.*

Proof. Since G is regular at 0, this follows from [19, 2.3.2, 4.1, 5.1.5].

Definition 13 *$V_\bullet G$ is called the V–filtration on G.*

The following criterion shall be used to compute the V–filtration on G.

Lemma 14 *Let $L \subset G$ be a $\tau\partial_\tau$–invariant $\mathbb{C}[\tau]$–lattice with*

$$\operatorname{spec}(-\tau\partial_\tau \in \operatorname{End}(L/\tau L)) \subset [\alpha, \alpha - 1)$$

for some α. Then $L = V_\alpha G$.

Proof. Let $\operatorname{spec}(-\tau\partial_\tau \in \operatorname{End}(L/\tau L)) = \{\underline{\alpha}\}$ with

$$\alpha \geq \alpha_1 > \cdots > \alpha_\nu > \alpha - 1$$

Let $\phi : L/\tau L \longrightarrow L$ be a $\mathbb{C}[\tau]$–basis of L and

$$C_{\alpha_i} := \phi(\ker((\tau\partial_\tau + \alpha_i)^\mu \in \operatorname{End}(L/\tau L)))$$

for all $i \in [1, \nu]$. Let

$$U_{\alpha_j - p} := \tau^p \bigoplus_{i=j}^{\nu} C_{\alpha_i} \oplus \tau^{p+1} L$$

for all $i \in [1, \nu]$ and $p \in \mathbb{Z}$. Then $U_\bullet$ is an increasing filtration on G by $\tau\partial_\tau$–invariant $\mathbb{C}[\tau]$–lattices. By construction, $\tau\partial_\tau + \alpha_i - p$ is nilpotent on $\operatorname{gr}^U_{\alpha_i - p} G$ and $U_{\alpha_i - p} = \tau^p U_{\alpha_i}$ for all $i \in [1, \nu]$ and $p \in \mathbb{Z}$. Since

$$\begin{aligned}
\partial_\tau U_{\alpha_j - p} &= \tau^{p-1}(\tau\partial_\tau + p - 1) \bigoplus_{i=j}^{\nu} C_{\alpha_i} \oplus \tau^p(\tau\partial_\tau + p) L \\
&\subset \tau^{p-1}(\tau\partial_\tau + p - 1) \bigoplus_{i=j}^{\nu} C_{\alpha_i} \oplus \tau^p(\tau\partial_\tau + p) \bigoplus_{i=1}^{j-1} C_{\alpha_i} + U_{\alpha_j - p}, \\
U_{\alpha_j - p + 1} &= \tau^{p-1} \bigoplus_{i=j}^{\nu} C_{\alpha_i} \oplus \tau^p \bigoplus_{i=1}^{j-1} C_{\alpha_i} \oplus \tau^{p+1} L \\
&\subset \tau^{p-1} \bigoplus_{i=j}^{\nu} C_{\alpha_i} \oplus \tau^p \bigoplus_{i=1}^{j-1} C_{\alpha_i} + U_{\alpha_j - p},
\end{aligned}$$

$\partial_\tau U_{\alpha_j - p} + U_{\alpha_j - p} = U_{\alpha_j - p + 1}$ for $p > \alpha_j + 1$ and hence $U_\bullet$ is $V_\bullet\mathbb{C}[\tau]\langle\partial_\tau\rangle$–good. Then, by Proposition 12, $U_\bullet G = V_\bullet G$ and hence $L = V_\alpha G$.

The following algorithm computes the V–filtration using the criterion in Lemma 14. For a given $\mathbb{C}[\theta]$–lattice with $\mathbb{C}[\theta]$–basis $\underline{\phi}$, $L := \langle\underline{\phi}\rangle\mathbb{C}[\tau]$ is a $\mathbb{C}[\tau]$–lattice with $\mathbb{C}[\tau]$-basis $\underline{\phi}$ and $-\tau\partial_\tau\underline{\phi} = \underline{\phi}B$ where $B = \tau A^{\underline{\phi}} \in \mathbb{C}[\tau, \theta]^{\mu \times \mu}$. By a saturation process of L with respect to $\tau\partial_\tau$, L is replaced by a $\tau\partial_\tau$–invariant

$\mathbb{C}[\tau]$–lattice and $\underline{\phi}$ is modified such that $B \in \mathbb{C}[\tau]^{\mu\times\mu}$. Then a sequence of basis transformations modifies $\underline{\phi}$ such that $\operatorname{spec}(B_0) \subset [\alpha, \alpha-1)$ for some α.

Algorithm 2

Input: *The matrix* $A = A^{\underline{\phi}} \in \mathbb{C}[\theta]^{\mu\times\mu}$ *for a basis* $\underline{\phi}$ *of a* t*–invariant* $\mathbb{C}[\theta]$*–lattice* $L \subset G$.

Output:
(a) *A matrix* $U \in \mathbb{C}[\theta]^{\mu\times\mu}$ *such that* $\underline{\phi}U$ *is a* $\mathbb{C}[\tau]$*–basis of* V_α *for some* α.
(b) *A matrix* $B = \sum_{i\geq 0} B_i\tau^i \in \mathbb{C}[\tau]^{\mu\times\mu}$ *such that* $-\tau\partial_\tau(\underline{\phi}U) = \underline{\phi}UB$ *and* $\operatorname{spec}(B_0) = \{\underline{\alpha}\}$ *with* $\alpha \geq \alpha_1 > \cdots > \alpha_\nu > \alpha - 1$.

(1) (a) *Set* $k := 0$ *and* $U_0 := E \in \mathbb{C}^{\mu\times\mu}$.
(b) *Until* $\{(\tau A - \tau\partial_\tau)(U_k)\} \subset \langle U_k\rangle\mathbb{C}[\tau]$ *do:*
  (i) *Set* $k := k+1$.
  (ii) *Compute* $U_k \in \mathbb{C}[\theta]^{\mu\times\mu}$ *with* $\deg(U_k) \leq k(\deg(A)-1)$ *such that*

$$\langle U_{k+1}\rangle\mathbb{C}[\tau] = \langle U_k\rangle\mathbb{C}[\tau] + \langle(\tau A - \tau\partial_\tau)(U_k)\rangle\mathbb{C}[\tau].$$

(c) *Set* $U := U_k$.

(2) (a) *Set* $B = \sum_{i\geq 0} B_i\tau^i := U^{-1}(\tau A - \tau\partial_\tau)(U) \in \mathbb{C}[\tau]^{\mu\times\mu}$.
(b) *Compute* $\{\underline{\alpha}\} := \operatorname{spec}(B_0)$ *and* $j \in [1,\nu]$ *such that*

$$\alpha_1 > \cdots > \alpha_j > \alpha_1 - 1 \geq \alpha_{j+1} > \cdots > \alpha_\nu.$$

(c) *If* $j = \nu$ *then return* U *and* B.
(d) *Compute* $U_0 \in \operatorname{GL}_\mu(\mathbb{C})$ *such that*

$$U_0^{-1}BU_0 = \begin{pmatrix} B^{1,1} & B^{1,2} \\ B^{2,1} & B^{2,2} \end{pmatrix}$$

where

$$\operatorname{spec}(B_0^{1,1}) = \{\alpha_1, \ldots, \alpha_j\},$$
$$\operatorname{spec}(B_0^{2,2}) = \{\alpha_{j+1}, \ldots, \alpha_\nu\},$$

$B_0^{1,2} = 0$, *and* $B_0^{2,1} = 0$.
(e) *Set* $U = \begin{pmatrix} U_1 & U_2 \end{pmatrix} := UU_0$ *and*

$$B = \begin{pmatrix} B^{1,1} & B^{1,2} \\ B^{2,1} & B^{2,2} \end{pmatrix} := U_0^{-1}BU_0.$$

(f) Set $U := \begin{pmatrix} U_1 & \tau^{-1}U_2 \end{pmatrix}$ *and*

$$B := \begin{pmatrix} B^{1,1} & \tau^{-1}B^{1,2} \\ \tau B^{2,1} & B^{2,2}+E \end{pmatrix}.$$

(g) Set $\alpha_i := \alpha_i + 1$ *for* $i = j+1, \ldots, \nu$.
(h) Reorder $\underline{\alpha}$ *and redefine* $j \in [1,\nu]$ *such that*

$$\alpha_1 > \cdots > \alpha_j > \alpha_1 - 1 \geq \alpha_{j+1} > \cdots > \alpha_\nu.$$

(i) Go to (2c).

Remark 15

(1) If $A = A_0 + \theta A_1$ *then* $U_k = U_0 = E$.
(2) If

$$B = \begin{pmatrix} B^{1,1} & B^{1,2} \\ 0 & B^{2,2} \end{pmatrix}$$

with $\operatorname{spec}(B_0^{1,1}) = \{\alpha_1, \ldots, \alpha_j\}$ *and* $\operatorname{spec}(B_0^{2,2}) = \{\alpha_{j+1}, \ldots, \alpha_\nu\}$ *then one can choose*

$$U_0 = \begin{pmatrix} E & U_0^{1,2} \\ 0 & E \end{pmatrix}.$$

Lemma 16 *Algorithm 2 terminates and is correct.*

Proof.

(1) By Lemma 7,

$$\begin{aligned} \langle \underline{\phi} U_{k+1}\rangle \mathbb{C}[\tau] &= \langle \underline{\phi} U_k\rangle \mathbb{C}[\tau] + \langle \underline{\phi} \circ (\tau A - \tau\partial_\tau)(U_k)\rangle \mathbb{C}[\tau] \\ &= \langle \underline{\phi} U_k\rangle \mathbb{C}[\tau] + \tau\langle \underline{\phi} \circ (A + \theta^2\partial_\theta)(U_k)\rangle \mathbb{C}[\tau] \\ &= \langle \underline{\phi} U_k\rangle \mathbb{C}[\tau] + \tau\partial_\tau \langle \underline{\phi} U_k\rangle \mathbb{C}[\tau] \end{aligned}$$

and hence $\{\langle \underline{\phi} U_k\rangle \mathbb{C}[\tau]\}_{k\geq 0}$ is an increasing sequence of finite $\mathbb{C}[\tau]$–modules. Since $\langle \underline{\phi} U_0\rangle \mathbb{C}[\tau] = \mathbb{C}[\tau]^\mu$, one can choose $U_k \in \mathbb{C}[\theta]^{\mu\times\mu}$. The V–filtration on G consists of finite and hence Noetherian $\tau\partial_\tau$–invariant $\mathbb{C}[\tau]$–modules. For some α, $\{\underline{\phi}\} \subset V_\alpha G$ and hence $\langle \underline{\phi} U_k\rangle \mathbb{C}[\tau] \subset V_\alpha$ for all $k \geq 0$. This implies that the sequence $\{\langle U_k\rangle \mathbb{C}[\tau]\}_{k\geq 0}$ is stationary. Then $\langle \underline{\phi} U\rangle \mathbb{C}[\tau] \subset G$ is a $\tau\partial_\tau$–invariant $\mathbb{C}[\tau]$–lattice.
(2) By Lemma 7, $-\tau\partial_\tau \circ \underline{\phi} = \tau t \circ \underline{\phi} = \underline{\phi} \circ (\tau A - \tau\partial_\tau)$ and hence

$$-\tau\partial_\tau \circ \underline{\phi} U = \underline{\phi} U \circ (B - \tau\partial_\tau).$$

The $\tau\partial_\tau$–invariance of the $\mathbb{C}[\tau]$–lattice $\langle \underline{\phi} U\rangle \mathbb{C}[\tau]$ is preserved since

$$\begin{pmatrix} U_1 & \tau^{-1}U_2 \end{pmatrix} = \begin{pmatrix} U_1 & U_2 \end{pmatrix} \begin{pmatrix} E & \\ & \tau^{-1}E \end{pmatrix},$$

$$\begin{pmatrix} B^{1,1} & \tau^{-1}B^{1,2}, \\ \tau B^{2,1} & B^{2,2}+E \end{pmatrix} = \begin{pmatrix} E & \\ & \tau E \end{pmatrix} \left(\begin{pmatrix} B^{1,1} & B^{1,2} \\ B^{2,1} & B^{2,2} \end{pmatrix} - \tau\partial_\tau \right) \begin{pmatrix} E & \\ & \tau^{-1}E \end{pmatrix},$$

and $B^{1,2}\tau^{-1} \in \mathbb{C}[\tau]^{j\times(\mu-j)}$. The index j is strictly increasing since

$$\begin{aligned} \operatorname{spec}(B_0) &= \operatorname{spec} \begin{pmatrix} B_0^{1,1} - E & \tau^{-1}B_0^{1,2} \\ 0 & B_0^{2,2} \end{pmatrix} \\ &= \{\alpha_1, \ldots, \alpha_j, \alpha_{j+1}+1, \ldots, \alpha_\nu + 1\} \end{aligned}$$

and hence the algorithm terminates. Then $L := \langle \underline{\phi} U\rangle \mathbb{C}[\tau] \subset G$ is a $\tau\partial_\tau$–invariant $\mathbb{C}[\tau]$–lattice with $\operatorname{spec}(-\tau\partial_\tau \in \operatorname{End}(L/\tau L)) \subset [\alpha, \alpha-1)$ for $\alpha := \alpha_1$. Hence, by Lemma 14, $L = V_\alpha$ and $\underline{\phi} U$ is a $\mathbb{C}[\tau]$–basis of $V_\alpha G$.

4 Spectrum

The spectrum with respect to the V–filtration shall be used to check equality of $\mathbb{C}[\theta]$–lattices.

Definition 17

(1) The spectrum $\operatorname{spec}(F_\bullet) : \mathbb{Q} \longrightarrow \mathbb{N}$ *of an increasing filtration* $F_\bullet$ *on a finite vector space* V *is defined by*

$$\operatorname{spec}(F_\bullet)(\alpha) := \dim(\operatorname{gr}_\alpha^{F_\bullet} V)$$

for all $\alpha \in \mathbb{Q}$*. The spectrum* $\operatorname{spec}(F_\bullet)$ *of a decreasing filtration* $F^\bullet$ *on* V *is defined analogously.*

(2) The spectrum of a $\mathbb{C}[\theta]$*–lattice* $L \subset G$ *is defined by*

$$\operatorname{spec}(L) := \operatorname{spec}(V_\bullet(L/\theta L)).$$

(3) The spectrum of f *is defined by*

$$\operatorname{spec}(f) := \operatorname{spec}(G_0).$$

The following algorithm computes the spectrum of a t–invariant $\mathbb{C}[\theta]$–lattice by computing a Gröbner basis compatible with the V–filtration.

Algorithm 3

Input: (a) *A matrix* $B = \sum_{i\geq 0} B_i \tau^i \in \mathbb{C}[\tau]^{\mu\times\mu}$ *such that* $-\tau\partial_\tau \underline{\phi} = \underline{\phi} B$ *for a* $\mathbb{C}[\tau]$*–basis* $\underline{\phi}$ *of* V_α *and* $\operatorname{spec}(B_0) = \{\underline{\alpha}\}$ *with* $\alpha \geq \alpha_1 > \cdots > \alpha_\nu > \alpha - 1$.

(b) *A matrix* $M \in \mathbb{C}[\tau,\theta]^{\mu\times\mu}$ *such that* $\underline{\phi}M$ *is a basis of a* t*–invariant* $\mathbb{C}[\theta]$*–lattice* $L \subset G$.

Output: *The spectrum* $\sigma = \operatorname{spec}(L) \in \mathbb{Q}^{\mathbb{N}}$.

(1) *Compute* $U_0 \in \mathrm{GL}_\mu(\mathbb{C})$ *such that*

$$U_0^{-1} B_0 U_0 = \begin{pmatrix} B_0^1 & & \\ & \ddots & \\ & & B_0^\nu \end{pmatrix}$$

where $B_0^i \in \mathbb{C}^{\mu_i\times\mu_i}$ *with* $\operatorname{spec}(B_0^i) = \{\alpha_i\}$ *for* $i \in [1,\nu]$.

(2) *Set* $\underline{\phi} = (\underline{\phi}^i)_{i\in[1,\nu]} := \underline{\phi} U_0$ *and* $M := U_0^{-1} M$.

(3) *Compute a minimal Gröbner basis*

$$M := \mathrm{GB}(M) \in \mathbb{C}[\theta]^{\mu\times\mu}$$

compatible with the ordering $>$ *on* $\{\theta^k \underline{\phi}^i \mid (k,i) \in \mathbb{Z}\times[1,\nu]\}$ *defined by*

$$(k,i) > (l,j) :\Leftrightarrow k > l \vee (k = l \wedge i > j)$$

for all $(k,i),(l,j) \in \mathbb{Z}\times[1,\nu]$.

(4) *Return* $\sigma \in \mathbb{Q}^{\mathbb{N}}$ *with*

$$\sigma(k+\alpha_i) := \#\Big(\{\operatorname{lead}(M)\}_{(k,i)}\Big)$$

for all $(k,i) \in \mathbb{Z}\times[1,\nu]$.

Lemma 18 *Algorithm 3 terminates and is correct.*

Proof. Since $-\tau\partial_\tau\underline{\phi} = \underline{\phi}B$,

$$-\tau\partial_\tau(\theta^l \underline{\phi}^j) \equiv \theta^q \underline{\phi}^j (B_0^j + q) \mod \bigoplus_{(k,i)\leq(l,j)} \langle \theta^k \underline{\phi}^i \rangle \mathbb{C}$$

with $\operatorname{spec}(B_0^j + l) = \{\alpha_j + l\}$. Then, by Lemma 14,

$$V_{\alpha_j+l} G = \bigoplus_{(i,k)\leq(j,l)} \theta^k \langle \underline{\phi}^i \rangle \mathbb{C}$$

and hence, since M is a minimal Gröbner basis,

$$\begin{aligned}
\operatorname{spec}(L)(\alpha_j + l) &= \dim_{\mathbb{C}} \operatorname{gr}^V_{\alpha+l}(L/\theta L) \\
&= \dim_{\mathbb{C}}\Big((\operatorname{gr}^V L/\theta \operatorname{gr}^V L)_{\alpha_j+l}\Big) \\
&= \dim_{\mathbb{C}}\Big(\langle \operatorname{lead}(M)\rangle \mathbb{C}[\theta]/\theta\langle \operatorname{lead}(M)\rangle\mathbb{C}[\theta])_{(q,j)}\Big) \\
&= \dim_{\mathbb{C}}\Big((\langle \operatorname{lead}(M)\rangle \mathbb{C})_{(l,j)}\Big) \\
&= \#\Big(\{\operatorname{lead}(M)\}_{(l,j)}\Big)
\end{aligned}$$

for all $(l, j) \in \mathbb{Z} \times [1, \nu]$.

The following lemma reduces the problem of equality of $\mathbb{C}[\theta]$–lattices to the problem of equality of filtrations on a finite vector space.

Definition 19 *A $\mathbb{C}[\theta]$–lattice $L \subset G$ defines an increasing filtration $L_\bullet$ on G by $\mathbb{C}[\theta]$–lattices $L_p := \tau^p L$ and a corresponding decreasing filtration $L^\bullet := L_{n-\bullet}$. We denote the filtrations defined by G_0 by $G_\bullet$ and $G^\bullet$.*

Lemma 20 *Let $L \subset G$ be a $\mathbb{C}[\theta]$–lattice. Then*

$$\operatorname{gr}^{n-p}_L \operatorname{gr}^V_\alpha G \underset{\tau^p}{\overset{\theta^p}{\rightleftarrows}} \operatorname{gr}^V_{\alpha+p} \operatorname{gr}^n_L G = \operatorname{gr}^V_{\alpha+p}(L/\theta L)$$

is an isomorphism for all $\alpha \in \mathbb{Q}$ and $p \in \mathbb{Z}$.

Proof. This follows from $\theta^p V_\alpha G = V_{\alpha+p} G$ and $L^{n-p} = \tau^p L$ for all $\alpha \in \mathbb{Q}$ and $p \in \mathbb{Z}$.

Definition 21 *The sum $\sum : \mathbb{N}^{\mathbb{Q}} \longrightarrow \mathbb{Q}$ is defined by*

$$\sum \sigma := \sum_{\alpha \in \mathbb{Q}} \alpha\sigma(\alpha)$$

for all $\sigma \in \mathbb{N}^{\mathbb{Q}}$.

The following lemma gives a criterion on the mean value of the spectrum to check equality of filtrations on a finite vector space.

Lemma 22 *Let $F_1^\bullet$ and $F_2^\bullet$ be decreasing filtrations on a finite vector space V with $F_2^\bullet \subset F_1^\bullet$. Then $\sum \operatorname{spec}(F_2^\bullet) \leq \sum \operatorname{spec}(F_1^\bullet)$ and equality implies that $F_2^\bullet = F_1^\bullet$.*

Proof. This is an elementary fact from linear algebra.

The following criterion on the mean value of the spectrum shall be used to check equality of $\mathbb{C}[\theta]$–lattices.

Lemma 23 *Let $L_2 \subset L_1 \subset G$ be $\mathbb{C}[\theta]$–lattices. Then $\sum \operatorname{spec}(L_1) \leq \sum \operatorname{spec}(L_2)$ and equality implies that $L_1 = L_2$.*

Proof. Since $L_2 \subset L_1$,

$$L_2^\bullet \operatorname{gr}^V_{[0,1)} G \subset L_1^\bullet \operatorname{gr}^V_{[0,1)} G$$

where $\operatorname{gr}^V_{[0,1)} G = \bigoplus_{0 \leq \alpha < 1} \operatorname{gr}^V_\alpha G$ and hence, by Lemma 22,

$$\sum \operatorname{spec}\Big(L_2^\bullet \operatorname{gr}^V_{[0,1)} G\Big) \leq \sum \operatorname{spec}\Big(L_1^\bullet \operatorname{gr}^V_{[0,1)} G\Big).$$

By Lemma 20, $\operatorname{spec}(L_i)(\alpha + p) = \operatorname{spec}\Big(L_i^\bullet \operatorname{gr}^V_\alpha G\Big)(n - p)$ and hence

$$\begin{aligned}
\sum \operatorname{spec}(L_i) &= \sum_{0 \leq \alpha < 1} \sum_{p \in \mathbb{Z}} (\alpha + n - p) \operatorname{spec}\Big(L_i^\bullet \operatorname{gr}^V_\alpha G\Big)(p) \\
&= n\mu + \sum_{0 \leq \alpha < 1} \alpha \dim_{\mathbb{C}}\Big(\operatorname{gr}^V_\alpha G\Big) - \sum \operatorname{spec}\Big(L_i^\bullet \operatorname{gr}^V_\alpha G\Big) \\
&= n\mu + \sum_{0 \leq \alpha < 1} \alpha \dim_{\mathbb{C}}\Big(\operatorname{gr}^V_\alpha G\Big) - \sum \operatorname{spec}\Big(L_i^\bullet \operatorname{gr}^V_{[0,1)} G\Big)
\end{aligned}$$

for $i = 1, 2$. This implies that

$$\sum \operatorname{spec}(L_2) - \sum \operatorname{spec}(L_1) = \sum \operatorname{spec}\Big(L_1^\bullet \operatorname{gr}^V_{[0,1)} G\Big) - \sum \operatorname{spec}\Big(L_2^\bullet \operatorname{gr}^V_{[0,1)} G\Big).$$

Let $x \in (L_1 \backslash L_2) \cap \Big(V_{\alpha+p} G \Big\backslash V_{<\alpha+p} G\Big)$ with $0 \leq \alpha < 1$ and minimal $\alpha + p$. Then, in particular, $x \notin \theta L_1$ and hence, by Lemma 20, $0 \neq [\tau^p x] \in \operatorname{gr}^{n-p}_{L_1} \operatorname{gr}^V_\alpha G$. Moreover, there is a $q \geq 1$ such that $\theta^q x \in L_2 \backslash \theta L_2$ and, again by Lemma 20, $0 \neq [\tau^p x] \in \operatorname{gr}^{n-p-q}_{L_2} \operatorname{gr}^V_\alpha G$. This implies that $L_2^{n-p} \operatorname{gr}^V_\alpha G \subsetneq L_1^{n-p} \operatorname{gr}^V_\alpha G$ and hence

$$L_2^\bullet \operatorname{gr}^V_{[0,1)} G \subsetneq L_1^\bullet \operatorname{gr}^V_{[0,1)} G.$$

Then the claim follows from Lemma 22.

The following theorem gives the mean value of the spectrum of G_0.

Theorem 24 (C. Sabbah [1, 11.1]) $\frac{1}{\mu} \sum \operatorname{spec}(G_0) = \frac{n+1}{2}$.

By Theorem 24, one can compute t on G_0 using Algorithm 1, 2, and 3 by increasing k until $\frac{1}{\mu} \sum \operatorname{spec}(L_k) = \frac{n+1}{2}$.

Our final goal is to compute a good basis of G_0. In terms of a good basis of G_0, the matrix of t has degree one and its degree one part determines the spectrum of f.

Definition 25 *Let $\underline{\phi}$ be a $\mathbb{C}[\theta]$–basis of a t–invariant $\mathbb{C}[\theta]$–lattice $L \subset G$. Then $\underline{\phi}$ is called good if $A^{\underline{\phi}} = A_0^{\underline{\phi}} + \theta A_1^{\underline{\phi}}$ where $A_0^{\underline{\phi}}, A_1^{\underline{\phi}} \in \mathbb{C}^{\mu\times\mu}$,*

$$A_1^{\underline{\phi}} = \begin{pmatrix} \alpha_1 & & \\ & \ddots & \\ & & \alpha_\mu \end{pmatrix}$$

and $\phi_i \in V_{\alpha_i}L$ for all $i \in [1,\mu]$.

Lemma 26 *Let $\underline{\phi}$ be a good basis of a t-invariant $\mathbb{C}[\theta]$–lattice $L \subset G$ and*

$$\begin{pmatrix} \alpha_1 & & \\ & \ddots & \\ & & \alpha_\mu \end{pmatrix} := A_1^{\underline{\phi}}.$$

Then $\mathrm{spec}(L)(\alpha) = \#\{i \in [1,\mu] \mid \alpha_i = \alpha\}$.

Proof. Since $\underline{\phi}$ is a $\mathbb{C}[\theta]$–basis of L and $\phi_i \in V_{\alpha_i}$ for all $i \in [1,\mu]$, $\big([\phi_i]\big)_{\alpha_i=\alpha}$ is a $\mathbb{C}$–basis of $\mathrm{gr}^V_\alpha(L/\theta L)$ and hence $\mathrm{spec}(L)(\alpha) = \#\{i \in [1,\mu] \mid \alpha_i = \alpha\}$.

5 Monodromy

Let T_∞ be the monodromy of M around the discriminant $\mathrm{D}(f) = f(\mathrm{C}(f))$ of f and $\widehat{T}_0$ be the monodromy of G at 0.

Theorem 27 (C. Sabbah [16, 1.10]) $T_\infty = \widehat{T}_0^{-1}$.

Using Theorem 27, the monodromy T_∞ can be read off from the matrix of t with respect to a good basis.

Proposition 28 *Let $\underline{\phi}$ be a good basis of a $\mathbb{C}[\theta]$–lattice $L \subset G$ and*

$$\begin{pmatrix} \alpha_1 & & \\ & \ddots & \\ & & \alpha_\mu \end{pmatrix} := A_1^{\underline{\phi}}.$$

Then

$$\exp\Big(-2\pi\mathrm{i}\Big(\mathrm{gr}_1^V\Big(A_0^{\underline{\phi}}\Big) + A_1^{\underline{\phi}}\Big)\Big)$$

is a matrix of T_∞ *where*

$$(V_\alpha(C))_{i,j} := \begin{cases} c_{i,j} & \text{if } \alpha_i \geq \alpha_j + \alpha, \\ 0 & \text{else,} \end{cases}$$

for $C = (c_{i,j})_{i,j} \in \mathbb{C}^{\mu\times\mu}$.

Proof. Since $\underline{\phi}$ is a $\mathbb{C}[\theta]$–basis of L and $\phi_i \in V_{\alpha_1}$ for all $i \in [1,\mu]$, $\Big([\phi_i]\Big)_{\alpha_i=\alpha}$ is a $\mathbb{C}$–basis of $\mathrm{gr}^V_\alpha(L/\theta L)$ and hence, by Lemma 20,

$$-\tau\partial_\tau = \partial_t t : \mathrm{gr}^p_L \mathrm{gr}^V_\alpha G \xrightarrow{\alpha\oplus N} \mathrm{gr}^p_L \mathrm{gr}^V_\alpha G \oplus \mathrm{gr}^{p-1}_L \mathrm{gr}^V_\alpha G$$

for all $\alpha \in \mathbb{Q}$ and $p \in \mathbb{Z}$ where θN is induced by

$$\mathrm{gr}^V_1 \mathrm{gr}^0_L t = \mathrm{gr}^V_1\Big(\underline{\phi} \circ A^{\underline{\phi}}_0 \circ \underline{\phi}^{-1}\Big) = \underline{\phi} \circ \mathrm{gr}^V_1\Big(A^{\underline{\phi}}_0\Big) \circ \underline{\phi}^{-1}.$$

Then, by [19, 6.0.1], $\exp\Big(2\pi\mathrm{i}\Big(\mathrm{gr}^V_1\Big(A^{\underline{\phi}}_0\Big) + A^{\underline{\phi}}_1\Big)\Big)$ is a matrix of $\widehat{T}_0$ and hence, by Theorem 27, $\exp\Big(-2\pi\mathrm{i}\Big(\mathrm{gr}^V_1\Big(A^{\underline{\phi}}_0\Big) + A^{\underline{\phi}}_1\Big)\Big)$ is a matrix of T_∞.

6 Good lattices

The following property is sufficient for the existence of a good basis of a $\mathbb{C}[\theta]$–lattice [1, 5.2]. Recall that a morphism $N : F^\bullet_1 V_1 \longrightarrow F^\bullet_2 V_2$ of filtered vector spaces $F^\bullet_i V_i$ for $i = 1,2$ is called strict if $N(V_1) \cap F^p_2 = N(F^p_2)$ for all $p \in \mathbb{Z}$.

Definition 29 *We call a* t*–invariant* $\mathbb{C}[\theta]$*–lattice* $L \subset G$ *good if*

$$(\tau\partial_\tau + \alpha)^p : L^\bullet \mathrm{gr}^V_\alpha G \longrightarrow L^{\bullet-p} \mathrm{gr}^V_\alpha G$$

is strict for all $\alpha \in \mathbb{Q}$ *and* $p \geq 1$.

The following theorem follows from the fact that

$$N := \bigoplus_{0\leq\alpha<1} (\tau\partial_\tau + \alpha)$$

is a morphism of a natural mixed Hodge structure on the moderate nearby cycles $\psi^{\mathrm{mod}}_\tau G = \bigoplus_{0\leq\alpha<1} \mathrm{gr}^V_\alpha G$ with Hodge filtration induced by $G_\bullet$ as defined in Definition 19 [1, 13.1].

Theorem 30 (C. Sabbah [1, 13.3]) G_0 *is a good* $\mathbb{C}[\theta]$*–lattice.*

The following lemma shall be used to construct an opposite filtration of $L^\bullet$ on $\mathrm{gr}^V G$ for a good lattice L.

Lemma 31 *Let V be a finite vector space, $F^\bullet$ a decreasing filtration on V with $F^p = 0$ for $p > m$, and $N \in \mathrm{End}(V)$ such that*

$$N^p : F^\bullet V \longrightarrow F^{\bullet - p} V$$

strict for all $p \geq 1$. Then $\sum_{q\geq 0} N^q(F^m) = \bigoplus_{q\geq 0} N^q(F^m)$ and

$$N^p : F^\bullet\Big(V\Big/\textstyle\sum_{q\geq 0} N^q(F^m)\Big) \longrightarrow F^{\bullet-p}\Big(V\Big/\textstyle\sum_{q\geq 0} N^q(F^m)\Big)$$

is strict for all $p \geq 1$.

Proof. If $x \in F^m$ with $N^{p+1}(x) \in \sum_{q=0}^{p} N^q(F^m) \subset F^{m-p}$ then

$$N^{p+1}(x) \in N^{p+1}(F^{m+1}) = 0$$

since N^{p+1} is strict and $F^{m+1} = 0$. Hence,

$$N^{p+1}(F^m) + \sum_{q=0}^{p} N^q(F^m) = N^{p+1}(F^m) \oplus \sum_{q=0}^{p} N^q(F^m)$$

and, by induction, $\sum_{q\geq 0} N^q(F^m) = \bigoplus_{q\geq 0} N^q(F^m)$.

Let $N^p(x) \in F^q + \sum_{r\geq 0} N^r(z_r)$ with $z_r \in F^m$ for all $r \geq 0$. If $m - p < q$ then $N^p\Big(x - \sum_{r>p} N^{r-p}(z_r)\Big) \in F^{m-p+1}$ and hence

$$N^p(x) \in N^p(F^{m+1}) + \sum_{r>p} N^r(F^m) \subset N^p(F^{q+p}) + \sum_{r\geq 0} N^r(F^m)$$

since N^p is strict and $F^{m+1} = 0$. If $m-p \geq q$ then $N^p\Big(x - \sum_{r>p} N^{r-p}(z_r)\Big) \in F^q$ and hence $N^p(x) \in N^p(F^{q+p}) + \sum_{r\geq 0} N^r(F^m)$ since N^p is strict. This implies that N^p is strict modulo $\sum_{r\geq 0} N^r(F^m)$ for all $p \geq 1$.

The following algorithm computes a $\mathbb{C}[\tau,\theta]$–basis of G compatible with the V–filtration refined by an opposite filtration of $L^\bullet$ on $\mathrm{gr}^V G$ for a good lattice L. This basis shall be used to compute a good basis of L.

Algorithm 4

Input: (a) *A matrix $B = \sum_{i\geq 0} B_i \tau^i \in \mathbb{C}[\tau]^{\mu\times\mu}$ such that $-\tau\partial_\tau \underline{\phi} = \underline{\phi} B$ for a $\mathbb{C}[\tau]$–basis $\underline{\phi}$ of V_α and $\mathrm{spec}(B_0) = \{\underline{\alpha}\}$ with $\alpha \geq \alpha_1 > \cdots > \alpha_\nu > \alpha - 1$.*

(b) A matrix $M \in \mathbb{C}[\tau,\theta]^{\mu\times\mu}$ *such that* $\underline{\phi}M$ *is a basis of a good* $\mathbb{C}[\theta]$*-lattice* $L \subset G$*.*

Output: *A matrix* $U = (U^{i,p})_{(i,p)\in[1,\nu]\times\mathbb{Z}} \in \mathrm{GL}_\mu(\mathbb{C})$ *such that* $(\underline{\phi}U^{i,q})_{q\geq p}$ *is a* $\mathbb{C}$*-basis of* $L^p \operatorname{gr}^V_{\alpha_i} G$ *and* $\{(\theta\partial_\theta - \alpha_i)(\underline{\phi}U^{i,p})\} \subset \{\underline{\phi}U^{i,p-1}\} + V_{\alpha-1}$ *for all* $(i,p) \in [1,\nu]\times\mathbb{Z}$*.*

(1) Compute $U_0 \in \mathrm{GL}_\mu(\mathbb{C})$ *such that*

$$U_0^{-1}B_0U_0 = \begin{pmatrix} B_0^1 & & \\ & \ddots & \\ & & B_0^\nu \end{pmatrix}$$

where $B_0^i \in \mathbb{C}^{\mu_i\times\mu_i}$ *with* $\mathrm{spec}(B_0^i) = \{\alpha_i\}$ *for* $i \in [1,\nu]$*.*

(2) Set $\underline{\phi} = (\underline{\phi}^i)_{i\in[1,\nu]} := \underline{\phi}U_0$ *and* $M := U_0^{-1}M$*.*

(3) Compute a Gröbner basis

$$M := \mathrm{GB}(M) \in \mathbb{C}[\theta]^{\mu\times\mu}$$

compatible with the ordering $>$ *on* $\{\theta^p\underline{\phi}^i \mid (p,i) \in \mathbb{Z}\times[1,\nu]\}$ *defined by*

$$(p,i) > (q,j) :\Leftrightarrow p > q \vee (p = q \wedge i > j)$$

for all $(p,i),(q,j) \in \mathbb{Z}\times[1,\nu]$*.*

(4) Set $(M^{p,i})_{(p,i)\in\mathbb{Z}\times[1,\nu]} := M$ *where*

$$\{\mathrm{lexp}(M^{p,i})\} = \{(p,i)\}$$

for all $(p,i) \in \mathbb{Z}\times[1,\nu]$*.*

(5) For $i = 1,\dots,r$ *do:*

(a) Compute $(F^{i,p})_{p\in\mathbb{Z}} \in \mathbb{C}^{\mu_i\times\mu_i}$ *such that*

$$F^{i,p} := (\tau^q \,\mathrm{lead}(M^{q,i}))_{q\leq n-p}.$$

(b) Set $N_i := B_0^i - \alpha_i$*.*

(c) Compute $U^i = (U^{i,p})_{p\in\mathbb{Z}} \in \mathbb{C}^{\mu_i\times\mu_i}$ *such that*

$$\langle F^{i,p}\rangle\mathbb{C} = \langle U^{i,q} \mid q \leq p\rangle\mathbb{C}, \quad \{N_iU^{i,p}\} \subset \{U^{i,p-1}\}$$

for all $p \in \mathbb{Z}$*.*

(6) Return

$$U = (U^{i,p})_{(i,p)\in[1,\nu]\times\mathbb{Z}} := U^0 \begin{pmatrix} U^1 & & \\ & \ddots & \\ & & U^\nu \end{pmatrix}.$$

Lemma 32 *Algorithm 4 terminates and is correct.*

Proof. Since $-\tau\partial_\tau\underline{\phi} = \underline{\phi}B$,

$$-\tau\partial_\tau(\theta^q\underline{\phi}^j) \equiv \theta^q\underline{\phi}^j(B_0^j + q) \mod V_{q+\alpha-1}$$

with $\mathrm{spec}(B_0^j + q) = \{\alpha_j + q\}$ and hence, by Lemma 14,

$$V_{\alpha_j+q}G = \bigoplus_{(p,i)\leq(q,j)} \theta^p\langle\underline{\phi}^i\rangle\mathbb{C}$$

for all $(q,j) \in \mathbb{Z}\times[1,\nu]$. Since M is a Gröbner basis, this implies that $\underline{\phi}M^{q,j} \in V_{\alpha_j+q}L$ for all $(q,j) \in \mathbb{Z}\times[1,\nu]$. Then, by Lemma 20,

$$L^\bullet \operatorname{gr}^V_{\alpha_i} G = \left(\langle\underline{\phi}^i F^{i,\bullet}\rangle\mathbb{C} + V_{\alpha_i}\right)\Big/V_{<\alpha_i} \subset \operatorname{gr}^V_{\alpha_i} G$$

and, since L is good and $(\theta\partial_\theta - \alpha_i)\underline{\phi}^i \equiv \underline{\phi}^i N_i \mod V_{\alpha-1}$,

$$N_i^p : \langle F^{i,\bullet}\rangle\mathbb{C} \longrightarrow \langle F^{i,\bullet-p}\rangle\mathbb{C}$$

is strict for all $i \in [1,\nu]$ and $p \geq 1$. Hence, by Lemma 31, one can compute $U^i = (U^{i,p})_{p\in\mathbb{Z}} \in \mathbb{C}^{\mu_i\times\mu_i}$ such that

$$\langle F^{i,p}\rangle\mathbb{C} = \langle U^{i,q} \mid q \geq p\rangle\mathbb{C}, \quad \{N_i U^{i,p}\} \subset \{U^{i,p-1}\}$$

for all $(i,p) \in [1,\nu]\times\mathbb{Z}$. Then

$$(\theta\partial_\theta - \alpha_i)(\underline{\phi}U^{i,p}) \equiv \underline{\phi}^i N_i U^{i,p} \mod V_{\alpha-1}$$

and hence $\{(\theta\partial_\theta - \alpha_i)(\underline{\phi}U^{i,p})\} \subset \{\underline{\phi}U^{i,p-1}\} + V_{\alpha-1}$ for all $(i,p) \in [1,\nu]\times\mathbb{Z}$.

7 Good bases

The following algorithm computes a good basis of a good lattice L by a simultaneous normal form computation and basis transformation. The computation requires a $\mathbb{C}[\tau,\theta]$–basis of G compatible with the V–filtration refined by an opposite filtration of $L^\bullet$ on $\operatorname{gr}^V G$.

Algorithm 5

Input: (a) *A matrix* $B \in \mathbb{C}[\tau]^{\mu\times\mu}$ *with* $\mathrm{spec}(B_0) = \{\underline{\alpha}\}$ *and* $\alpha \geq \alpha_1 > \cdots > \alpha_\nu > \alpha - 1$ *such that* $-\tau\partial_\tau\underline{\phi} = \underline{\phi}B$ *for a* $\mathbb{C}[\tau]$*–basis* $\underline{\phi}$ *of* V_α

(b) *A matrix* $M \in \mathbb{C}[\tau,\theta]^{\mu\times\mu}$ *such that* $\underline{\phi}M$ *is a basis of a good* $\mathbb{C}[\theta]$*–lattice* $L \subset G$.

(c) *An indexing* $\underline{\phi} = (\underline{\phi}^{i,p})_{(i,p)\in[1,\nu]\times\mathbb{Z}}$ *such that* $(\underline{\phi}^{i,q})_{q\geq p}$ *is a* $\mathbb{C}$*–basis of* $L^p \operatorname{gr}^V_{\alpha_i} G$ *and* $\{(\theta\partial_\theta - \alpha_i)(\underline{\phi}^{i,p})\} \subset \{\underline{\phi}^{i,p-1}\} + V_{<\alpha_i}$ *for all* $(i,p) \in [1,\nu]\times\mathbb{Z}$.

Output: *A matrix* $M \in \mathbb{C}[\tau,\theta]^{\mu\times\mu}$ *such that* $\underline{\phi}M$ *is a good basis of* L.

(1) Compute a minimal Gröbner basis

$$M := \mathrm{GB}(M) \in \mathbb{C}[\theta]^{\mu\times\mu}$$

compatible with the ordering $>$ *on* $\{\theta^k\underline{\phi}^{i,p} \mid (k,i,p) \in \mathbb{Z}\times[1,\nu]\times\mathbb{Z}\}$ *defined by*

$$(k,i,p) > (l,j,q) :\Leftrightarrow k > l \vee (k = l \wedge (i > j \vee (i = j \wedge p > q)))$$

for all $(k,i,p),(l,j,q) \in \mathbb{Z}\times[1,\nu]\times\mathbb{Z}$.

(2) Set $(M^{k,i})_{(k,i)\in\mathbb{Z}\times[1,\nu]} := M$ *where*

$$\{\mathrm{lexp}(M^{k,i})\} = \{(k,i,n-k)\}$$

for all $(k,i) \in \mathbb{Z}\times[1,\nu]$.

(3) Compute $(A^{k,i}_{s,l,j})_{(s,j,l)\in\mathbb{Z}\times[1,\nu]\times\mathbb{Z}}$ *and*

$$\begin{aligned}\Phi^{k,i}_{s,l,j} &:= \theta(B-(\alpha_i+k)+\theta\partial_\theta)M^{k,i} - M^{1+k,i}A^{k,i}_{0,1+k,i}\\ &- \sum_{\substack{(l',j')<(1+k,i)\\(s+l,j,n-l)<(l',j',n-l')}} M^{l',j'}A^{k,i}_{0,l',j'} - \theta^s M^{l,j}A^{k,i}_{s,l,j}\end{aligned}$$

such that $\mathrm{lexp}(\Phi^{k,i}_{s,l,j}) < (s+l,j,n-l)$ *for all* $(k,i) \in \mathbb{Z}\times[1,\nu]$ *for decreasing* $(s+l,j,n-l)$ *until* $\Phi^{k,i}_{s,l,j} = 0$ *or* $A^{k,i}_{s,l,j} \neq 0$ *and* $s \geq 1$.

(4) If $\Phi^{k,i}_{s,l,j} = 0$ *then return* $M := (M^{k,i})_{(k,i)\in\mathbb{Z}\times[1,\nu]}$.

(5) Choose $(k,i) \in \mathbb{Z}\times[1,\nu]$ *and* $(s,j,l) \in \mathbb{Z}\times[1,\nu]\times\mathbb{Z}$ *with* $A^{k,i}_{s,l,j} \neq 0$ *and* $s \geq 1$ *such that* $(s+l,j,n-l)$ *is maximal.*

(6) Set $c^{k,i}_{s,l,j} := (1+k+\alpha_i-s-l-\alpha_j)^{-1}$ *and*

$$M^{k,i} := M^{k,i} + c^{k,i}_{s,l,j}\theta^{s-1}M^{l,j}A^{k,i}_{s,l,j}$$

(7) Go to (3).

Lemma 33 *Algorithm 5 terminates and is correct.*

Proof. By Lemma 14,

$$V_{\alpha_j+q}G = \bigoplus_{(p,i)\leq(q,j)} \theta^p\langle\underline{\phi}^i\rangle\mathbb{C}$$

for all $(q,j) \in \mathbb{Z}\times[1,\nu]$. Let $0 \neq \overline{m} \in \langle M\rangle\mathbb{C}[\theta]$ with $\mathrm{lexp}(\overline{m}) = (k,i,p)$. Then

$$\tau^k\underline{\phi}\overline{m} \in \langle\underline{\phi}^{i,q} \mid q \leq p\rangle\mathbb{C} + V_{<\alpha_i}G$$

and, since $\underline{\phi}\overline{m} \in \langle\underline{\phi}M\rangle\mathbb{C}[\theta] = L$ and $L^{n-k}\,\mathrm{gr}^V_{\alpha_i}G = \langle\underline{\phi}^{i,q} \mid q \geq n-k\rangle\mathbb{C}$,

$$\tau^k\underline{\phi}\overline{m} \in \langle\underline{\phi}^{i,q} \mid q \geq n-k\rangle\mathbb{C} + V_{<\alpha_i}G.$$

In particular, $p \geq n-k$. Moreover, $\tau^k \underline{\phi}\operatorname{lead}(\overline{m}) \in \operatorname{gr}_L^p \operatorname{gr}_{\alpha_i}^V G$ and hence, by Lemma 20,

$$\theta^{n-p-k}\underline{\phi}\operatorname{lead}(\overline{m}) \in \operatorname{gr}_L \operatorname{gr}^V L = \langle \underline{\phi}\operatorname{lead}(M)\rangle \mathbb{C}[\theta].$$

In particular, if $p > n-k$ then $\operatorname{lead}(\overline{m}) \in \theta\langle \operatorname{lead}(M)\rangle\mathbb{C}[\theta]$. Since M is a minimal Gröbner basis, this implies that

$$\{\operatorname{lexp}(M^{k,i})\} = \{(k,i,n-k)\}, \quad M^{k,i} \equiv \operatorname{lead}(M^{k,i}) \mod \text{terms} < (k,i)$$

for all $(k,i) \in \mathbb{Z}\times[1,\nu]$. In particular, $\{\underline{\phi}M^{k,i}\} \subset V_{k+\alpha_i}$ for all $(k,i) \in \mathbb{Z}\times[1,\nu]$.

By Lemma 7, $t \circ \underline{\phi} = \theta(-\tau\partial_\tau)\circ\underline{\phi} = \underline{\phi}\circ\theta(B+\theta\partial_\theta)$. Since

$$\{\theta(\theta\partial_\theta - (k+\alpha_i))(\theta^k\underline{\phi}^{i,n-k})\} \subset \{\theta^{1+k}\underline{\phi}^{i,n-(1+k)}\} \mod V_{<1+k+\alpha_i},$$

there is a matrix $A^{i,k}_{0,1+k,i}$ such that

$$\theta(B-(k+\alpha_i)+\theta\partial_\theta)M^{k,i} \equiv M^{1+k,i}A^{i,k}_{0,1+k,i} \mod \text{terms} < (1+k,i)$$

and hence there are matrices $A^{i,k}_{s,l,j}$ such that

$$\theta(B-(k+\alpha_i)+\theta\partial_\theta)M^{k,i} - M^{1+k,i}A^{i,k}_{0,1+k,i} = \sum_{(s'+l',j')<(1+k,i)} \theta^{s'}M^{l',j'}A^{k,i}_{s',l',j'}$$

for all $(k,i) \in \mathbb{Z}\times[1,\nu]$. Choose $(k,i) \in \mathbb{Z}\times[1,\nu]$ and $(s,j,l) \in \mathbb{Z}\times[1,\nu]\times\mathbb{Z}$ such that $(s+l,j,n-l)$ is maximal with $A^{k,i}_{s,l,j} \neq 0$ and $s \geq 1$. In particular, $(s+l,j) < (1+k,i)$ and hence $1+k+\alpha_i-s-l-\alpha_j > 0$ and $c^{k,i}_{s,l,j} > 0$ is defined. Moreover, since

$$\{(\theta\partial_\theta - (\alpha_j+l))(\theta^l\underline{\phi}^{j,n-l})\} \subset \{\theta^l\underline{\phi}^{j,n-(1+l)}\} \mod V_{<\alpha_j+l},$$

$$\begin{aligned}
\Phi^{k,i}_{s,l,j} &= \theta(B-(k+\alpha_i)+\theta\partial_\theta)\Big(M^{k,i} - c^{k,i}_{s,l,j}\theta^{s-1}M^{l,j}A^{k,i}_{s,l,j}\Big) \\
&\quad - M^{1+k,i}A^{i,k}_{0,1+k,i} - \sum_{\substack{(l',j')<(1+k,i)\\(s+l,j,n-l)<(l',j',n-l')}} M^{l',j'}A^{k,i}_{0,l',j'} \\
&\equiv \theta^s M^{l,j}A^{k,i}_{s,l,j} + c^{k,i}_{s,l,j}\theta(\theta\partial_\theta - k - \alpha_i)\theta^{s-1}M^{l,j}A^{k,i}_{s,l,j} \\
&\equiv \theta^s M^{l,j}A^{k,i}_{s,l,j} + c^{k,i}_{s,l,j}\theta^s(\theta\partial_\theta + s - 1 - k - \alpha_i)M^{l,j}A^{k,i}_{s,l,j} \\
&\equiv \theta^s M^{l,j}A^{k,i}_{s,l,j} + c^{k,i}_{s,l,j}\theta^s(s+l+\alpha_j-1-k-\alpha_i)M^{l,j}A^{k,i}_{s,l,j} \\
&\equiv 0 \mod \text{terms} < (s+l,j,n-l)
\end{aligned}$$

and hence $(s+l,j,n-l)$ is strictly decreasing until $\Phi^{k,i}_{s,l,j} = 0$. Then the algorithm terminates and $\underline{\phi}M = (\underline{\phi}M^{k,i})_{(k,i)\in\mathbb{Z}\times[1,\nu]}$ is a $\mathbb{C}[\theta]$–basis of L with

$$t(\underline{\phi}M^{k,i}) = \underline{\phi}M^{1+k,i}A^{i,k}_{0,1+k,i} + \sum_{(l',j')<(1+k,i)} \underline{\phi}M^{l',j'}A^{k,i}_{0,l',j'} + \theta(k+\alpha_i)\underline{\phi}M^{k,i}$$

and $\{\underline{\phi} M^{k,i}\} \subset V_{k+\alpha_i}$ for all $(k,i) \in \mathbb{Z} \times [1,\nu]$. Hence, $\underline{\phi} M$ is a good basis of L.

The following algorithm combines Algorithms 1, 2, 3, 4, and 5 to compute a good basis of G_0.

Algorithm 6

Input: *A cohomologically tame polynomial* $f \in \mathbb{C}[\underline{x}]$.
Output: *(a) A vector* $\underline{\phi} \in \mathbb{C}[\underline{x},\theta]^\mu$ *such that* $[\underline{\phi}]$ *is a good basis of* G_0.
(b) The matrix $A = A^{[\underline{\phi}]} \in \mathbb{C}[\theta]^{\mu\times\mu}$ *of* t *with respect to* $[\underline{\phi}]$.

(1) Set $k := \deg(f)$.
(2) Compute $\underline{\phi} \in \mathbb{C}[\underline{x},\theta]^\mu$ *and* $A \in \mathbb{C}[\theta]^{\mu\times\mu}$ *by Algorithm 1.*
(3) Compute $U \in \mathbb{C}[\theta]^{\mu\times\mu}$ *and* $B \in \mathbb{C}[\tau]^{\mu\times\mu}$ *by Algorithm 2.*
(4) Set $\underline{\phi} := \underline{\phi} U$, $B := U^{-1}(B - \tau\partial_\tau)U \in \mathbb{C}[\tau]^\mu$, *and* $M := U^{-1} \in \mathbb{C}[\tau,\theta]^\mu$.
(5) Compute σ *by Algorithm 3.*
(6) If $\frac{1}{\mu}\sum\sigma > \frac{n+1}{2}$ *then set* $k := k+1$ *and go to (2).*
(7) Compute $U = (U^{i,p})_{(i,p)\in[1,\nu]\times\mathbb{Z}} \in \mathrm{GL}_\mu(\mathbb{C})$ *by Algorithm 4.*
(8) Set $\underline{\phi} := (\underline{\phi} U^{i,p})_{(i,p)\in[1,\nu]\times\mathbb{Z}}$, $B := U^{-1}(B - \tau\partial_\tau)U \in \mathbb{C}[\tau]^{\mu\times\mu}$, *and* $M := U^{-1}M$.
(9) Compute $M \in \mathbb{C}[\tau,\theta]^{\mu\times\mu}$ *by Algorithm 5.*
(10) Set $\underline{\phi} := \underline{\phi} M$ *and* $A := M^{-1}\theta(B - \tau\partial_\tau)M \in \mathbb{C}[\theta]^{\mu\times\mu}$.
(11) Return $\underline{\phi}$ *and* A.

Proposition 34 *Algorithm 6 terminates and is correct.*

Proof. Let $L_k \subset G_0$ be computed by Algorithm 1. Then $L_k = G_0$ for $k \gg 0$ and k is strictly increasing while $\frac{1}{\mu}\sum\sigma > \frac{n+1}{2}$. By Lemma 23 and Theorem 24, $L = G_0$ if and only if $\frac{1}{\mu}\sum\sigma = \frac{1}{\mu}\sum\mathrm{spec}(L) = \frac{1}{\mu}\sum\mathrm{spec}(G_0) = \frac{n+1}{2}$. This implies that $L_k = G_0$ after finitely many steps. By Theorem 30, $L := L_k = G_0$ is a good lattice as required by algorithms 4 and 5. Hence, the algorithm terminates and is correct.

Remark 35 *In the local situation, one can replace the algorithms [8, 7.4–5] by the algorithms 4 and 5 to avoid the linear algebra computation [8, 7.4]. This modified algorithm is implemented in the* SINGULAR *[20] library* `gmssing.lib` *[21].*

8 Examples

Algorithm 6 is implemented in the SINGULAR [20] library `gmspoly.lib` [22]. Using this implementation, we compute a good basis $\underline{\phi}$ of G_0 for several examples. By Lemma 26, the diagonal of $A_1^{\underline{\phi}}$ determines the spectrum of f. Using Proposition 28, we read off the monodromy T_∞ around the discriminant of f from $A^{\underline{\phi}}$. First, we compute two convenient and Newton non–degenerate examples [10].

Example 36 *Let* $f = x^2 + y^2 + x^2y^2$. *Then* SINGULAR *computes*

$$\underline{\phi} = \left(1, xy, y, x, x^2 + \frac{1}{2}\right)$$

and

$$A^{[\underline{\phi}]} = \begin{pmatrix} -\frac{1}{2} & 0 & 0 & 0 & \frac{1}{4} \\ 0 & -1 & 0 & 0 & 0 \\ 0 & 0 & -1 & 0 & 0 \\ 0 & 0 & 0 & -1 & 0 \\ 1 & 0 & 0 & 0 & -\frac{1}{2} \end{pmatrix} + \theta \begin{pmatrix} \frac{1}{2} & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & \frac{3}{2} \end{pmatrix}.$$

The monodromy T_∞ *has a* 2×2 *Jordan block with eigenvalue* -1.

Example 37 *Let* $f = x + y + z + x^2y^2z^2$. *Then* SINGULAR *computes*

$$\underline{\phi} = \left(1, \theta^2 x - 3\theta x^2 + x^3, \frac{5}{2}x, 10\theta^2 x^2 - \frac{25}{2}\theta x^3 + \frac{5}{2}x^4, -\frac{25}{4}\theta x + \frac{25}{4}x^2\right)$$

and

$$A^{[\underline{\phi}]} = \begin{pmatrix} 0 & 0 & 0 & -\frac{25}{8} & 0 \\ 0 & 0 & 0 & 0 & \frac{125}{8} \\ 1 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 \end{pmatrix} + \theta \begin{pmatrix} \frac{1}{2} & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & \frac{3}{2} & 0 & 0 \\ 0 & 0 & 0 & 2 & 0 \\ 0 & 0 & 0 & 0 & \frac{5}{2} \end{pmatrix}.$$

The monodromy T_∞ *has a* 2×2 *Jordan block with eigenvalue* 1 *and a* 3×3 *Jordan block with eigenvalue* -1.

Finally, we compute a non–convenient and Newton degenerate but tame [23, 3] example.

Example 38 *Let* $f = x(x^2+y^3)^2 + x$. *Then* SINGULAR *computes*

$$
\begin{aligned}
\underline{\phi} = \Big(&1, 623645y, \frac{8645}{24}x, -2470\big(y^3+x^2\big),\\
&-11339\big(y^4+x^2y\big), 475\big(2\theta y^3 - 5\theta x^2 + 6x^3\big), y^2, 3\theta^2y^2 + 4y^5,\\
&6670\big(\theta y^4 - 10\theta x^2y + 6x^3y\big), 8\theta^2y^3 - 20\theta^2x^2 - 15\theta x^3 + 18x^4 + 3,\\
&-4365515\big(35\theta^2y^4 - 350\theta^2x^2y - 300\theta x^3y + 180x^4y + 24y\big), \frac{623645}{6}xy,\\
&-8645\big(\theta x + 2y^3 - 4x^2\big), -124729\big(5\theta xy + y^4 - 5x^2y\big)\Big)
\end{aligned}
$$

and $A^{[\underline{\phi}]} = A_0^{[\underline{\phi}]} + \theta A_1^{[\underline{\phi}]}$ *where*

$$
A_0^{[\underline{\phi}]} = \begin{pmatrix}
0 & 0 & 0 & 0 & 0 & -380 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0\\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & -\frac{32}{4675} & 0 & 0 & 0 & 0 & 0\\
\frac{96}{43225} & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & \frac{-288}{216125} & 0 & 0 & 0 & 0\\
0 & 0 & -\frac{7}{180} & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0\\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & -\frac{11}{9} & 0 & 0\\
0 & 0 & 0 & -\frac{52}{75} & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & \frac{728}{75} & 0\\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0\\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0\\
0 & 0 & 0 & 0 & -\frac{17}{75} & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & \frac{187}{15}\\
0 & 0 & 0 & 0 & 0 & \frac{380}{3} & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0\\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & -\frac{4}{98175} & 0 & 0 & 0 & 0 & 0\\
0 & \frac{24}{5} & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & \frac{2016}{5} & 0 & 0 & 0\\
0 & 0 & \frac{1}{180} & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0\\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & \frac{1}{9} & 0 & 0
\end{pmatrix}
$$

and

$$
A_1^{[\underline{\phi}]} = \begin{pmatrix}
\frac{1}{3} & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0\\
0 & \frac{7}{15} & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0\\
0 & 0 & \frac{2}{3} & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0\\
0 & 0 & 0 & \frac{11}{15} & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0\\
0 & 0 & 0 & 0 & \frac{13}{15} & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0\\
0 & 0 & 0 & 0 & 0 & \frac{14}{15} & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0\\
0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0\\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0\\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & \frac{16}{15} & 0 & 0 & 0 & 0 & 0\\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & \frac{17}{15} & 0 & 0 & 0 & 0\\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & \frac{19}{15} & 0 & 0 & 0\\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & \frac{4}{3} & 0 & 0\\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & \frac{23}{15} & 0\\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & \frac{5}{3}
\end{pmatrix}.
$$

The monodromy T_∞ *is unipotent with eigenvalues*

$$
\begin{aligned}
&\mathrm{e}^{-2\pi\mathrm{i}\frac{1}{3}}, \mathrm{e}^{-2\pi\mathrm{i}\frac{7}{15}}, \mathrm{e}^{-2\pi\mathrm{i}\frac{2}{3}}, \mathrm{e}^{-2\pi\mathrm{i}\frac{11}{15}}, \mathrm{e}^{-2\pi\mathrm{i}\frac{13}{15}}, \mathrm{e}^{-2\pi\mathrm{i}\frac{14}{15}}, 1,\\
&1, \mathrm{e}^{-2\pi\mathrm{i}\frac{16}{15}}, \mathrm{e}^{-2\pi\mathrm{i}\frac{17}{15}}, \mathrm{e}^{-2\pi\mathrm{i}\frac{19}{15}}, \mathrm{e}^{-2\pi\mathrm{i}\frac{4}{3}}, \mathrm{e}^{-2\pi\mathrm{i}\frac{23}{15}}, \mathrm{e}^{-2\pi\mathrm{i}\frac{5}{3}}.
\end{aligned}
$$

References

[1] C. Sabbah, Hypergeometric periods for a tame polynomial, arXiv.org math.AG/9805077.

[2] E. Brieskorn, Die Monodromie der isolierten Singularitäten von Hyperflächen, Manuscr. Math. 2 (1970) 103–161.

[3] M. Sebastiani, Preuve d'une conjecture de Brieskorn, Manuscr. Math. 2 (1970) 301–308.

[4] J. Steenbrink, Mixed Hodge structure on the vanishing cohomology, in: Real and complex singularities, Nordic summer school, Oslo, 1976, pp. 525–562.

[5] F. Pham, Singularités des systèmes de Gauss–Manin, Vol. 2 of Progr. in Math., Birkhäuser, 1979.

[6] A. Varchenko, Asymptotic Hodge structure in the vanishing cohomology, Math. USSR Izv. 18 (3) (1982) 496–512.

[7] M. Saito, On the structure of Brieskorn lattices, Ann. Inst. Fourier Grenoble 39 (1989) 27–72.

[8] M. Schulze, The differential structure of the Brieskorn lattice, in: A. Cohen, et al. (Eds.), Mathematical Software — ICMS 2002, World Sci., 2002, pp. 136–146.

[9] M. Schulze, A normal form algorithm for the Brieskorn lattice, to appear in J. Symb. Comp. .

[10] A. Douai, Très bonnes bases du réseau de Brieskorn d'un polynôme modéré, Bull. Soc. Math. France 127 (1999) 255–287.

[11] A. Khovanskii, A. Varchenko, Asymptotics of integrals over vanishing cycles and the Newton polyhedron, Sov. Math. Docl. 32 (1985) 122–127.

[12] A. Douai, Équations aux différences finies, intégrales de fonctions multiformes et polyèdres de Newton, Comp. math. 87 (1993) 311–355.

[13] J. Briançon, M. Granger, P. Maisonobe, M. Miniconi, Algorithme de calcul du polynôme de Bernstein, Ann. Inst. Fourier 3 (1989) 553–609.

[14] A. Borel, et al. (Eds.), Algebraic D–modules, 2nd Edition, Vol. 2 of Persp. in Math., Acad. Press, 1987.

[15] D. Eisenbud, Commutative Algebra with a View toward Algebraic Geometry, Vol. 150 of Grad. Texts in Math., Springer, 1996.

[16] C. Sabbah, Monodromy at infinity and Fourier transform, Publ. RIMS, Kyoto Univ. 33 (1998) 643–685.

[17] C. Sabbah, Déformations isomonodromiques et variétés de Frobenius, EDP Sciences, 2002.

[18] G. Greuel, G. Pfister, A SINGULAR Introduction to Commutative Algebra, Springer, 2002.

[19] C. Sabbah, $\mathcal{D}$-modules et cycles évanescents, in: J.-M. Aroca, T. Sanchez-Giralda, J.-L. Vicente (Eds.), Deuxième conférence de La Rabida, Géométrie algebrique et applications III, Vol. 24 of Travaux en cours, Hermann, Paris, 1987, pp. 53–98.

[20] G.-M. Greuel, G. Pfister, H. Schönemann, SINGULAR 2.0.5, A Computer Algebra System for Polynomial Computations, Centre for Computer Algebra, University of Kaiserslautern, `http://www.singular.uni-kl.de` (2004).

[21] M. Schulze, `gmssing.lib`, SINGULAR library, Centre for Computer Algebra, University of Kaiserslautern, `http://www.singular.uni-kl.de` (2004).

[22] M. Schulze, `gmspoly.lib`, SINGULAR library, Centre for Computer Algebra, University of Kaiserslautern, `http://www.singular.uni-kl.de` (2004).

[23] S. Broughton, Milnor numbers and the topology of polynomial hypersurfaces, Inv. Math. 92 (1988) 217–241.