

INVERSE LIMITS OF MACAULAY'S INVERSE SYSTEMS

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ABSTRACT. Generalizing a result of Masuti and the second author, we describe inverse limits of Macaulay's inverse systems for Cohen–Macaulay factor algebras of formal power series or polynomial rings over an infinite field. On the way we find a strictness result for filtrations defined by regular sequences. It generalizes both a lemma of Uli Walther and the Rees isomorphism.

INTRODUCTION

Let \mathbb{K} be a field, and let P be either the (standard graded) polynomial ring $\mathbb{K}[x_1, \dots, x_n]$ or the formal power series ring $\mathbb{K}[[x_1, \dots, x_n]]$ (with trivial grading). The injective hull E of \mathbb{K} over P_0 defines a duality $-^\vee = {}^*\mathrm{Hom}_{P_0}(-, E)$ between Artinian and finitely generated (graded) P -modules. In particular, this yields an antiisomorphism $-^\perp$ of the lattices of (graded) ideals $I \triangleleft P$ and (graded) P -submodules W of $D = P^\vee$ (see (2.1)). The ideals I for which P/I is Artinian correspond to finitely generated submodules $W = I^\perp$. In this case and for the polynomial ring the correspondence was proved by Macaulay at the beginning of the 20th century. The dual I^\perp is known as the inverse system of I (see [Mac94]).

Around 1960 Macaulay's correspondence turned out to be a special case of Matlis duality (see [Mat58; Gab60]). Later it was rediscovered and further developed (see for instance [Ems78; Iar94; Ger96; GS98; Kle07; CI12]). Recent applications concern the n -factorial conjecture (see [Hai94]), Waring's problem (see [Ger96]), the geometry of the punctual Hilbert scheme of Gorenstein schemes (see [IK99]), the analytic classification of Artinian Gorenstein rings (see [ER12]), the cactus rank (see [RS13]), and the Kaplansky–Serre problem (see [RŠ14]).

Elias and Rossi (see [ER17]) described the first generalization of Macaulay's inverse system in the positive-dimensional case. Their result which applies to Gorenstein algebras was extended by Masuti and the second author (see [MT18]) to the case of level algebras. We give

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a more conceptual description of their construction in terms of inverse limits. We show how to drop the level-hypothesis by identifying the various socles in the inverse system (see Corollary 1.9). This fact is deduced from a general strictness result for filtrations defined by regular sequences, which generalizes both a lemma of Walther (see [Wal17, Lem. 6.5]) and the Rees isomorphism. The full generality of this result is not needed for our application.

Our main result (see Theorem 2.8) gives an explicit description of inverse limits of Macaulay's inverse systems obtained by dividing out powers of a linear regular sequence. It applies to (graded) Cohen–Macaulay factor algebras of formal power series (or polynomial) rings over an infinite field (see Lemma 2.2 for a more intrinsic description of these types of algebras).

1. STRICT FILTRATIONS BY REGULAR SEQUENCES

We underline vectors and denote (component-wise) residue classes by an overline. We apply maps component-wise to vectors. All rings considered are commutative unitary. We use the ideal symbol \triangleleft . By an *R-sequence* we mean a regular sequence in R .

Let R be a ring. Any ideal $I \triangleleft R$ defines an *exhaustive decreasing filtration*

$$R = I^0 \supset I \supset I^2 \supset I^3 \supset \dots$$

on R denoted by I^\bullet . It is called *separated* if $\bigcap_{k \in \mathbb{N}} I^k = 0$. The I -order of $p \in R$ is

$$\text{ord}_I(p) = \max \{k \in \mathbb{N} \mid p \in I^k\} \in \mathbb{N} \cup \{\infty\}.$$

We abbreviate $R_I := R/I$. The *associated I-graded ring*

$$\text{gr}_I R = \bigoplus_{l \in \mathbb{N}} I^l / I^{l+1}$$

is a homogeneous graded R_I -algebra. There is an *I-symbol map*

$$\sigma_I: R \setminus \bigcap_{k \in \mathbb{N}} I^k \rightarrow \text{gr}_I R, \quad p \mapsto \bar{p} \in \text{gr}_I^{\text{ord}_I(p)} R = I^{\text{ord}_I(p)} / I^{\text{ord}_I(p)+1}.$$

Any ideal $J \triangleleft R$ induces a filtration $\text{gr}_I J^\bullet$ on $\text{gr}_I R$, where

$$(1.1) \quad \begin{aligned} \text{gr}_I^l J^k &= (J^k \cap I^l + I^{l+1}) / I^{l+1} \subset \text{gr}_I^l R, \\ \text{gr}_I J^k &= \bigoplus_{l \in \mathbb{N}} \text{gr}_I^l J^k \subset \text{gr}_I R. \end{aligned}$$

We refer to any filtration induced by J as a *J-filtration*. If J is generated by $\underline{f} = f_1, \dots, f_r \in R$, then we use \underline{f} as a shortcut for the J -filtration.

Remark 1.1. Let $\sigma_{\underline{g}}(p), \sigma_{\underline{g}}(q) \in \text{gr}_{\underline{g}} R$. Then

$$\text{ord}_{\underline{g}}(pq) \geq \text{ord}_{\underline{g}} p + \text{ord}_{\underline{g}} q,$$

with equality equivalent to $pq \notin I^{\text{ord}_{\underline{g}}p + \text{ord}_{\underline{g}}q + 1}$. It follows that

$$\sigma_{\underline{g}}(p)\sigma_{\underline{g}}(q) = \begin{cases} \sigma_{\underline{g}}(pq) & \text{if } \text{ord}_{\underline{g}}(pq) = \text{ord}_{\underline{g}}p + \text{ord}_{\underline{g}}q, \\ 0 & \text{otherwise.} \end{cases}$$

Let $\underline{g} = (g_1, \dots, g_s) \in R^s$ and denote by $\underline{Y} = (Y_1, \dots, Y_s)$ corresponding indeterminates of degree 1.

Theorem 1.2 (Rees). *The Rees map of graded $R_{\underline{g}}$ -algebras*

$$(1.2) \quad \begin{aligned} \varphi_{\underline{g}}: R_{\underline{g}}[\underline{Y}] &\longrightarrow \text{gr}_{\underline{g}} R, \\ Y_i &\longmapsto \sigma_{\underline{g}}(g_i), \end{aligned}$$

is an isomorphism if \underline{g} is an R -sequence (see [BH93, Thm. 1.1.8]). \square

Remark 1.3. If $P \in R[\underline{Y}]_l$ such that $\text{ord}_{\underline{g}}(P(\underline{g})) = l$, then, using Remark 1.1,

$$\sigma_{\underline{g}}(P(\underline{g})) = \overline{P}(\sigma_{\underline{g}}(\underline{g})) = \varphi_{\underline{g}}(\overline{P}),$$

where $P \mapsto \overline{P}$ under $R[\underline{Y}] \rightarrow R_{\underline{g}}[\underline{Y}]$.

Remark 1.4. If (1.2) is an isomorphism, then the components of $\sigma_{\underline{g}}(\underline{g})$ are regular on $\text{gr}_{\underline{g}} R$. With Remark 1.1, it follows that

$$\sigma_{\underline{g}}(p\underline{g}^{\underline{m}}) = \sigma_{\underline{g}}(p)\sigma_{\underline{g}}(\underline{g}^{\underline{m}}) = \sigma_{\underline{g}}(p)\sigma_{\underline{g}}(\underline{g})^{\underline{m}},$$

for all $\sigma_{\underline{g}}(p) \in \text{gr}_{\underline{g}} R$ and $\underline{m} \in \mathbb{N}^s$. By Theorem 1.2, this holds if \underline{g} is an R -sequence.

Let $\underline{f} = (f_1, \dots, f_r) \in R^r$, and set

$$\underline{h} = (h_1, \dots, h_t) = \underline{f}, \underline{g} \in R^r \times R^s = R^t.$$

Denote by $\underline{X} = (X_1, \dots, X_r)$ indeterminates of degree 1 corresponding to \underline{f} , and set

$$\underline{Z} = (Z_1, \dots, Z_t) = \underline{X}, \underline{Y}.$$

For $i \in \{1, \dots, u\}$, let

$$0 \neq \underline{m}_i = (\underline{k}_i, \underline{l}_i) \in \mathbb{N}^r \times \mathbb{N}^s = \mathbb{N}^t$$

be the rows of a matrix

$$(1.3) \quad M = (KL).$$

We denote by $\underline{h}^M, \underline{f}^K, \underline{g}^L \in R^u$ the vectors with respective entries $\underline{h}^{\underline{m}_i}, \underline{f}^{\underline{k}_i}, \underline{g}^{\underline{l}_i} \in R$. Consider the R -linear map

$$\begin{aligned} \underline{h}^M: R^u &\longrightarrow R, \\ \underline{e}_i &\longmapsto \underline{h}^{\underline{m}_i}, \end{aligned}$$

with image $\langle \underline{h}^M \rangle$. Assigning degrees $\deg e_i = |l_i|$ to the generators defines a \underline{g} -filtration $\bigoplus_{i=1}^u \langle \underline{g} \rangle^{\bullet - l_i}$ on R^u and turns \underline{h}^M into a filtered map. It fits into a commutative diagram of filtered R -linear maps

$$\begin{array}{ccc} (R^u, \bigoplus_{i=1}^u \langle \underline{g} \rangle^{\bullet - l_i}) & \xrightarrow{\underline{h}^M} & (R, \langle \underline{g} \rangle^{\bullet}) \\ \downarrow \underline{h}^M & & \uparrow \\ (\langle \underline{h}^M \rangle, \sum_{i=1}^u \langle \underline{g} \rangle^{\bullet - l_i} \underline{h}^{m_i}) & \rightarrow & (\langle \underline{h}^M \rangle, \langle \underline{g} \rangle^{\bullet} \cap \langle \underline{h}^M \rangle). \end{array}$$

The bottom map is the identity of $\langle \underline{h}^M \rangle$, but its source and target carry respectively the image and preimage filtration from the source and target of the (horizontal) map \underline{h}^M . If it identifies the two filtrations, then \underline{h}^M is said to be \underline{g} -strict. The vertical maps are \underline{g} -strict by construction.

The following proposition gives a generalized Rees isomorphism.

Proposition 1.5. *Suppose that both $\underline{h} = \underline{f}, \underline{g}$ and $\underline{g}, \underline{f}$ are R -sequences, and that the \overline{f} -filtration on $R_{\underline{g}}$ is separated and complete. Then \underline{h}^M is \underline{g} -strict. In particular, the Rees map (1.2) induces an isomorphism of graded $R_{\underline{g}}$ -algebras*

$$\begin{array}{ccc} R_{\underline{g}}[\underline{Y}] / \langle \overline{f}^K \underline{Y}^L \rangle & \xrightarrow[\cong]{\overline{\varphi}_{\underline{g}}} & \text{gr}_{\underline{g}}(R) / \langle \sigma_{\underline{g}}(\underline{h}^M) \rangle & \xrightarrow[\cong]{} & \text{gr}_{\underline{g}}(R / \langle \underline{h}^M \rangle), \\ & & \overline{\sigma}_{\underline{g}}(x) & \longleftarrow & \longrightarrow & \sigma_{\underline{g}}(\overline{x}), \end{array}$$

where $\overline{f}^K \underline{Y}^L$ denotes the vector with entries $\overline{f}^{k_i} \underline{Y}^{l_i}$.

The proof of Proposition 1.5 relies on the following lemma proved by Uli Walther for $k = 1$ (see [Wal17, Lem. 6.5]). He assumes that R is a domain, and that $\underline{g}, \underline{f}$ is an R -sequence in every order. However, his proof needs only that $\underline{f}, \underline{g}$ is an R -sequence.

Lemma 1.6. *Suppose that \underline{g} and $\underline{f}, \underline{g}$ are R -sequences. Let $P \in R[\underline{Y}]_l$ such that $P(\underline{g}) \in \langle \underline{f} \rangle^k$. Then $P(\underline{g}) = Q(\underline{g})$ for some $Q \in \langle \underline{f} \rangle^k[\underline{Y}]_l$. In particular, the Rees map (1.2) induces a filtered isomorphism*

$$\varphi_{\underline{g}}: (R_{\underline{g}}[\underline{Y}], \langle \overline{f} \rangle^{\bullet}[\underline{Y}]) \longrightarrow (\text{gr}_{\underline{g}} R, \text{gr}_{\underline{g}} \langle \underline{f} \rangle^{\bullet}).$$

Proof. We proceed by induction on k . The claim is vacuous for $k = 0$. By Walther's lemma (see [Wal17, Lem. 6.5]), we may assume that $P = \sum_i P_i f_i \in \langle \underline{f} \rangle[\underline{Y}]_l$ with $P_i \in R[\underline{Y}]_l$, and hence $P(\underline{g}) = \sum_i P_i(\underline{g}) f_i \in \langle \underline{f} \rangle^k$. The following proof of Proposition 1.5 relies only on the particular claim, which reduces to the Rees isomorphism in case $r = 0$. Applying Proposition 1.5 with $r = 0$, \underline{f} playing the role of \underline{g} and $\underline{m}_i = \underline{e}_i$, it follows that $P_i(\underline{g}) \in \langle \underline{f} \rangle^{k-1}$. By induction hypothesis, $P_i(\underline{g}) = Q_i(\underline{g})$ for

some $Q_i \in \langle \underline{f} \rangle^{k-1} [\underline{Y}]_l$. Then $P(\underline{g}) = Q(\underline{g})$ for $Q = \sum_i Q_i f_i \in \langle \underline{f} \rangle^k [\underline{Y}]_l$. This proves the first claim.

Suppose now that \underline{g} is an R -sequence. Then $\varphi_{\underline{g}}$ is an isomorphism by Theorem 1.2. Clearly $\varphi_{\underline{g}}(\langle \underline{f} \rangle^\bullet [\underline{Y}]) \subset \text{gr}_{\underline{g}} \langle \underline{f} \rangle^\bullet$ (see (1.1)). For the converse inclusion, take $\sigma_{\underline{g}}(x) \in \text{gr}_{\underline{g}}^l \langle \underline{f} \rangle^k$. Then $x = P(\underline{g}) \in \langle \underline{f} \rangle^k$ for some $P \in R[\underline{Y}]_l$. By the first claim, we may assume that $P \in \langle \underline{f} \rangle^k [\underline{Y}]_l$. Then $R[\underline{Y}] \rightarrow R_{\underline{g}}[\underline{Y}]$ maps $P \mapsto \bar{P} \in \langle \underline{f} \rangle^k [\underline{Y}]_l$ with $y = \sigma_{\underline{g}}(x) = \sigma_{\underline{g}}(P(\underline{g})) = \varphi_{\underline{g}}(\bar{P})$ by Remark 1.3. This shows that $\varphi_{\underline{g}}(\langle \underline{f} \rangle^k [\underline{Y}]_l) = \text{gr}_{\underline{g}}^l \langle \underline{f} \rangle^k$ and the particular claim follows. \square

A second ingredient of the proof of Proposition 1.5 is the following general relation of strict and graded exactness.

Lemma 1.7. *Let A be a filtered ring, and let*

$$C: N' \xrightarrow{\alpha'} N \xrightarrow{\alpha} N''$$

be a filtered complex of A -modules with associated graded complex $\text{gr } C$.

- (a) If C is strict exact, then $\text{gr } C$ is exact (see [Sjö73, Lem. 1.(a)]).
- (b) If $\text{gr } C$ is exact and the filtration on N is exhaustive, then α is strict (see [Sjö73, Lem. 1.(b)]).
- (c) If the filtration on N' is complete and the filtration on N is exhaustive and separated, then C is strict exact if and only if $\text{gr } C$ is exact (see [Sjö73, Lem. 1.(e)]). \square

Proof of Proposition 1.5. Set $U = \{(i, j) \mid 1 \leq i < j \leq u\}$ and consider the R -linear map

$$\begin{aligned} R^U &\longrightarrow R^u, \\ \underline{e}_{i,j} &\longmapsto \underline{h}^{m_j - \min\{m_i, m_j\}} \underline{e}_i - \underline{h}^{m_i - \min\{m_i, m_j\}} \underline{e}_j, \end{aligned}$$

where $\underline{\min}$ denotes the component-wise minimum. Assign to the generators bidegrees

$$\deg(\underline{e}_i) = (|L_i|, |\underline{k}_i|), \quad \deg(\underline{e}_{i,j}) = (|L_i| + |L_j|, |\underline{k}_i| + |\underline{k}_j|).$$

With component-wise \underline{f} - and \underline{g} -filtrations,

$$C: R^U \longrightarrow R^u \xrightarrow{\underline{h}^M} R$$

becomes a bifiltered complex of free R -modules. By Lemma 1.7.(b), it suffices to show that the \underline{g} -graded complex $\text{gr}_{\underline{g}} C$ is exact. This can be checked on graded pieces. By Lemma 1.7.(c), it suffices to show that the \underline{f} -filtration $\text{gr}_{\underline{g}} \langle \underline{f} \rangle^\bullet$ is separated and complete on each graded piece of $\text{gr}_{\underline{g}} C$, and that the associated graded complex $\text{gr}_{\underline{f}} \text{gr}_{\underline{g}} C$ is exact.

By Lemma 1.6, the Rees isomorphism $\varphi_g: R_g[\underline{Y}] \rightarrow \text{gr}_g R$ identifies the induced \underline{f} -filtration $\text{gr}_g \langle \underline{f} \rangle^\bullet$ on $\text{gr}_g R$ with the \overline{f} -filtration on the coefficient ring R_g . We denote by $\text{gr}_f \varphi_g$ the associated graded isomorphism. The graded pieces of $\text{gr}_g R$, and hence of $\text{gr}_g C$, are finite direct sums of R_g . By hypothesis, the \overline{f} -filtration is separated and complete on each summand. It follows that the induced \underline{f} -filtration is separated and complete on each graded piece of $\text{gr}_g C$.

By Theorem 1.2 applied to the regular R_g -sequence \overline{f} and Lemma 1.6, there is a bigraded isomorphism of $R_{\underline{h}}$ -algebras

$$(1.4) \quad R_{\underline{h}}[\underline{Z}] \cong (R_g)_{\overline{f}}[\underline{X}][\underline{Y}] \xrightarrow[\cong]{\varphi_{\overline{f}}[\underline{Y}]} \text{gr}_{\overline{f}}(R_g)[\underline{Y}] \xrightarrow[\cong]{\text{gr}_f \varphi_g} \text{gr}_f \text{gr}_g R.$$

Let now $\underline{m} = (k, l) \in \mathbb{N}^r \times \mathbb{N}^s = \mathbb{N}^t$. Note that $\overline{f}^k \neq 0$ by Remark 1.4 applied to the R_g -sequence \overline{f} . By R_g -linearity of φ_g and Remark 1.4,

$$(1.5) \quad \varphi_g(\overline{f}^k \underline{Y}^l) = \overline{f}^k \varphi_g(\underline{Y}^l) = \sigma_g(\overline{f}^k) \sigma_g(\underline{Y}^l) = \sigma_g(\overline{f}^k \underline{Y}^l) = \sigma_g(\underline{h}^{\underline{m}}).$$

It follows that isomorphism (1.4) maps

$$\underline{Z}^{\underline{m}} = \underline{X}^k \underline{Y}^l \longmapsto \sigma_{\overline{f}}(\overline{f}^k) \underline{Y}^l \longmapsto \sigma_{\overline{f}}(\varphi_g(\overline{f}^k \underline{Y}^l)) = \sigma_{\overline{f}}(\sigma_g(\underline{h}^{\underline{m}})).$$

The isomorphism (1.4) thus turns $\text{gr}_f \text{gr}_g C$ into the exact complex (see [Eis95, Lem. 15.1])

$$\begin{array}{ccccc} R_g[\underline{Z}]^U & \longrightarrow & R_g[\underline{Z}]^u & \longrightarrow & R_g[\underline{Z}], \\ \underline{e}_{i,j} \longmapsto & \underline{Z}^{m_j - \min\{m_i, m_j\}} \underline{e}_i - \underline{Z}^{m_i - \min\{m_i, m_j\}} \underline{e}_j, & & & \\ & & \underline{e}_i \longmapsto & \longrightarrow & \underline{Z}^{m_i}, \end{array}$$

which proves the first claim.

With $R/\langle \underline{h}^M \rangle$ equipped with the image \underline{g} -filtration,

$$R^u \xrightarrow{\underline{h}^M} R \longrightarrow R/\langle \underline{h}^M \rangle \longrightarrow 0$$

is an exact complex of \underline{g} -strict R -linear maps. Then the corresponding \underline{g} -graded complex is exact by Lemma 1.7.(a), and hence

$$\text{gr}_g(R/\langle \underline{h}^M \rangle) \cong \text{gr}_g(R)/\text{gr}_g \langle \underline{h}^M \rangle \cong \text{gr}_g(R)/\langle \sigma(\underline{h}^M) \rangle.$$

With (1.5), this proves the particular claim. \square

We now specialize to the case where $\underline{g} = \underline{h}$ and R is Noetherian * local graded with * maximal ideal \mathfrak{m}_R . Denote the (homogeneous) socle of R by

$$(1.6) \quad \text{soc } R = \text{ann}_R \mathfrak{m}_R.$$

We assume that \underline{h} has homogeneous components making $\langle \underline{h} \rangle \triangleleft R$ a graded ideal. Then $\langle \underline{h} \rangle \subset \mathfrak{m}_R$ and the filtration $\langle \underline{h} \rangle^\bullet$ is separated by Krull intersection theorem (see [Nor53, §3.1, Thm. 1]) and Nakayama lemma (see [BH93, Ex. 1.5.24]). With R also $\text{gr}_{\underline{h}}^0 R = R_{\underline{h}}$ is Noetherian * local graded with * maximal ideal $\mathfrak{m}_{R_{\underline{h}}} = \mathfrak{m}_R / \langle \underline{h} \rangle$. The $R_{\underline{h}}$ -algebra $\text{gr}_{\underline{h}} R$ is now bigraded with unique bigraded maximal ideal

$$(1.7) \quad \mathfrak{m}_{\text{gr}_{\underline{h}} R} = \mathfrak{m}_{R_{\underline{h}}} + \langle \sigma_{\underline{h}}(\underline{h}) \rangle.$$

For any bigraded algebra S with unique bigraded maximal ideal \mathfrak{m}_S , we define the *(bihomogeneous) socle* as in (1.6).

The rows of the matrix M from (1.3) generate a monoid ideal

$$\mathbb{M} = \langle m_1, \dots, m_u \rangle \subset \mathbb{N}^t.$$

By Dickson's lemma (see [Dic13]), every monoid ideal is finitely generated.

Definition 1.8. Let $\mathbb{M} \subset \mathbb{N}^t$ be a monoid ideal. By the *socle* of \mathbb{M} we mean the subset

$$\text{soc } \mathbb{M} = \{ \underline{n} \in \mathbb{N}^t \setminus \mathbb{M} \mid \underline{n} + (\mathbb{N}^t \setminus \{0\}) \subset \mathbb{M} \} \subset \mathbb{N}^t.$$

For $d \in \mathbb{N}$ we write $\text{soc}_d \mathbb{M} = \{ \underline{m} \in \text{soc } \mathbb{M} \mid |\underline{m}| = d \}$.

By definition,

$$(1.8) \quad \text{ann}_{R[\underline{Z}]/\langle \underline{Z}^M \rangle}(\overline{\underline{Z}}) = \bigoplus_{\underline{m} \in \text{soc } \mathbb{M}} R \overline{\underline{Z}}^{\underline{m}} \cong R^{\text{soc } \mathbb{M}}.$$

This is a Noetherian R -module if R is a Noetherian ring. In particular, $\text{soc } \mathbb{M}$ is a finite set. Using (1.8) and $\mathfrak{m}_{R[\underline{Z}]/\langle \underline{Z}^M \rangle} = \mathfrak{m}_R + \langle \underline{Z} \rangle$, we find

$$(1.9) \quad \text{soc}(R[\underline{Z}]/\langle \underline{Z}^M \rangle) = \bigoplus_{\underline{m} \in \text{soc } \mathbb{M}} \text{soc}(R) \overline{\underline{Z}}^{\underline{m}} \cong \text{soc}(R)^{\text{soc } \mathbb{M}}.$$

Corollary 1.9. *Suppose that R is a Noetherian * local graded ring, and that \underline{h} a (component-wise) homogeneous R -sequence. Then the symbol map (extended by zero)*

$$\begin{aligned} R/\langle \underline{h}^M \rangle &\longrightarrow \text{gr}_{\overline{\underline{h}}}(R/\langle \underline{h}^M \rangle), \\ \overline{x} &\longmapsto \sigma_{\overline{\underline{h}}}(\overline{x}), \end{aligned}$$

identifies socles.

Proof. Proposition 1.5 yields an isomorphism of free $R_{\underline{h}}$ -modules

$$(1.10) \quad \begin{array}{ccc} \text{ann}_{\text{gr}_{\underline{h}}(R)/\langle \sigma_{\underline{h}}(\underline{h}^M) \rangle}(\overline{\sigma_{\underline{h}}(\underline{h})}) & \xrightarrow{\cong} & \text{ann}_{\text{gr}_{\overline{\underline{h}}}(R/\langle \underline{h}^M \rangle)}(\sigma_{\overline{\underline{h}}}(\overline{\underline{h}})) \\ \parallel & & \parallel \\ \bigoplus_{\underline{m} \in \text{soc } \mathbb{M}} R_{\underline{h}} \overline{\sigma_{\underline{h}}(\underline{h}^{\underline{m}})} & & \bigoplus_{\underline{m} \in \text{soc } \mathbb{M}} R_{\underline{h}} \sigma_{\overline{\underline{h}}}(\overline{\underline{h}^{\underline{m}}}), \\ \overline{\sigma_{\underline{h}}(x)} & \longmapsto & \sigma_{\overline{\underline{h}}}(\overline{x}). \end{array}$$

Since $\sigma_{\bar{h}}(\bar{h}) \in \mathfrak{m}_{\text{gr}_{\bar{h}}(R/\langle \underline{h}^M \rangle)}$ by (1.7), it follows that

$$\text{soc}(\text{gr}_{\bar{h}}(R/\langle \underline{h}^M \rangle)) \subset \bigoplus_{\underline{m} \in \text{soc } \mathbb{M}} R_{\underline{h}} \sigma_{\bar{h}}(\bar{h}^{\underline{m}}).$$

Moreover, the $R_{\underline{h}}$ -linear surjection

$$R/\langle \underline{h}^M \rangle \supset \sum_{\underline{m} \in \text{soc } \mathbb{M}} R_{\underline{h}} \bar{h}^{\underline{m}} \longrightarrow \bigoplus_{\underline{m} \in \text{soc } \mathbb{M}} R_{\underline{h}} \sigma_{\bar{h}}(\bar{h}^{\underline{m}}) \cong R_{\underline{h}}^{\text{soc } \mathbb{M}}$$

onto the free $R_{\underline{h}}$ -module must be an isomorphism, and hence

$$R/\langle \underline{h}^M \rangle \supset \bigoplus_{\underline{m} \in \text{soc } \mathbb{M}} R_{\underline{h}} \bar{h}^{\underline{m}} \cong \bigoplus_{\underline{m} \in \text{soc } \mathbb{M}} R_{\underline{h}} \sigma_{\bar{h}}(\bar{h}^{\underline{m}}) \subset \text{gr}_{\bar{h}}(R/\langle \underline{h}^M \rangle)$$

is an isomorphism of free $R_{\underline{h}}$ -modules. The action of the respective graded and bigraded maximal ideal on these modules reduces to that of $\mathfrak{m}_{R_{\underline{h}}}$. Therefore, it remains to show that

$$\text{soc}(R/\langle \underline{h}^M \rangle) \subset \bigoplus_{\underline{m} \in \text{soc } \mathbb{M}} R_{\underline{h}} \bar{h}^{\underline{m}}.$$

To this end, let $0 \neq \bar{x} \in \text{soc}(R/\langle \underline{h}^M \rangle)$ of \bar{h} -order $d = \text{ord}_{\bar{h}}(\bar{x})$. In particular, $x \langle \underline{h} \rangle \subset \langle \underline{h}^M \rangle$ since $\underline{h} \in \mathfrak{m}_R$. By Remark 1.4 and Proposition 1.5, taking symbols yields

$$\sigma_{\underline{h}}(x) \sigma_{\underline{h}}(\underline{h}) = \sigma_{\underline{h}}(x \underline{h}) \in \text{gr}_{\underline{h}} \langle \underline{h}^M \rangle = \langle \sigma_{\underline{h}}(\underline{h})^M \rangle.$$

Then, by (1.10), $\sigma_{\bar{h}}(\bar{x}) \in \text{ann}_{\text{gr}_{\bar{h}}(R/\langle \underline{h}^M \rangle)}(\sigma_{\bar{h}}(\bar{h}))$ can be written as

$$\sigma_{\bar{h}}(\bar{x}) = \sum_{\underline{m} \in \text{soc}_d \mathbb{M}} \bar{x}_{\underline{m}} \sigma_{\bar{h}}(\bar{h}^{\underline{m}}) \in \text{gr}_{\bar{h}}(R/\langle \underline{h}^M \rangle),$$

where $\bar{x}_{\underline{m}} \in R_{\underline{h}}$. With $x' = x - \sum_{\underline{m} \in \text{soc}_d \mathbb{M}} x_{\underline{m}} \underline{h}^{\underline{m}}$, this means that

$$\bar{x}' = \bar{x} - \sum_{\underline{m} \in \text{soc}_d \mathbb{M}} \bar{x}_{\underline{m}} \bar{h}^{\underline{m}} \in \langle \bar{h} \rangle^{d+1} \triangleleft R/\langle \underline{h}^M \rangle,$$

and hence $\text{ord}_{\bar{h}}(\bar{x}') > d = \text{ord}_{\bar{h}}(\bar{x})$ if $\bar{x}' \neq 0$. By (1.8), $\underline{Z}^m \langle \underline{Z} \rangle \in \langle \underline{Z}^M \rangle \triangleleft R[\underline{Z}]$. Substituting $\underline{Z} = \underline{h}$ gives $\underline{h}^m \langle \underline{h} \rangle \subset \langle \underline{h}^M \rangle$, and hence $x' \langle \underline{h} \rangle \subset \langle \underline{h}^M \rangle$. Since $\text{soc } \mathbb{M}$ is finite, iterating yields

$$\bar{x} = \sum_{\underline{m} \in \text{soc } \mathbb{M}} \bar{x}_{\underline{m}} \bar{h}^{\underline{m}}.$$

The remaining inclusion follows. □

2. MACAULAY'S INVERSE SYSTEM

Let P be a Noetherian * local graded ring with * maximal ideal \mathfrak{m}_P . Then P_0 is Noetherian local with maximal ideal $\mathfrak{m}_{P_0} = (\mathfrak{m}_P)_0$. We assume that P is * complete, which means that P_0 is complete. A P -module M is called * Artinian if every descending chain of graded P -submodules is stationary. Using Nakayama lemma (see [BH93, Ex. 1.5.24]), we define its *socle degree* to be the nilpotency index

$$\text{socdeg } M = \inf \{k \in \mathbb{Z} \mid \mathfrak{m}_P^k M = 0\} - 1 \in \mathbb{N} \cup \{-\infty\}.$$

Denote by $E_{P_0}(P_0/\mathfrak{m}_{P_0})$ the injective hull of the residue field of P_0 .

Theorem 2.1 (Graded Matlis duality). *The dualizing functor*

$$-\vee = {}^* \text{Hom}_{P_0}(-, E_{P_0}(P_0/\mathfrak{m}_{P_0}))$$

*defines an antiequivalence between the categories of * Artinian and finitely generated graded P -modules (see [BH93, Thm. 3.6.17]).* \square

With $D = P^\vee$, the functor $-\vee$ induces an antiisomorphism of lattices

$$(2.1) \quad \begin{array}{ccc} \{I \triangleleft P \text{ graded ideal}\} & \xleftrightarrow{\perp} & \{W \subset D \text{ graded } P\text{-submodule}\}, \\ I & \xrightarrow{\quad\quad\quad} & I^\perp = (P/I)^\vee, \\ (D/W)^\vee = W^\perp & \xleftarrow{\quad\quad\quad} & W, \end{array}$$

where P/I is * Artinian if and only if W is finitely generated.

Let \mathbb{K} be field, and let $\underline{x} = x_1, \dots, x_n$ be indeterminates, where $n \in \mathbb{N} \setminus \{0\}$. Denote by P either the (standard graded) polynomial ring $\mathbb{K}[\underline{x}]$ or the formal power series ring $\mathbb{K}[[\underline{x}]]$, both with $\mathfrak{m}_P = \langle \underline{x} \rangle$. In both cases, D identifies as a \mathbb{K} -vector space with a polynomial ring $\mathbb{K}[\underline{X}]$ in indeterminates $\underline{X} = X_1, \dots, X_n$ with P -module structure given by (see [Eli18, Thm. 2.3.2])

$$(2.2) \quad \underline{x}^n \cdot \underline{X}^m = \begin{cases} \underline{X}^{m-n} & \text{if } m \geq n, \\ 0 & \text{otherwise.} \end{cases}$$

Note that $(\mathfrak{m}_P^k)^\perp = D_{<k} = \bigoplus_{j=0}^{k-1} D_j$ with $\dim_{\mathbb{K}} D_{<k} < \infty$, for all $k \in \mathbb{N}$. With (2.1), it follows that

$$(2.3) \quad \max \deg I^\perp = \text{socdeg}(P/I), \quad \dim_{\mathbb{K}} I^\perp < \infty,$$

if P/I is * Artinian.

In the case where \mathbb{K} is infinite and P/I is Cohen–Macaulay, the following lemma is the starting point for our explicit description of I^\perp .

Lemma 2.2. *Suppose that R is a Noetherian * complete * local homogeneous graded algebra with coefficient field \mathbb{K} . Then $R \cong P/I$ where $I \triangleleft \mathbb{K}[[\underline{y}]][\underline{z}] = P$ with $P_0 = \mathbb{K}[[\underline{y}]]$ and indeterminates \underline{z} of degree 1. Suppose that $P = \mathbb{K}[\underline{x}]$ or $P = \mathbb{K}[[\underline{x}]]$, \mathbb{K} is infinite, and that R is*

Cohen–Macaulay of dimension d . Then, after a \mathbb{K} -linear change of coordinates, $\underline{x} = \underline{y}, \underline{z}$ and \underline{z} maps to an R -sequence of length d .

Proof. By hypothesis, R_0 is Noetherian and R_1 is a finite R_0 -module (see [BH93, Prop. 1.5.4]). Then $R_0 \cong \mathbb{K}[[y]]/I_0$ by Cohen structure theorem and the first claim follows. Suppose now that $P = \mathbb{K}[\underline{x}]$ or $P = \mathbb{K}[[\underline{x}]]$. If \mathbb{K} is infinite and $\text{grade}(\mathfrak{m}_R, R) > 0$, then the \mathbb{K} -vector space $\langle \underline{x} \rangle_{\mathbb{K}} \cong \mathfrak{m}_R/\mathfrak{m}_R^2$ is not the finite union of proper subspaces $\bigcup_{\mathfrak{p} \in \text{Ass } R} (\mathfrak{p} + \mathfrak{m}_R^2)/\mathfrak{m}_R^2$. Then some \mathbb{K} -linear combination of \underline{x} is regular on R and the second claim follows by induction (see [BH93, Prop. 1.5.12]). \square

Let $d \in \{0, \dots, n\}$ and partition $\underline{x} = \underline{y}, \underline{z}$ into sets of indeterminates $\underline{y} = y_1, \dots, y_{n-d}$ and $\underline{z} = z_1, \dots, z_d$. Partition $\underline{X} = \underline{Y}, \underline{Z}$ correspondingly into sets of indeterminates $\underline{Y} = Y_1, \dots, Y_{n-d}$ and $\underline{Z} = Z_1, \dots, Z_d$. The indeterminates $\underline{X}, \underline{Y}, \underline{Z}$ are not related to the ones denoted by the same symbols in §1. Consider the inverse system over \mathbb{N}^d defined by

$$\underline{n} \mapsto D, \quad \underline{n} \leq \underline{m} \mapsto z^{\underline{m}-\underline{n}} \in \text{End}_P(D)$$

with limit $\varprojlim D = \mathbb{K}[\underline{Y}][[\underline{Z}]]$.

Notation 2.3. Consider the P -submodules

$$V_{\underline{m}}^{j,k} = \langle \underline{X}^k \mid |k| \leq |\underline{m}| + k, \underline{k} = (l, \underline{n}), n_j < m_j - 1 \rangle_P \subset D$$

where $j \in \{1, \dots, d\}$, $k \in \mathbb{N}$ and $\underline{m} \in \mathbb{N}^d$.

Remark 2.4. By definition, $V_{\underline{m}}^{j,k}$ is an intersection of P -modules

$$V_{\underline{m}}^{j,k} = \langle \underline{X}^k \mid |k| \leq |\underline{m}| + k \rangle_P \cap \langle \underline{Y}^l \underline{Z}^n \mid n_j < m_j - 1 \rangle_P$$

and applying the lattice antiisomorphism (2.1) yields

$$\begin{aligned} (V_{\underline{m}}^{j,k})^\perp &= \langle \underline{X}^k \mid |k| \leq |\underline{m}| + k \rangle_P^\perp + \langle \underline{Y}^l \underline{Z}^n \mid n_j < m_j - 1 \rangle_P^\perp \\ &= \langle \underline{x} \rangle^{|\underline{m}|+k+1} + \langle z_j^{m_j-1} \rangle. \end{aligned}$$

Definition 2.5. Let $d \in \{0, \dots, n\}$, and let $H \subset \varprojlim D$ be a finite \mathbb{K} -vector subspace. Denote by $H_{\underline{m}}$ its image in the copy of D assigned to $\underline{m} \in \mathbb{N}^d$ and consider the P -submodule

$$(2.4) \quad W_{\underline{m}} = \langle H_{\underline{m}} \rangle_P \subset D.$$

We call H a *limit inverse system of dimension* $\dim H = d$, *type* $\text{type } H = r \in \mathbb{N} \setminus \{0\}$ and *socle degree* $\text{socdeg } H = s \in \mathbb{N}$ if

- (a) $\dim_{\mathbb{K}} H = r$,
- (b) $\min\{\underline{m} \in \mathbb{N}^d \mid H_{\underline{m}} \neq 0\} = \underline{1}$,
- (c) $\max \deg H_{\underline{m}} = |\underline{m}| + s - d$ and
- (d) $W_{\underline{m}} \cap V_{\underline{m}}^{j,s-d} \subset W_{\underline{m}-\underline{e}_j}$,

for all $\underline{m} \in \mathbb{N}^d$ and $j \in \{1, \dots, d\}$. We consider $H \simeq H'$ as equivalent if $W_{\underline{m}} = W'_{\underline{m}}$, for all $\underline{m} \in \mathbb{N}^d$.

Remark 2.6.

(a) Condition 2.5.(b) implies that for all $\underline{m} \in \mathbb{N}^d$ and $i \in \{1, \dots, d\}$, $z_i^{m_i} \cdot H_{\underline{m}} = 0$, and hence $\max \deg_{z_i} H_{\underline{m}} = m_i - 1$. In particular, $\max \deg_{\underline{z}} H_{\underline{m}} = |\underline{m}| - d$.

(b) Condition 2.5.(c) can be substituted by $\max \deg H_{\underline{1}} = s$ and $\max \deg H_{\underline{m}} \leq |\underline{m}| + s - d$, for all $\underline{m} \in \mathbb{N}^d$.

For any $I \triangleleft P$ and $\underline{m} \in \mathbb{N}^d$, we set

$$(2.5) \quad I_{\underline{m}} = I + \langle z_1^{m_1}, \dots, z_d^{m_d} \rangle \triangleleft P, \quad R_{\underline{m}} = P/I_{\underline{m}}.$$

Lemma 2.7. *Any $I \triangleleft P$ can be recovered from (2.5) as*

$$I = \bigcap_{\underline{n} \in \mathbb{N}^d} I_{\underline{n}}.$$

Proof. This is a consequence of Krull intersection theorem. \square

Theorem 2.8. *Let $d \in \{0, \dots, n\}$, $r \in \mathbb{N} \setminus \{0\}$ and $s \in \mathbb{N}$. Then there is a bijection between*

- (a) *the set of (graded) ideals $I \triangleleft P$ such that $R = P/I$ is Cohen–Macaulay, $\dim R = d$, $\underline{z} = z_1, \dots, z_d$ maps to an R -sequence, type $R = r$ and $\text{socdeg } R_{\underline{1}} = s$ and*
- (b) *the set of limit inverse systems $H \subset \varprojlim D$ with $\dim H = d$, type $H = r$ and $\text{socdeg } H = s$ modulo equivalence.*

The map from (a) to (b) is defined by setting (see (2.5))

$$(2.6) \quad W_{\underline{m}} = I_{\underline{m}}^{\perp} = R_{\underline{m}}^{\vee} \subset D$$

and taking $H \subset \varprojlim_{\underline{n} \in \mathbb{N}^d} W_{\underline{n}}$ the image of a \mathbb{K} -linear section of the canonical surjection

$$(2.7) \quad \varprojlim D \supset \varprojlim_{\underline{n} \in \mathbb{N}^d} W_{\underline{n}} \longrightarrow \varprojlim_{\underline{n} \in \mathbb{N}^d} (W_{\underline{n}} \otimes \mathbb{K}) \cong W_{\underline{1}} \otimes \mathbb{K} \cong \text{soc}(R_{\underline{1}})^{\vee},$$

where the inverse systems are defined by $\underline{n} \leq \underline{m} \mapsto z^{\underline{m}-\underline{n}}$. The map from (b) to (a) is defined by setting (see (2.4))

$$(2.8) \quad I = \bigcap_{\underline{n} \in \mathbb{N}^d} W_{\underline{n}}^{\perp}.$$

Lemma 2.9.

- (a) *Let I be in the set 2.8.(a) and $W_{\underline{m}}$ as in (2.6). Then $R_{\underline{m}}$ is Artinian, and hence $\dim_{\mathbb{K}} W_{\underline{m}} < \infty$.*
- (b) *Let H in the set 2.8.(b) and $W_{\underline{m}}$ as in (2.4). Then*

$$\max \deg W_{\underline{m}} = |\underline{m}| + s - d,$$

and hence $\dim_{\mathbb{K}} W_{\underline{m}} < \infty$.

Proof.

(a) Since $\langle \underline{z} \rangle = \sqrt{\langle z_1^{m_1}, \dots, z_d^{m_d} \rangle}$ and \underline{z} maps to an R -sequence of length $d = \dim R$, $\dim R_{\underline{m}} = \dim R_{\underline{1}} = 0$. Then $R_{\underline{m}}$ is Artinian by Hopkins theorem, and hence $\dim_{\mathbb{K}} W_{\underline{m}} < \infty$ by (2.3).

(b) This follows from Definition 2.5.(c) and (2.2). \square

Lemma 2.10. *Let I be in the set 2.8.(a) and $W_{\underline{m}}$ as in (2.6).*

(a) *There is a canonical surjection (2.7).*

Let $H \subset \varprojlim_{\underline{n} \in \mathbb{N}^d} W_{\underline{n}}$ be the image of a \mathbb{K} -linear section of the surjection (2.7).

(b) *The P -module $W_{\underline{m}}$ is minimally generated by $H_{\underline{m}}$, for all $\underline{m} \in \mathbb{N}^d$. In particular, (2.4) holds true.*

(c) *The \mathbb{K} -vector space H is in the set 2.8.(b).*

Proof. In the following, $\underline{n}, \underline{m} \in \mathbb{N}^d$ with $\underline{n} \leq \underline{m}$.

(a) Consider the surjection of direct systems represented by

$$(2.9) \quad \begin{array}{ccc} R & \xrightarrow{z^{\underline{m}-\underline{n}}} & R \\ \downarrow & & \downarrow \\ R_{\underline{n}} & \xrightarrow{z^{\underline{m}-\underline{n}}} & R_{\underline{m}}. \end{array}$$

Applying $-\vee$ yields an inclusion of inverse systems represented by

$$\begin{array}{ccc} D & \xleftarrow{z^{\underline{m}-\underline{n}}} & D \\ \uparrow & & \uparrow \\ \bigcup W_{\underline{n}} & \xleftarrow{z^{\underline{m}-\underline{n}}} & \bigcup W_{\underline{m}}. \end{array}$$

Left-exactness of the inverse limit then yields the inclusion in (2.7).

We now apply §1 with $\underline{g} = \underline{h} = z$ and M the matrix with diagonal $\underline{m} \in \mathbb{N}^d$. Then $\text{soc } M = \{\underline{m} - \underline{1}\}$ in Definition 1.8.

By Proposition 1.5, Corollary 1.9 and (1.9), the bottom map in (2.9) identifies (homogeneous) socles. This yields an inclusion of direct systems represented by

$$(2.10) \quad \begin{array}{ccc} R_{\underline{n}} & \xrightarrow{z^{\underline{m}-\underline{n}}} & R_{\underline{m}} \\ \uparrow & & \uparrow \\ \text{soc } R_{\underline{n}} & \xrightarrow{\cong} & \text{soc } R_{\underline{m}}. \end{array}$$

By Lemma 2.9.(a), $R_{\underline{m}}$ is Artinian, and hence (see [Eli18, Prop. 2.4.3])

$$(2.11) \quad \text{soc}(R_{\underline{m}})^\vee \cong I_{\underline{m}}^\perp / \mathfrak{m}_P \cdot I_{\underline{m}}^\perp \cong W_{\underline{m}} \otimes \mathbb{K}.$$

With $\underline{m} = \underline{1}$, this is the second isomorphism in (2.7).

Applying (2.11) to the bottom row of (2.10), this yields a trivial inverse system represented by

$$\begin{array}{ccc} \text{soc}(R_n)^\vee & \xleftarrow{\cong} & \text{soc}(R_m)^\vee \\ \downarrow \cong & & \downarrow \cong \\ W_n \otimes \mathbb{K} & \xleftarrow[\cong]{z^{m-n}} & W_m \otimes \mathbb{K}, \end{array}$$

and hence the first isomorphism in (2.7).

Consider the short exact sequence of inverse systems represented by

$$(2.12) \quad 0 \rightarrow \mathfrak{m}_P \cdot W_m \rightarrow W_m \rightarrow W_m \otimes \mathbb{K} \rightarrow 0.$$

Since $\dim_{\mathbb{K}} W_m < \infty$ by Lemma 2.9.(a), the left inverse system in (2.12) satisfies the Mittag-Leffler condition. Therefore, the inverse limit preserves exactness when applied to (2.12). This yields the surjection in (2.7).

(b) Any \mathbb{K} -linear section σ of (2.7) with image H fits into a diagram

$$\begin{array}{ccccc} & & \sigma & & \\ & & \cong & & \\ & \xleftarrow{\quad} & & \xrightarrow{\quad} & \\ H & \xleftarrow{\quad} & \varprojlim_{n \in \mathbb{N}^d} W_n & \xrightarrow{\quad} & \varprojlim_{n \in \mathbb{N}^d} (W_n \otimes \mathbb{K}) \xrightarrow{\cong} W_{\underline{1}} \otimes \mathbb{K} \\ & \downarrow \cong & \downarrow & & \downarrow \cong \\ H_{\underline{m}} & \hookrightarrow & W_{\underline{m}} & \xrightarrow{\quad} & W_{\underline{m}} \otimes \mathbb{K} \\ & & \cong & & \cong \end{array}$$

and the claim follows by Nakayama lemma (see [BH93, Ex. 1.5.24]). \square

(c) By construction, $H \cong H_{\underline{1}} \cong \text{soc}(R_{\underline{1}})^\vee$. Using that $-^\vee$ preserves length, this gives condition 2.5.(a) (see [BH93, Lem. 1.2.19]),

$$(2.13) \quad \dim_{\mathbb{K}} H = \dim_{\mathbb{K}} H_{\underline{1}} = \dim_{\mathbb{K}} \text{soc } R_{\underline{1}} = \text{type } R = r.$$

For $\underline{m} \not\geq \underline{1}$, $R_{\underline{m}} = 0$, and hence $H_{\underline{m}} \subset W_{\underline{m}} = 0$. With (2.13), condition 2.5.(b) follows. Part (b) with $\underline{m} = \underline{1}$ gives $\langle H_{\underline{1}} \rangle = W_{\underline{1}} = I_{\underline{1}}^\perp$, and hence $\max \deg H_{\underline{1}} = \text{socdeg } R_{\underline{1}} = s$ by (2.2) and (2.3). Condition 2.5.(c) follows by (2.2). By (2.1) and Remark 2.4,

$$\begin{aligned} (W_{\underline{m}} \cap V_{\underline{m}}^{j,s-d})^\perp &= W_{\underline{m}}^\perp + (V_{\underline{m}}^{j,s-d})^\perp \\ &= I + \langle z_1^{m_1}, \dots, z_d^{m_d} \rangle + \langle \underline{x} \rangle^{|\underline{m}|+s-d+1} + \langle z_j^{m_j-1} \rangle \\ &\supset I + \langle z_1^{m_1}, \dots, z_j^{m_j-1}, \dots, z_d^{m_d} \rangle = W_{\underline{m}-\underline{e}_j}^\perp. \end{aligned}$$

Condition 2.5.(d) follows with (2.1).

Lemma 2.11. *Let H be in the set 2.8.(b) and I as in (2.8).*

(a) *There is an equality $I_{\underline{m}} = W_{\underline{m}}^\perp$, for all $\underline{m} \in \mathbb{N}^d$.*

(b) *The sequence \underline{z} maps to an R -sequence.*

(c) *The ring $R = P/I$ is Cohen-Macaulay with $\dim R = d$.*

Proof.

(a) Let $j \in \{1, \dots, d\}$ and $\underline{m} \in \mathbb{N}^d$. By Lemma 2.9.(b), $W_{\underline{m}+\underline{e}_j} \subset D$ is a finitely generated (graded) P -submodule. Then $P/W_{\underline{m}+\underline{e}_j}^\perp$ is Artinian by (2.1). By (2.3) and Lemma 2.9.(b),

$$\text{socdeg}(P/W_{\underline{m}+\underline{e}_j}^\perp) = \max \deg W_{\underline{m}+\underline{e}_j} = |\underline{m} + \underline{e}_j| + s - d,$$

and hence

$$W_{\underline{m}+\underline{e}_j}^\perp \supset \langle \underline{x} \rangle^{|\underline{m}+\underline{e}_j|+s-d+1}.$$

Using Definition 2.5.(d), (2.1) and Remark 2.4, it follows that

$$\begin{aligned} W_{\underline{m}}^\perp &\subset (W_{\underline{m}+\underline{e}_j} \cap V_{\underline{m}+\underline{e}_j}^{j,s-d})^\perp \\ &= W_{\underline{m}+\underline{e}_j}^\perp + (V_{\underline{m}+\underline{e}_j}^{j,s-d})^\perp \\ &= W_{\underline{m}+\underline{e}_j}^\perp + \langle \underline{x} \rangle^{|\underline{m}+\underline{e}_j|+s-d+1} + \langle z_j^{m_j} \rangle \\ &= W_{\underline{m}+\underline{e}_j}^\perp + \langle z_j^{m_j} \rangle. \end{aligned}$$

This already implies that (see [MT18, Prop. 10, Claim 1])

$$W_{\underline{m}}^\perp \subset I + \langle z_1^{m_1}, \dots, z_d^{m_d} \rangle.$$

The opposite inclusion holds true since $I \subset W_{\underline{m}}^\perp$ by definition and $z_i^{m_i} \cdot H_{\underline{m}} = 0$, for all $i \in \{1, \dots, d\}$, by Remark 2.6.(a).

(b) By dualizing surjections $\underline{z}^{\underline{m}-\underline{n}}: W_{\underline{m}} \rightarrow W_{\underline{n}}$ for suitable $\underline{m}, \underline{n} \in \mathbb{N}^d$ with $\underline{n} \leq \underline{m}$, one shows that \underline{z} maps to a weak R -sequence (see [MT18, Prop. 10, Claim 2]). Since $W_{\underline{1}} \neq 0$ by Definition 2.5.(b), $I_{\underline{1}} = W_{\underline{1}}^\perp \neq R$ by part (a) with $\underline{m} = \underline{1}$ and (2.1). Thus, $R_{\underline{1}} \neq 0$ and \underline{z} maps to an R -sequence.

(c) By part (a) with $\underline{m} = \underline{1}$, the ring $R_{\underline{1}}$ is Artinian, and hence $\dim R_{\underline{1}} = 0$ by Hopkins theorem. With (b), it follows that $R_{\underline{m}_R}$, and hence R is Cohen–Macaulay with $\dim R = d$ (see [BH93, Ex. 2.1.27.(c)]). \square

Proof of Theorem 2.8. This follows from (2.1), Lemmas 2.7, 2.10 and 2.11. \square

Example 2.12. Let us consider the irreducible algebroid curve

$$R = \mathbb{C}[[t^6, t^7, t^{11}, t^{13}]].$$

Note that R is not quasi-homogeneous. We write $R \cong P/I$ where

$$\begin{aligned} P &= \mathbb{C}[[x, y, z, w]], \\ I &= \langle w - xy, yz - x^3, xz^2 - y^4, z^3 - x^2y^3, y^5 - x^4z \rangle. \end{aligned}$$

The element $x \in P$ maps to the regular element $t^6 \in R$. It can be checked that $\text{type } R = 2$ and $P/(I + \langle x \rangle)$ has Hilbert–Samuel function

(1, 2, 2, 1, 1). It follows that R is not level. Using SINGULAR (see [Dec+18]), we compute the socles of R_m (see (2.5)) up to $m = 3$:

$$\begin{aligned}\text{soc } R_1 &= \langle \bar{z}^2, \bar{y}^3 \rangle, \\ \text{soc } R_2 &= \langle \bar{x}\bar{z}^2, \bar{x}\bar{y}^3 \rangle, \\ \text{soc } R_3 &= \langle \bar{x}^2\bar{z}^2, \bar{x}^2\bar{y}^3 \rangle.\end{aligned}$$

They fit into the commutative diagram (see (2.10))

$$\begin{array}{ccccc} R_1 & \xrightarrow{x} & R_2 & \xrightarrow{x} & R_3 \\ \uparrow & & \uparrow & & \uparrow \\ \text{soc } R_1 & \xrightarrow{\cong} & \text{soc } R_2 & \xrightarrow{\cong} & \text{soc } R_3. \end{array}$$

Using a SINGULAR library by Elias (see [Eli15]), we compute the limit inverse system H associated to I by Theorem 2.8 up to $m = 7$:

$$\begin{aligned}H_1 &= \langle Y^3, Z^2 \rangle_{\mathbb{K}}, \\ H_2 &= \langle XY^3 + Y^2W, XZ^2 + Y^4 \rangle_{\mathbb{K}}, \\ H_3 &= \langle X^2Y^3 + XY^2W + YW^2 + Z^3, X^2Z^2 + XY^4 + Y^3W \rangle_{\mathbb{K}}, \\ H_4 &= \langle X^3Y^3 + X^2Y^2W + XYW^2 + XZ^3 + Y^4Z + W^3, \\ &\quad X^3Z^2 + X^2Y^4 + XY^3W + YZ^3 + Y^2W^2 \rangle_{\mathbb{K}}, \\ H_5 &= \langle X^4Y^3 + X^3Y^2W + X^2YW^2 + X^2Z^3 + XY^4Z + XW^3 + Y^3ZW, \\ &\quad X^4Z^2 + X^3Y^4 + X^2Y^3W + XYZ^3 + XY^2W^2 + Z^3W + YW^3 + Y^5Z \rangle_{\mathbb{K}}, \\ H_6 &= \langle X^5Y^3 + X^3Z^3 + X^4Y^2W + X^3YW^2 + X^2Y^4Z + X^2W^3 + XY^3ZW + Y^2ZW^2 + YZ^4, \\ &\quad X^5Z^2 + X^4Y^4 + X^3Y^3W + X^2YZ^3 + X^2Y^2W^2 + XZ^3W + XYW^3 + XY^5Z + \\ &\quad Y^4ZW + W^4 \rangle_{\mathbb{K}}, \\ H_7 &= \langle X^6Y^3 + X^5Y^2W + X^4Z^3 + X^4YW^2 + X^3Y^4Z + X^3W^3 + X^2Y^3ZW + XY^2ZW^2 + \\ &\quad XYZ^4 + Z^4W + YZW^3 + Y^5Z^2, \\ &\quad X^6Z^2 + X^5Y^4 + X^4Y^3W + X^3YZ^3 + X^3Y^2W^2 + X^2Z^3W + X^2YW^3 + X^2Y^5Z + \\ &\quad XY^4ZW + XW^4 + Y^2Z^4 + Y^3ZW^2 \rangle_{\mathbb{K}}.\end{aligned}$$

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