

LOCAL ANALYSIS OF GRAUERT–REMMERT-TYPE NORMALIZATION ALGORITHMS

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ABSTRACT. Normalization is a fundamental ring-theoretic operation; geometrically it resolves singularities in codimension one. Existing algorithmic methods for computing the normalization rely on a common recipe: successively enlarge the given ring in form of an endomorphism ring of a certain (fractional) ideal until the process becomes stationary. While Vasconcelos' method uses the dual Jacobian ideal, Grauert–Remmert-type algorithms rely on so-called test ideals.

For algebraic varieties, one can apply such normalization algorithms globally, locally, or formal analytically at all points of the variety. In this paper, we relate the number of iterations for global Grauert–Remmert-type normalization algorithms to that of its local descendants.

We complement our results by a study of ADE singularities. All intermediate singularities occurring in the normalization process are determined explicitly. Besides ADE singularities the process yields simple space curve singularities from the list of Frühbis-Krüger.

INTRODUCTION

Normalization of rings is an important concept in commutative algebra, with applications in algebraic geometry and singularity theory. In fact, geometrically, normalization leads to a desingularization in codimension one. In particular, in the case of curves, it is the same as desingularization.

Given a reduced Noetherian ring A for which the normalization \bar{A} is a finite A -module, an obvious approach for finding \bar{A} is to successively enlarge A in form of an endomorphism ring of a fractional ideal. Specific instances of such algorithms require a recipe for choosing the fractional ideal, supported by a suitable normality criterion. The latter must ensure that the endomorphism ring is strictly larger than A exactly if A is not normal. For a reduced affine K -algebra A , where K is a perfect field, two approaches of this type have proven to work:

Vasconcelos' algorithm [Vas91] (see also [Vas98, §6]) is based on a regularity criterion of Lipman (see [Lip69]). It first replaces A by its (S_2) -ification $\text{End}_A(\omega_A)$, which is computed via a Noether normalization, and then iteratively by $\text{End}_A(J_A^\vee)$, where J_A is the Jacobian ideal of A .

De Jong's algorithm [dJ98, DdJGP99] is based on the normality criterion of Grauert and Remmert [GR71, Anhang, §3.3, Satz 7], see [GLS10] for a historical account. Here, the idea is to choose a so-called *test ideal* J (for example, the radical of the Jacobian ideal), and to replace A by $\text{End}_A(J)$, repeating the procedure if necessary. A variant of this algorithm with improved efficiency is described in [GLS10];

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we refer to this version as the *GLS normalization algorithm*. The performance of the GLS algorithm can be further enhanced by applying it to suitable strata of the singular locus and combining the individual results, see [BDL⁺13]. This technique is particularly useful for parallel computing. The parallel GLS algorithm is the fastest normalization algorithm known to date. An implementation is available in the computer algebra system SINGULAR [DGPS12].

From a more theoretical point of view, algorithmic normalization and its key players, endomorphism rings of fractional ideals, leave many questions to be answered. We give an incomplete list of research related to this topic: de Jong [dJ98, p. 275] briefly discusses relations of endomorphism rings with respect to products of ideals. Pham and Vasconcelos [PV08] analyse the complexity of general normalization algorithms in the graded case. Buchweitz et. al. [BEGvB09] give examples of free discriminants of versal families whose normalization is obtained in one step. Granger et. al. [GMS12] explicitly describe certain natural partial normalizations of finite Coxeter arrangements and their discriminants. Granger and Schulze [GS12] relate the minimal step size of the Vasconcelos algorithm to quasihomogeneity for Gorenstein curves.

In this paper, we focus on the GLS algorithm and study the number of iterations in terms of localization and completion of the given ring and with respect to the inclusion of test ideals.

In Section 1, we introduce the objects and operations relevant to algorithmic normalization and notice that they are compatible with localization. To obtain similar results for completion, we consider the class of excellent semilocal rings. For such rings, normalization is finite and commutes with completion.

Based on these preparations, in Section 2, we show that the GLS algorithm behaves well with respect to localization and completion. That is, these operations preserve the property of being a test ideal and commute with forming endomorphism rings. By considering test ideals relative to a subset of the spectrum of A , we prepare the ground for the study of stratified normalization in the next section.

In Section 3, we stratify the singular locus of A and consider test ideals relative to the individual strata. We show that the number of steps of the global algorithm (using the radical ideal of the singular locus) is at most the maximal number of steps among the strata. We prove equality in the case where A is equidimensional and satisfies Serre's condition (S_2) . Being Cohen–Macaulay, local complete intersections or curves satisfy these conditions. As a side result, in the equidimensional case, we show that the GLS algorithm preserves (S_2) at each step.

In Section 4, we prove formulas for the number of steps of the GLS algorithm for plane curve singularities of type ADE¹. Even more, we explicitly determine the singularity types that occur in the process, describing the respective value semigroups on our way. The occurring types are ADE singularities and simple space curve singularities from the list of Frühbis-Krüger [FK99].

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1. NORMALIZATION AND COMPLETION

In the following, all rings are commutative with one and \mathbb{N} includes 0.

Let A be a reduced Noetherian ring. We write $Q(A)$ for the total ring of fractions of A , which is again a reduced Noetherian ring.

Definition 1.1. The *normalization* of A , written \bar{A} , is the integral closure of A in $Q(A)$. We call A *normalization-finite* if \bar{A} is a finite A -module, and we call A

¹Based on experiments using SINGULAR, these formulas are also given in [Mor13].

normal if $A = \bar{A}$. We denote by

$$N(A) = \{P \in \text{Spec}(A) \mid A_P \text{ is not normal}\}$$

the *non-normal locus* of A , and by

$$\text{Sing}(A) = \{P \in \text{Spec}(A) \mid A_P \text{ is not regular}\}$$

the *singular locus* of A .

Remark 1.2. Note that $N(A) \subset \text{Sing}(A)$. Equality holds if A is of pure dimension one. Indeed, a Noetherian local ring of dimension one is normal if and only if it is regular (see [dJP00, Thm. 4.4.9]).

Remark 1.3. Recall that a ring is reduced if and only if it satisfies Serre's conditions (R_0) and (S_1) ; it is normal if and only if satisfies (R_1) and (S_2) (see [HS06, §4.5]).

Definition 1.4. The *conductor* of A in \bar{A} is $\mathcal{C}_A = \text{Ann}_A(\bar{A}/A)$.

Remark 1.5. Note that \mathcal{C}_A is the largest ideal of A which is also an ideal of \bar{A} . In particular, A is normal if and only if $\mathcal{C}_A = A$.

Definition 1.6. If M is an A -module, we write $M^{-1} = \text{Hom}_A(M, A)$ for the *dual module*, and call M *reflexive* if the canonical map $M \rightarrow (M^{-1})^{-1}$ is an isomorphism.

Remark 1.7. Any map $\varphi \in \bar{A}^{-1} = \text{Hom}_A(\bar{A}, A)$ is multiplication by $\varphi(1) \in A$. We may, thus, identify $\bar{A}^{-1} = \mathcal{C}_A$ by taking φ to $\varphi(1)$.

If A is normalization-finite, satisfies (S_2) , and is Gorenstein in codimension 1, then \mathcal{C}_A is reflexive (see [GS11, Lem. 2.8]).

Notation 1.8. If I is an ideal of a ring R , we write $V(I) = \{P \in \text{Spec}(R) \mid P \supseteq I\}$ for the *vanishing locus of I in $\text{Spec}(R)$* .

Remark 1.9. As usual, if P is a prime of a ring R , and M is an R -module, we write M_P for the localization of M at $R \setminus P$. Recall that if $R \subseteq R'$ is a ring extension, and M' is an R' -module, then the localization of M' at $R \setminus P \subset R'$ coincides with M'_P if M' is considered as an R -module.

Proposition 1.10. *As above, let A be a reduced Noetherian ring. Then:*

- (1) *A is normalization-finite if and only if \mathcal{C}_A contains a non-zero-divisor.*
- (2) *$N(A) \subseteq V(\mathcal{C}_A)$, with equality if A is normalization-finite.*
- (3) *If A is normalization-finite, then $(\mathcal{C}_A)_P = \mathcal{C}_{A_P}$ for all $P \in \text{Spec}(A)$.*

Proof. (1) Note that a common denominator of a finite set of generators for \bar{A} over A is a non-zero-divisor in \mathcal{C}_A .

(2) See [GP08, Lem. 3.6.3].

(3) See [ZS75, Ch. V, §5].

□

By part (1) of the preceding proposition, if A is normalization-finite, then \bar{A} is a fractional ideal in the following sense:

Definition 1.11. A *fractional ideal* of A is a finite A -submodule of $Q(A)$ containing a non-zero-divisor of A .

Lemma 1.12. *Let M and N be fractional ideals of A . Then, independently of the choice of a non-zero-divisor $g \in M$ of A , we may identify $\text{Hom}_A(M, N)$ with a fractional ideal of A by means of*

$$\text{Hom}_A(M, N) \hookrightarrow Q(A), \quad \varphi \mapsto \varphi(g)/g.$$

In particular, any $\varphi \in \text{Hom}_A(M, N)$ is multiplication by $\varphi(g)/g$.

Proof. See, for example, [GLS10, Lem. 3.1]. \square

Our applications will be geometric in nature: The rings under discussion will be finite K -algebras, where K is a field, localizations of such algebras, the semi-local rings appearing in the normalization process of a localization, or the rings obtained by completing one of the semi-local rings. As we recall in what follows, all these rings are normalization-finite.

Notation 1.13. The Jacobson radical of a ring R will be denoted by \mathfrak{m}_R .

Theorem 1.14. *Let R be a Noetherian ring, and let $\widehat{}$ be completion at an ideal I of R . Then \widehat{R} is Noetherian, and we have:*

- (1) *If M is a finite R -module, then the natural map $M \otimes_R \widehat{R} \rightarrow \widehat{M}$ is an isomorphism, and \widehat{R} is flat over R .*
- (2) *If I is contained in \mathfrak{m}_R , then the natural map $R \rightarrow \widehat{R}$ is an inclusion, and \widehat{R} is faithfully flat over R .*

Proof. See [Bou98, Ch. III, §3.4 Thm. 3, Props. 8, 9]. Note that in case (2), R together with the I -adic topology is a Zariski ring. \square

Remark 1.15. If R is a semi-local Noetherian ring, then $\widehat{}$ will always stand for the completion at \mathfrak{m}_R . In this case, \widehat{R} is again a semi-local Noetherian ring, and if $R \subset R'$ is a finite ring extension, then R' is semi-local and Noetherian as well, and the completion of any R' -module M' at $\mathfrak{m}_{R'}$ coincides with that at \mathfrak{m}_R if M' is regarded as an R -module. See [Bou98, Ch. III, §3.1, Cor.; Ch. IV, §2.5, Cor. 3] for details.

Grothendieck's notion of an *excellent ring* provides the general framework for the rings considered here. Referring to [Mat80] for the defining properties of an excellent ring (which include that the ring is Noetherian), we recall a number of consequences of these properties.

Theorem 1.16 (Grothendieck). *Let R be an excellent ring. Then all localizations of R and all finitely generated R -algebras are excellent.*

Proof. See [Mat80, Thms. 73, 77]. \square

Remark 1.17. Complete semi-local Noetherian rings are excellent (for a proof see [Mat80, (28.P); Thms. 68, 74]). In particular, any field K and, hence, any localization of any finitely generated K -algebra are excellent.

Theorem 1.18 (Grothendieck). *Let A be a reduced excellent ring. Then:*

- (1) *The completion \widehat{A} of A at any ideal of A is reduced. If A is normal, then \widehat{A} is normal.*
- (2) *A is normalization-finite.*
- (3) *If A is semi-local, then $\overline{\widehat{A}} = \widehat{\overline{A}}$. In particular,*
 - (a) *$\overline{\widehat{A}} = \widehat{\overline{A}} = \overline{A} \otimes_A \widehat{A}$ is a finite \widehat{A} -module, and*
 - (b) *if A is complete, then \overline{A} is complete.*

Proof. For (1), see [Mat80, Thm. 79].

For (2), note that excellent rings are Nagata rings (see [Mat80, Thm. 78]). Hence, if P is any prime ideal of A , then A/P is normalization-finite, so that $\overline{A/P}$ is in particular finite over A . The result follows from the *splitting of normalization* (see [dJP00, Thm. 1.5.20]): If P_1, \dots, P_s are the minimal primes of R , then

$$\overline{A} \cong \prod_{i=1}^s \overline{A/P_i}.$$

For (3), note that \overline{A} is finite over A by (2) and, hence, an excellent ring by Theorem 1.16. We conclude from (1) that $\widehat{\overline{A}}$ is normal. Now consider the inclusions $A \subseteq \overline{A} \subseteq Q(A)$ which by the flatness of completion give rise to the inclusions $\widehat{A} \subseteq \widehat{\overline{A}} \subseteq Q(A) \otimes_A \widehat{A}$, where $\widehat{\overline{A}}$ is finite over \widehat{A} . Since every non-zero-divisor of A is a non-zero-divisor of \widehat{A} , we may regard $Q(A) \otimes_A \widehat{A}$ as a subring of $Q(\widehat{A})$. Since $\widehat{\overline{A}}$ is normal, we conclude that $\widehat{\overline{A}} = \widehat{A}$. \square

Corollary 1.19. *If A is a Noetherian complete local domain, then \overline{A} is a Noetherian complete local domain.*

Proof. It is clear from Remark 1.17 that A is excellent. Hence, by Theorem 1.18, A is normalization-finite and \overline{A} is complete. Taking Remark 1.15 into account, we conclude that \overline{A} is a Noetherian complete semi-local ring and, thus, a product of local rings (see [Bou98, Ch. III, §2.13, Cor.]). Since \overline{A} is also a domain, it must be a local ring. \square

Corollary 1.20. *If J is an ideal of an excellent ring R , and $\widehat{}$ is completion at an arbitrary ideal of R , then*

$$\sqrt{\widehat{J}} = \widehat{\sqrt{J}}.$$

In particular, if J is radical, then \widehat{J} is radical as well.

Proof. By the flatness of completion, we first deduce from $J \subseteq \sqrt{J}$ that $\widehat{J} \subseteq \widehat{\sqrt{J}}$, and second from Theorem 1.18.(1) that $\widehat{\sqrt{J}}$ is radical. Thus, $\sqrt{\widehat{J}} \subseteq \widehat{\sqrt{J}}$. On the other hand, there is an m with $(\sqrt{J})^m \subseteq J$. Again by flatness, this implies that $(\widehat{\sqrt{J}})^m \subseteq \widehat{J}$, that is, $\widehat{\sqrt{J}} \subseteq \sqrt{\widehat{J}}$. \square

Proposition 1.21. *Suppose $R \rightarrow S$ is a flat homomorphism of rings, M and N are R -modules, and M is finitely generated. Then there is a unique S -module isomorphism*

$$\mathrm{Hom}_R(M, N) \otimes_R S \cong \mathrm{Hom}_S(M \otimes_R S, N \otimes_R S)$$

which takes $\varphi \otimes 1$ to $\varphi \otimes \mathrm{id}_S$.

Proof. See [Eis95, Prop. 2.10]. \square

Corollary 1.22. *If A is a reduced excellent semi-local ring, then*

$$\widehat{\mathcal{C}}_A = \mathcal{C}_{\widehat{A}}.$$

Proof. Use Remark 1.7, Theorem 1.14.(1), Proposition 1.21, and Theorem 1.18.(3). \square

2. ALGORITHMIC NORMALIZATION

Throughout this section, A denotes a reduced excellent ring. In particular, A is normalization-finite.

Definition 2.1. A *test ideal* at a subset $W \subseteq \mathrm{Spec}(A)$ is an ideal $J \subseteq A$ satisfying the following conditions:

- (1) J contains a non-zero-divisor of A ,
- (2) J is a radical ideal, and
- (3) $V(\mathcal{C}_{A_P}) \subseteq V(J_P)$ for all $P \in W$.

A *test ideal* for A is an ideal $J \subseteq A$ satisfying (1), (2), and

- (3') $V(\mathcal{C}_A) \subseteq V(J)$.

Remark 2.2. Since we assume that A is normalization-finite,

$$V(\mathcal{C}_A) = N(A) \subseteq \text{Sing}(A).$$

Hence the vanishing ideal J of $\text{Sing}(A)$ is a valid test ideal for A . Indeed, J contains a non-zero-divisor since A is reduced and, hence, regular in codimension zero, so that $J \otimes_A Q(A) = Q(A)$.

In what follows, if $J \subseteq A$ is any ideal containing a non-zero-divisor of A , we regard A as a subring of $\text{End}_A(J) = \text{Hom}_A(J, J)$ by sending $a \in A$ to multiplication by a , and $\text{End}_A(J)$ as a fractional ideal of A as in Lemma 1.12. Then

$$A \subseteq \text{End}_A(J) \subseteq \overline{A} \tag{2.1}$$

(see [GP08, Lem. 3.6.1]).

Proposition 2.3 (Grauert and Remmert Criterion). *Let J be a test ideal for A . Then A is normal if and only if $A = \text{End}_A(J)$.*

Proof. See [GR71], [GP08, Prop. 3.6.5]. \square

Before describing the normalization algorithm arising from the Grauert and Remmert criterion, we discuss how the criterion behaves with respect to localization and completion. We first address the test ideals:

Lemma 2.4. *Let $J \subset A$ be an ideal, and let $W \subset \text{Spec}(A)$. Then:*

- (1) *If J is a test ideal at W , and $P \in W$, then J_P is a test ideal for A_P .*
- (2) *If $W \supseteq N(A)$, then J is a test ideal for A if and only if it is a test ideal at W .*

Proof. Condition (3) of Definition 2.1 means that condition (3') of the definition holds for the rings A_P together with the ideals J_P , $P \in W$. The first statement of the lemma follows since conditions (1) and (2) of Definition 2.1 carry over from J to J_P : Use the flatness of localization and that localization commutes with passing to radicals, respectively. Taking into account that $(\mathcal{C}_A)_P = \mathcal{C}_{A_P}$ since A is assumed normalization-finite (see Proposition 1.10.(3)), the same reasoning shows the second statement of the lemma. Indeed, $J \subseteq \sqrt{\mathcal{C}_A}$ if and only if $J_P \subseteq (\sqrt{\mathcal{C}_A})_P$ for all $P \in \text{Spec}(A)$, and $\mathcal{C}_P = A_P$ if $P \notin N(A)$. \square

Lemma 2.5. *If A is semi-local and J is a test ideal for A , then \widehat{J} is a test ideal for \widehat{A} .*

Proof. If $g \in J$ is a non-zero-divisor of A , then $g \otimes_A 1 \in J \otimes_A \widehat{A} = \widehat{J}$ is a non-zero-divisor of \widehat{A} by the flatness of completion. Moreover, by Corollary 1.20, \widehat{J} is radical and $\widehat{J} \subseteq \widehat{\sqrt{\mathcal{C}_A}} = \sqrt{\widehat{\mathcal{C}_A}} = \sqrt{\mathcal{C}_{\widehat{A}}}$, where we use the assumption $J \subseteq \sqrt{\mathcal{C}_A}$, and where the last equality holds by Corollary 1.22. \square

Next, in the two corollaries of Proposition 1.21 below, we treat the endomorphism rings appearing in the Grauert and Remmert criterion.

Corollary 2.6. *Let J be an ideal of A . Then, for all $P \in \text{Spec}(A)$,*

$$(\text{End}_A(J))_P = \text{End}_{A_P}(J_P).$$

Further, $A = \text{End}_A(J)$ if and only if $A_P = \text{End}_{A_P}(J_P)$ for all $P \in \text{Spec}(A)$.

Proof. Apply Proposition 1.21 and use that equality is a local property. \square

Corollary 2.7. *If A is semi-local, and J is an ideal of A , then*

$$\widehat{\text{End}_A(J)} = \text{End}_{\widehat{A}}(\widehat{J}),$$

and $A = \text{End}_A(J)$ if and only if $\widehat{A} = \text{End}_{\widehat{A}}(\widehat{J})$.

Proof. Apply Proposition 1.21 and use that \widehat{A} is faithfully flat over A . \square

The Grauert and Remmert criterion allows us to compute \overline{A} by successively enlarging the given ring by an endomorphism ring. This is based on the following result from [GLS10, Prop. 3.2]:

Lemma 2.8. *Let $A \subseteq A' \subseteq \overline{A}$ be an intermediate ring, and let J be a test ideal for A . Then $\sqrt{JA'}$ is a test ideal for A' . \square*

Given any radical ideal $J \subseteq A$ containing a non-zerodivisor, we inductively define radical ideals and intermediate rings by setting $A_0 = A$,

$$J_i = \sqrt{JA_i}, \text{ and } A_{i+1} = \text{End}_{A_i}(J_i) \subseteq Q(A).$$

Here, with A , also $Q(A)$ and, hence, all the A_i are reduced. Since we assume that A is Noetherian and normalization-finite, we get, thus, a finite chain of extensions of reduced Noetherian rings

$$A = A_0 \subsetneq \cdots \subsetneq A_{i-1} \subsetneq A_i \subsetneq \cdots \subsetneq A_n = A_{n+1} \subseteq \overline{A}.$$

Notation 2.9. We write $n(A, J) = n$ for the number of steps above.

Note that if J is a test ideal for A , then each J_i is a test ideal for A_i , so that $A_n = \overline{A}$ by the Grauert and Remmert criterion. More generally, by the proof of [BDL⁺13, Prop. 3.3], we have:

Proposition 2.10. *Let $A \subseteq A' \subseteq \overline{A}$ be an intermediate ring. Let $W \subseteq \text{Spec}(A)$, let J be a test ideal at W , and let $J' = \sqrt{JA'}$. If*

$$A' \cong \text{End}_{A'}(J'),$$

then A'_P is normal for each $P \in W$.

We now relate the behaviour of the global version of the normalization algorithm to that of its local version:

Proposition 2.11. *Let $J \subseteq A$ be a radical ideal containing a non-zerodivisor of A , and let $W \subseteq \text{Spec}(A)$. Then*

$$\begin{aligned} n(A, J) &\geq \max_{P \in W} n(A_P, J_P) \\ &= \max_{P \in W} n(\widehat{A}_P, \widehat{J}_P). \end{aligned}$$

Equality holds if either $V(J) \subseteq W$ or $N(A) \subseteq W$.

Proof. Let $P \in \text{Spec}(A)$. If $P \notin V(J)$, then $J_P = A_P$, so $n(A_P, J_P) = 0$. Furthermore, as is clear from (2.1), this number is also zero if $P \notin N(A)$. Hence, all statements of the proposition will follow once we show that

- $n(A, J) = \max_{P \in \text{Spec}(A)} n(A_P, J_P)$, and
- $n(A_P, J_P) = n(\widehat{A}_P, \widehat{J}_P)$ for all $P \in \text{Spec}(A)$.

To establish these equalities, given $P \in \text{Spec}(A)$, we inductively set $B_0 = A_P$, $C_0 = \widehat{A}_P$,

$$B_{i+1} = \text{End}_{B_i}(\sqrt{J_P B_i}), \text{ and } C_{i+1} = \text{End}_{C_i}(\sqrt{\widehat{J}_P C_i}).$$

Since equality is a local property, and completion is faithfully flat in the semi-local case, it is enough to show that $B_i = (A_i)_P$ and $C_i = \widehat{B}_i$ for all i and P . For this, we do induction, assuming that our claim is true for i : We first note that

$$J_P(A_i)_P = (JA_i)_P \text{ and } \widehat{J}_P \widehat{B}_i = \widehat{J_P B_i}.$$

Then, applying Corollary 2.6, we get

$$B_{i+1} = \text{End}_{(A_i)_P} \left(\sqrt{J_P(A_i)_P} \right) = \text{End}_{(A_i)_P} \left(\left(\sqrt{JA_i} \right)_P \right) = (A_{i+1})_P,$$

and Corollaries 1.20 and 2.7 give

$$C_{i+1} = \text{End}_{\widehat{B}_i} \left(\sqrt{\widehat{J_P B_i}} \right) = \text{End}_{\widehat{B}_i} \left(\widehat{\sqrt{J_P B_i}} \right) = \widehat{B_{i+1}}. \quad \square$$

In the proposition, the maximum exists even though W may be infinite. On the other hand, if $\text{Sing}(A)$ is finite, then $n(A, J)$ can be read off from just finitely many values $n(A_P, J_P) = n(\widehat{A_P}, \widehat{J_P})$. To obtain some sort of general analogue of this fact, we discuss a convenient stratification of $\text{Sing}(A)$.

3. BOUNDS FOR STRATIFIED NORMALIZATION

Let again A be a reduced excellent ring. If $P \in \text{Sing}(A)$, we denote by

$$L_P = \bigcap_{P \supseteq \tilde{P} \in \text{Sing}(A)} \tilde{P}$$

the intersection of (minimal) primes contained in P . We stratify $\text{Sing}(A)$ according to the values of the function $P \mapsto L_P$. That is, if

$$\mathcal{L} = \{L_P \mid P \in \text{Sing}(A)\}$$

denotes the (finite) set of all possible values, then the strata are the sets

$$W_L = \{P \in \text{Sing}(A) \mid L_P = L\}, \quad L \in \mathcal{L}.$$

We write $\text{Strata}(A) = \{W_L \mid L \in \mathcal{L}\}$ for the (finite) set of all strata.

By construction, the singular locus is the disjoint union of the strata. For $W \in \text{Strata}(A)$, write L_W for the constant value of $P \mapsto L_P$ on W .

Lemma 3.1. *If $W \in \text{Strata}(A)$, then L_W is a test ideal at W .*

Proof. By construction, L_W is radical. If $J = \bigcap_{P \in \text{Sing}(A)} P$ is the vanishing ideal of $\text{Sing}(A)$, then $(L_W)_P = J_P \subseteq \sqrt{C_{A_P}}$ for all $P \in W$: the equality holds by construction of L_W , and the inclusion since J is a test ideal for A (see Remark 2.2). From the latter, we also get that L_W contains a non-zerodivisor since $J \subseteq L_W$. \square

Considering the ideal $J = L_W$ and proceeding as in the previous section, we obtain a chain of rings

$$A = A_0 \subsetneq \cdots \subsetneq A_{i-1} \subsetneq A_i \subsetneq \cdots \subsetneq A_n = A_{n+1} \subseteq \overline{A},$$

where, by Lemma 3.1 and Proposition 2.10, $(A_n)_P$ is normal and, hence, equal to $\overline{(A_n)_P} = \overline{A_P}$ for all $P \in W$.

By considering all strata and combining the resulting rings, we get \overline{A} :

Proposition 3.2 ([BDL⁺13]). *Suppose $\text{Sing}(A) = \bigcup_{i=1}^s W_i$. For $i = 1, \dots, s$, let an intermediate ring $A \subseteq A^{(i)} \subseteq \overline{A}$ be given such that $(A^{(i)})_P = \overline{A_P}$ for each $P \in W_i$. Then*

$$\sum_{i=1}^s A^{(i)} = \overline{A}.$$

We know from Remark 1.3 how the Serre conditions characterize reduced and normal rings, respectively. Since we will make explicit use of these conditions in what follows, we recall their definition:

Definition 3.3. Let R be a Noetherian ring, and let $i \geq 0$ be an integer.

We say that R satisfies *Serre's condition* (R_i) if for all $P \in \text{Spec}(R)$ with $\dim R_P = \text{height } P \leq i$, R_P is a regular local ring.

We say that R satisfies *Serre's condition* (S_i) if for all $P \in \text{Spec}(R)$,

$$\text{depth } R_P \geq \min\{i, \dim R_P\}.$$

Notation 3.4. If R is a ring, and I is an ideal of R or an R -module, then $\text{Ass}(I)$ denotes the set of associated ideals of I .

The following is well-known:

Lemma 3.5. *Let R be a local ring with maximal ideal \mathfrak{m}_R . Then*

$$\text{End}_R(\mathfrak{m}_R) = \begin{cases} R & \text{if } R \text{ is regular,} \\ \text{Hom}_R(\mathfrak{m}_R, R) & \text{otherwise.} \end{cases}$$

Proof. Assuming the contrary, there is a surjection $\mathfrak{m}_R \twoheadrightarrow R$ which splits as R is trivially projective. But then $\mathfrak{m}_R = Rx \oplus I$ for a non-zerodivisor x of R and an ideal $I \subset R$, and $xI \subseteq Rx \cap I = 0$ implies $I = 0$ and hence that R is regular. \square

Lemma 3.6. *Let $J \subseteq A$ be any radical ideal containing a non-zerodivisor and assume that A is (S_2) . Then:*

- (1) *The ring $\text{End}_A(J)$ is (S_2) .*
- (2) *If A_P is regular for all $P \in \text{Ass}(J)$ with $\text{height } P = 1$, then $\text{End}_A(J) = A$.*

Proof. (1) Fix an arbitrary $Q \in \text{Spec}(\text{End}_A(J))$ and let $P = Q \cap A$. Then, by [BH93, Prop. 1.2.10.(a)] and the proof of [AK70, III, Prop. 3.16],

$$\begin{aligned} \text{depth}(\text{End}_A(J)_Q) &\geq \text{grade}(P(\text{End}_A(J))_P, (\text{End}_A(J))_P) \\ &= \text{depth}_{A_P}(\text{End}_{A_P}(J_P)). \end{aligned}$$

On the other hand, since $A_P \subseteq \text{End}_A(J)_P$ is an integral ring extension,

$$\dim A_P = \dim(\text{End}_A(J)_P) \geq \dim(\text{End}_A(J)_Q).$$

Hence, by (S_2) for A , it is enough to show that

$$\text{depth}_{A_P}(\text{End}_{A_P}(J_P)) \geq \min\{2, \text{depth } A_P\}. \quad (3.1)$$

We distinguish two cases.

If $\dim(A_P/J_P) = 0$, then $J_P = \mathfrak{m}_P$. Hence (3.1) follows from Lemma 3.5 and [BH93, Exc. 1.4.19].

If $\dim(A_P/J_P) = 1$, then

$$\text{depth}(A_P/J_P) \geq 1 \quad (3.2)$$

by (S_1) for the reduced ring A_P/J_P . On the other hand, using again [BH93, Exc. 1.4.19], we get

$$\text{depth}_{A_P}(\text{End}_A(J_P)) \geq \min\{2, \text{depth } J_P\}.$$

To estimate $\text{depth } J_P$, we apply the Depth Lemma (see [BH93, Prop. 1.2.9]) to the short exact sequence

$$0 \rightarrow J_P \rightarrow A_P \rightarrow A_P/J_P \rightarrow 0$$

and obtain

$$\text{depth } J_P \geq \min\{\text{depth } A_P, \text{depth}(A_P/J_P) + 1\}.$$

Combined with (3.2), this proves (3.1).

(2) Consider the exact sequence

$$0 \rightarrow A \rightarrow \text{End}_A(J) \rightarrow B \rightarrow 0 \quad (3.3)$$

with cokernel B . We show that $\text{Ass}(B) = \emptyset$ and, hence, that $B = 0$. For this, let $P \in \text{Spec}(A)$.

If $P \notin V(J)$, then $J_P = A_P$, so $B_P = 0$. If $P \notin \text{Sing}(A)$, then A_P is normal and $J_P \subseteq A_P = \mathcal{C}_{A_P}$ is a test ideal for A_P , so again $B_P = 0$ by the Grauert and Remmert criterion. We conclude that if $P \notin V(J) \cap \text{Sing}(A)$, then $P \notin \text{Ass}(B)$.

If $P \in V(J) \cap \text{Sing}(A)$, then $\dim A_P \geq 2$ (otherwise, $\text{height } P = \dim A_P = 1$, so $P \in \text{Ass}(J)$; by assumption, A_P would be regular, a contradiction). By (3.1) and (S_2) for A , this implies

$$\text{depth}_{A_P}(\text{End}_{A_P}(J_P)) \geq 2.$$

Localizing (3.3) at P and applying the Depth Lemma, this gives

$$\text{depth}_{A_P}(B_P) \geq \min\{\text{depth}_{A_P}(\text{End}_{A_P}(J_P)), \text{depth}(A_P) - 1\} \geq 1$$

using once more (S_2) for A . We conclude again that $P \notin \text{Ass}(B)$. \square

Proposition 3.7. *Suppose A is equidimensional and satisfies (S_2) , and let J be a test ideal for A . If $J' \subseteq A$ is a radical ideal with $\text{Ass}(J') \subseteq \text{Ass}(J)$, then*

$$n(A, J') \leq n(A, J).$$

Proof. If $J' = A$, then $n(A, J') = 0 \leq n(A, J)$.

Now let $J' \subsetneq A$. Inductively, set $A_0 = B_0 = A$,

$$A_{i+1} = \text{End}_{A_i}(\sqrt{JA_i}), \text{ and } B_{i+1} = \text{End}_{B_i}(\sqrt{J'B_i}).$$

By assumption and Lemma 3.6.(1), all rings A_i, B_i satisfy (S_2) . Let $n = n(A, J)$. Then $A_n = A_{n+1} = \overline{A}$ by the Grauert and Remmert criterion, and we must show that $B_n = B_{n+1}$.

We use Lemma 3.6.(2). Since A is equidimensional and $A \subset B_n$ is an integral extension, also B_n is equidimensional. Fix $Q \in \text{Ass}(\sqrt{J'B_n})$ with $\text{height } Q = 1$, and set $P = Q \cap A$. Then $J' \subseteq P$ and $\text{height } P = 1$ by equidimensionality. We conclude that $P \in \text{Ass}(J') \subseteq \text{Ass}(J)$ and, hence, that $J'_P = J_P$ since J and J' are radical. From the proof of Proposition 2.11, it follows that

$$(A_i)_P = (B_i)_P, \text{ and } \sqrt{J_P(A_i)_P} = \sqrt{J'_P(B_i)_P},$$

for $i = 0, \dots, n$. In particular, $(B_n)_P = (A_n)_P = \overline{A_P}$ and, hence, $(B_n)_Q$ are regular. This proves that the hypothesis of Lemma 3.6.(2) applied to $\sqrt{J'B_n}$ and B_n is satisfied. Hence, as desired,

$$B_{n+1} = \text{End}_{B_n}(\sqrt{J'B_n}) = B_n. \quad \square$$

Corollary 3.8. *Let J be the vanishing ideal of $\text{Sing}(A)$. Then*

$$n(A, J) \leq \max_{W \in \text{Strata}(A)} n(A, L_W).$$

If A is equidimensional and satisfies (S_2) , then equality holds.

Proof. Recall from the proof of Lemma 3.1 that $J_P = (L_W)_P$ for all $P \in W$. Hence, by Proposition 2.11,

$$\begin{aligned} n(A, J) &= \max_{P \in \text{Sing}(A)} n(A_P, J_P) \\ &= \max_{W \in \text{Strata}(A)} \max_{P \in W} n(A_P, (L_W)_P) \\ &\leq \max_{W \in \text{Strata}(A)} n(A, L_W). \end{aligned}$$

This gives the first statement of the corollary. The second statement follows from the first one and Proposition 3.7: If A satisfies (S_2) , then

$$n(A, L_W) \leq n(A, J). \quad \square$$

Remark 3.9. If A is equidimensional and satisfies (S_2) , then by Proposition 3.7

$$n(A, J) = \max_{L \in \min \mathcal{L}} n(A, L)$$

where $\min \mathcal{L}$ denotes the set of minimal elements of \mathcal{L} .

Remark 3.10. It would be interesting to know whether the assumptions of the second statement of the corollary can be weakened. On the other hand, in its present form, the statement applies already to interesting classes of examples such as local complete intersections.

4. PLANE CURVES

In this section, by a curve we mean a reduced excellent ring A of pure dimension one. In particular, A is Noetherian and normalization-finite, and it has a finite singular locus, which coincides with its non-normal locus by Remark 1.2.

Remark 4.1. If A is local and non-normal, then there is a unique test ideal J for A , namely $J = \mathfrak{m}_A$. In this case, we write $n(A) = n(A, \mathfrak{m}_A)$.

With this notation, Proposition 2.11 reduces to:

Corollary 4.2. *Let A be a non-normal curve. Then, if J is any test ideal for A , we have*

$$n(A, J) = \max_{P \in \text{Sing}(A)} n(\widehat{A}_P).$$

In particular, $n(A) := n(A, J)$ does not depend on the choice of J .

If A is regular, we write $n(A) = 0$.

Corollary 4.3. *Let A be a non-normal curve with $\text{Sing}(A) = \{P_1, \dots, P_r\}$. For each i , let a curve B_i and a prime $Q_i \in \text{Sing}(B_i)$ be given such that*

$$\widehat{A}_{P_i} \cong (\widehat{B_i})_{Q_i}.$$

Then

$$n(A) = \max_{i=1, \dots, r} n(B_i, Q_i).$$

Proof. We fix an index i and write $B = B_i$ and $Q = Q_i$. If $Q' \in \text{Sing}(B)$ is different from Q , then $Q_{Q'} = B_{Q'}$. Hence

$$n(B, Q) = \max_{Q' \in \text{Sing}(B)} n(B_{Q'}, Q_{Q'}) = n(B_Q, Q_Q) = n(\widehat{B_Q}),$$

and the claim follows by Corollary 4.2. □

In the following, A will be an *algebroid curve* over $K = \mathbb{C}$, that is, a reduced complete local Noetherian K -algebra A of dimension one. Then A is excellent by Remark 1.17.

Notation 4.4. We write \mathfrak{m}_A for the maximal ideal of A and

$$A' = \text{End}_A(\mathfrak{m}_A).$$

Then \mathfrak{m}_A is a test ideal for A by Remark 4.1 and $A \subset A' = \mathfrak{m}_A^{-1}$ by Lemma 3.5.

Remark 4.5. Recall that an algebroid curve A is Gorenstein if and only if every fractional ideal of A is reflexive or, equivalently, if $\dim_K(A'/A) = 1$ (see [HK71, p. 19, Bsp. (b)] and [Ber62, Satz 1]). It follows that each local ring of a plane curve singularity is Gorenstein (see [HK71, Satz 1.46.(d)]).

Let P_1, \dots, P_s be the associated primes of A , and let $A_i = A/P_i$, $i = 1, \dots, s$, be the branches of A . We have inclusions

$$A \hookrightarrow \prod_{i=1}^s A_i \hookrightarrow \prod_{i=1}^s \overline{A}_i \cong \overline{A},$$

where, by Corollary 1.19 and Remark 1.2, each \overline{A}_i is a complete discrete valuation ring, say with valuation $\nu_i: \overline{A}_i \rightarrow \mathbb{N} \cup \{\infty\}$ and uniformizing parameter t_i . Then $\overline{A}_i \cong K[[t_i]]$, and evaluating ν_i at a power series means to take its order.

We write the elements of \overline{A} as $a = (a_1, \dots, a_s)$, with all the $a_i \in \overline{A}_i$, and consider the *valuation map*

$$\nu: \overline{A} \rightarrow (\mathbb{N} \cup \{\infty\})^s, \quad a \mapsto (\nu_1(a_1), \dots, \nu_s(a_s)).$$

For monomials in $K[[t_1]] \times \dots \times K[[t_s]] \cong \overline{A}$, we use multi-index notation: If $\alpha \in \mathbb{N}^s$, we write α_i for the i th component of α , and $t^\alpha = (t_1^{\alpha_1}, \dots, t_s^{\alpha_s})$.

Definition 4.6. The *semigroup* of a fractional ideal I of A is defined as

$$\Gamma_I = \{(\nu_1(a_1), \dots, \nu_s(a_s)) \mid a = (a_1, \dots, a_s) \in I, a_i \neq 0 \text{ for all } i\} \subseteq \mathbb{N}^s.$$

We call I *multigraded* with respect to t if it is invariant under the action

$$T \times \overline{A} \rightarrow \overline{A}, \quad (\lambda, t^\alpha) \mapsto (\lambda_1^{\alpha_1} t_1^{\alpha_1}, \dots, \lambda_s^{\alpha_s} t_s^{\alpha_s})$$

of the torus $T = \prod_{i=1}^s T_i$ where $T_i = K^*$.

Lemma 4.7. *If I is multigraded, then $\alpha \in \Gamma_I$ implies that $K^s \cdot t^\alpha \subseteq I$.*

Proof. The T_i -actions on \overline{A} commute. Therefore, for any $N \geq 0$, $(I + \mathfrak{m}_A^N)/\mathfrak{m}_A^N$ decomposes as a direct sum of eigenspaces $K^s t^\alpha$ indexed by a subset of $\prod_{i=1}^s X(T_i) \cong \mathbb{Z}^s$, where $X(-)$ denotes the character group. Let $at^\alpha \neq 0$ be an element of such an eigenspace, fix j and define $\lambda \in T$ by $\lambda_j^{\alpha_j} = 2$ and $\lambda_i = 1$ for $i \neq j$. Then $\lambda a - a = a_j t_j^{\alpha_j} e_j$ and hence $t_j^{\alpha_j} e_j \in I + \mathfrak{m}_A^N$ for all j and N . Since \overline{A} is complete, $\bigcap_N (I + \mathfrak{m}_A^N) = I$, so the claim follows. \square

Remark 4.8. Let I be a fractional ideal. Suppose that $t_i^{\alpha_i} e_i \in I$ for all i . Then $K^s \cdot t^\alpha \subseteq I$.

Remark 4.9 (Properties of the Conductor). Recall that the conductor \mathcal{C}_A is the largest ideal of A which is also an ideal of $\overline{A} \cong \prod_{i=1}^s \overline{A}_i$. It is, hence, generated by a monomial, say $\mathcal{C}_A = \langle t^\gamma \rangle = \langle t_1^{\gamma_1} \rangle \times \dots \times \langle t_s^{\gamma_s} \rangle$, where for each i ,

$$\gamma_i = \min \{\alpha_i \mid \alpha + \mathbb{N}^s \subseteq \Gamma_A\}.$$

In particular, \mathcal{C}_A is multigraded, and it follows from Lemma 4.7 that if $\alpha \in \Gamma_{\mathcal{C}_A}$, then $K^s \cdot t^\alpha \subseteq \mathcal{C}_A$.

Notation 4.10. Set $\tau = \gamma - \mathbf{1}$, and for any $\alpha \in \mathbb{Z}^s$, write

$$\begin{aligned} \Delta(\alpha) &= \bigcup_{i=1}^s \Delta_i(\alpha), \text{ where} \\ \Delta_i(\alpha) &= \{\beta \in \mathbb{N}^s \mid \alpha_i = \beta_i \text{ and } \alpha_j < \beta_j \text{ if } j \neq i\}. \end{aligned}$$

Note that for $s = 1$, we have $\Delta(\alpha) = \{\alpha\}$.

The following theorem generalizes a result of Kunz [Kun70] from irreducible to reducible algebroid curves:

Theorem 4.11 ([DdlM88]). *The algebroid curve A is Gorenstein if and only if for all $\alpha \in \mathbb{Z}^s$, the following symmetry condition is satisfied:*

$$\alpha \in \Gamma_A \Leftrightarrow \Delta(\tau - \alpha) \cap \Gamma_A = \emptyset.$$

Lemma 4.12. *Let A be any algebroid curve. Then:*

- (1) $\Delta(\tau) \cap \Gamma_A = \emptyset$.
- (2) $\Gamma_A \subset \{0\} \cup \Gamma_{\overline{\mathfrak{m}_A}}$.
- (3) *Let A be Gorenstein with s branches. If $s = 2$, then $\tau \in \Gamma_A$. If $s \geq 3$, then $\tau + \bigcup_{i=1}^s \mathbb{N}e_i \subseteq \Gamma_A$.*

Proof.

- (1) See [DdlM88, (1.9) Cor. (i)].
- (2) This follows from $\mathfrak{m}_A \subset \overline{\mathfrak{m}_A}$.
- (3) This follows from Theorem 4.11 using (2). □

Remark 4.13. Let A be Gorenstein with $s \geq 2$ branches. Taking $\alpha = \tau \in \Gamma_A$ in Theorem 4.11, we get $\Delta(\mathbf{0}) \cap \Gamma_A = \emptyset$.

Lemma 4.14. *If A is any algebroid curve, and $I \subseteq J$ are fractional ideals of A , then $I = J$ if and only if $\Gamma_I = \Gamma_J$.*

Proof. There is only one direction to show: Suppose that $\Gamma_I = \Gamma_J$. Let $g \in I$ be a non-zero-divisor of A . Since $I = J$ if and only if $(t^\delta/g)I = (t^\delta/g)J$, we may assume that $g = t^\delta$. Now let $b \in J$. By adding a K -multiple of t^δ to b , we may assume that $\nu(b) \in \Gamma_J = \Gamma_I$. Hence, there is an $a \in I$ with $\nu(a) = \nu(b)$. If $b \in \mathcal{C}_A$, then $b \in I$, and we are done. Otherwise, there is some j with $\nu(b_j) < \delta_j$. Choose a scalar $c \in K$ with $\nu(b_j - ca_j) > \nu(b_j)$. Setting $b^{(1)} = b - (c, \dots, c) \cdot a$, we have $\nu(b^{(1)}) > \nu(b)$ with respect to the natural partial ordering on \mathbb{N}^s . Continuing in this way, after finitely many steps, we arrive at an element $b^{(m)} \in \mathcal{C}_A$, and conclude that $b \in I$. □

Recall that we write $A' = \text{End}_A(\mathfrak{m}_A)$, and that we think of this as a fractional ideal of A as in Lemma 1.12.

Lemma 4.15. $\Gamma_{A'} \subseteq \bigcap_{m \in \mathfrak{m}_A} (\Gamma_A - \nu(m))$.

Proof. If $a' \in A'$ and $m \in \mathfrak{m}_A$, then $a'm \in \mathfrak{m}_A \subseteq A$. □

Lemma 4.16. $\overline{A} \cdot t^\tau \subseteq A'$.

Proof. We have

$$(\overline{A} \cdot t^\tau) \cdot \mathfrak{m}_A \subseteq (\overline{A} \cdot t^{\gamma-1}) \cdot (\overline{A} \cdot t^1) = \overline{A} \cdot t^\gamma = \mathcal{C}_A \subseteq A,$$

so the result follows from Lemma 3.5. □

In the following, $A = K[[x, y]] = K[[X, Y]]/\langle f \rangle$ will be a plane algebroid curve with a singularity of type ADE .

Proposition 4.17. *If A is of type A_n then A' is of type A_{n-2} . In particular, $n(A) = \lfloor \frac{n+1}{2} \rfloor$.*

Proof. We may assume that $f = X^2 - Y^{n+1}$.

- (1) Suppose $n = 2k$ is even. Then f is irreducible, $A \rightarrow K[[t]] = \overline{A}$, $x \mapsto t^{n+1}$, $y \mapsto t^2$ is the normalization, and $\gamma = 2k$. In accordance with Lemma 4.12.(1), $t^\tau \notin A$. On the other hand, by Lemma 4.16, $t^\tau \in A'$, and by Lemma 4.15, $\Gamma_{A'} \subseteq \Gamma_A \cup \{\tau\} = \Gamma_{A+K \cdot t^\tau}$. Hence, by Lemma 4.14,

$$A' = A + K \cdot t^\tau = K[[x', y]] \cong K[[X, Y]] / \langle X^2 - Y^{n-1} \rangle,$$

where $x' = t^\tau$. See Figure 1.

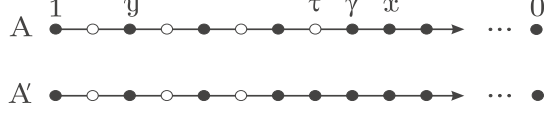


FIGURE 1. Normalization steps for a singularity of type A_8 .

- (2) Suppose $n = 2k - 1$ is odd. Then there are two branches,

$$f = (X - Y^k) \cdot (X + Y^k),$$

and $A \rightarrow K[[t_1]] \times K[[t_2]] = \overline{A}$, $x \mapsto (t_1^k, -t_2^k)$, $y \mapsto (t_1, t_2)$ is the normalization.

The properties of the conductor discussed in Remark 4.9 allow us to determine γ : First, $x = (t_1^k, -t_2^k) \in \mathcal{C}_A = \langle t^\gamma \rangle$ since, for all $i, j \geq 0$,

$$(2t_1^{k+i}, -2t_2^{k+j}) = y^i (x + y^k) + y^j (x - y^k) \in A;$$

then $\gamma = (k, k)$ since \mathcal{C}_A is multigraded and $(t_1^{k-1}, 0) \notin A$ and $(0, t_2^{k-1}) \notin A$.

Considering the powers $y^j \in A$, $j = 0, \dots, k-1$, we deduce from Theorem 4.11 that Γ_A is as shown in Figure 2 (cf. Lemma 4.12 and Remark 4.13 for $\tau = \nu(y^{k-1})$ and $\mathbf{0} = \nu(y^0)$, respectively).

Applying Lemmas 4.16, 4.15, and 4.14 as in part (1) of the proof, we get $\mathcal{C}_{A'} = \langle t^{\gamma'} \rangle$, where $\gamma' = \tau = (k-1, k-1)$, and

$$A' = A + \overline{A} \cdot t^\tau.$$

See again Figure 2. Setting $x' = (t_1^{k-1}, -t_2^{k-1})$, we have $x' \in \mathcal{C}_{A'} \setminus A$ by Remark 4.9. As $\dim_K(A'/A) = 1$ by Remark 4.5, it follows that

$$A' = A + K \cdot x' = K[[x', y]] \cong K[[X, Y]] / \langle X^2 - Y^{n-1} \rangle.$$

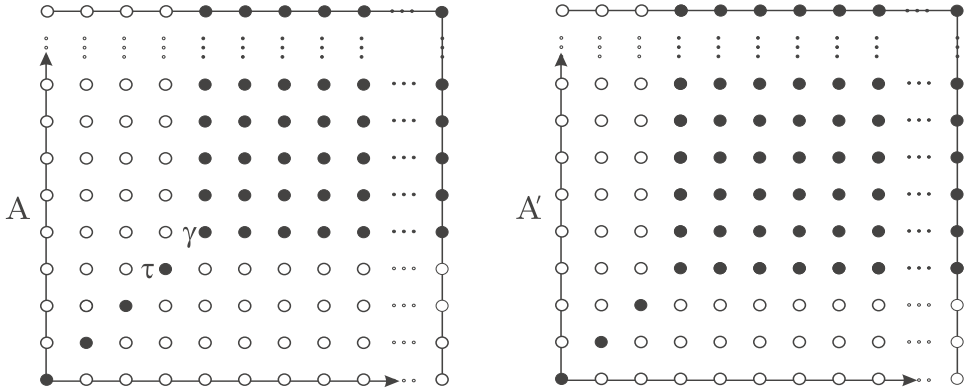


FIGURE 2. Normalization steps for a singularity of type A_7 .

□

Proposition 4.18. *Suppose A is of type D_n , where $n \geq 4$. Then A' is a simple, non-Gorenstein space curve singularity of type $A_{n-3} \vee L$, that is, a transversal union of an A_{n-3} -singularity and a line L as in [FK99, Tab. 2a]. Moreover, A'' is the disjoint union of an A_{n-5} -singularity and a line for $n \geq 6$, and it is smooth for $n = 4, 5$. In particular, $n(A) = \lfloor \frac{n}{2} \rfloor$.*

Proof. We may assume that $f = X(Y^2 - X^{n-2})$.

(1) Suppose $n = 2k + 3 \geq 5$ is odd. Then $A \rightarrow K[[t_1]] \times K[[t_2]] = \bar{A}$, $x \mapsto (0, t_2^2)$, $y \mapsto (t_1, t_2^{2k+1})$ is the normalization.

(A) Considering the elements $x^j y^i = (0, t_2^{i(2k+1)+2j}) \in A$ for $i = 0, 1$ and $j \geq 1$, and $y^i(y^2 - x^{2k+1}) = (t_1^{2+i}, 0) \in A$ for $i \geq 0$, we deduce from Remarks 4.8 and 4.9 that $(t_1^2, t_2^{2k+2}) \in \mathcal{C}_A = \langle t^\gamma \rangle$. Then $\gamma = (2, 2k+2)$ by Remark 4.9 since $(t_1, 0) \notin A$ and $(0, t_2^{2k+1}) \notin A$. Hence, $\tau = (1, 2k+1) = \nu(y)$ and $\Gamma_A \cap (\tau + \mathbb{N}^2)$ is of the form shown in Figure 3 by Lemma 4.12.(1).

Next, for $i \geq 1$ and $j = 1, \dots, k$, considering the elements $x^j + y^i = (t_1^i, t_2^{2j} + t_2^{i(n-2)}) \in A$, we get $(i, 2j) \in \Gamma_A$. Since $y \equiv (t_1, 0) \pmod{\bar{A} \cdot (0, t_2^{2k+1})}$, we conclude from Remark 4.13 that Γ_A is as depicted in Figure 3.

(A') To determine A' , we argue as in part (2) of the proof of the previous Proposition 4.17. This gives $\mathcal{C}_{A'} = \langle t^{\gamma'} \rangle$, where $\gamma' = (1, 2k)$, and

$$A' = A + K \cdot z = K[[x, y, z]] \cong K[[X, Y, Z]]/I',$$

where $z = (t_1, -t_2^{2k+1})$ and

$$I' = \langle X, Y - Z \rangle \cap \langle Y + Z, Z^2 - X^{n-2} \rangle.$$

Indeed, for the latter, note that x, y, z satisfy the relations given by the generators of I' , and that I' is a radical ideal with the right number of components of the right codimension. The coordinate change $(X, Y, Z) \mapsto (X, Y + Z, -Y + Z)$ turns I' into the ideal of maximal minors of the matrix

$$\begin{pmatrix} Z & Y & X^{n-3} \\ 0 & X & Y \end{pmatrix}.$$

Hence, A' is of the claimed type. Note that $(t_1, 0) = (y + z)/2 \in A'$ and $(0, t_2^{2j}) \in A'$, $j = 1, \dots, k-1$, and hence $\mathfrak{m}_{A'}$ is multigraded by Remark 4.8.

(A'') By Lemma 4.16, $\bar{A} \cdot t^{\gamma'} \subseteq A''$. In particular, $(1, 0) \in A''$. Applying again Remark 4.8, we see that $\sum_{j=1}^{k-1} K^2 \cdot (1, t_2^{2j}) \subseteq A''$. Furthermore, by the very definition of $A'' = \text{End}_{A'}(\mathfrak{m}_{A'}) \subset \bar{A}$, and since $\mathfrak{m}_{A'}$ is multigraded, $\sum_{i=0}^{\infty} K^2 \cdot (t_1^i, 1) \subseteq A''$. It follows that A'' is multigraded, with $\mathcal{C}_{A''} = \langle t^{\gamma''} \rangle$, where $\gamma'' = (0, 2k-2)$, and with $\Gamma_{A''}$ as in Figure 3. We conclude that A'' admits a product decomposition

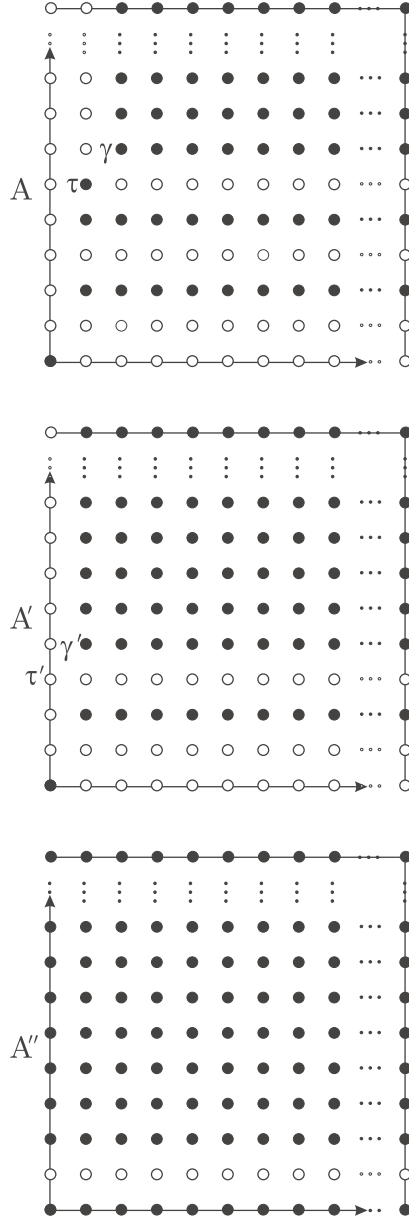
$$A'' = K[[t_1]] \times K[[x', y']],$$

where $x' = t_2^2$ and $y' = t_2^{2k-1}$. Moreover,

$$K[[x', y']] \cong K[[X, Y]]/\langle Y^2 - X^{2k-1} \rangle$$

is of type A_{n-5} for $n \geq 7$, and it is smooth for $n = 5$.

(2) Suppose $n = 2k + 2$ is even. Then $f = X(Y^2 - X^{n-2}) = X(Y - X^k)(Y + X^k)$ and $A \rightarrow K[[t_1]] \times K[[t_2]] \times K[[t_3]] = \bar{A}$, $x \mapsto (0, t_2, t_3)$, $y \mapsto (t_1, t_2^k, -t_3^k)$ is the normalization.

FIGURE 3. Normalization steps for a singularity of type D_7 .

(A) For $j = 1, \dots, k-1$ and $i \geq 1$, we have $x^j + y^i = (t_1^i, t_2^j + t_2^{ki}, t_3^j + (-1)^i t_3^{ki})$. Since $y \equiv (t_1, 0, 0) \pmod{\bar{A} \cdot (0, t_2^k, t_3^k)}$, it follows that

$$\{\alpha \in \Gamma_A \mid \alpha_2 < k, \alpha_3 < k\} = \{\mathbf{0}\} \cup \bigcup_{i \geq 1, j=1, \dots, k-1} (i, j, j).$$

Next, set

$$\begin{aligned} a &= y - x^k = (t_1, 0, -2t_3^k), \\ b &= y + x^k = (t_1, 2t_2^k, 0). \end{aligned}$$

Then, for $j \geq 0$, we have

$$\begin{aligned} y^2 - x^{2k} &= (t_1^2, 0, 0), \\ x^{j+1} \cdot a &= (0, 0, -2t_3^{k+j+1}), \\ x^{j+1} \cdot b &= (0, 2t_2^{k+j+1}, 0). \end{aligned}$$

Hence, $(t_1^2, t_2^{k+1}, t_3^{k+1}) \in \mathcal{C}_A = \langle t^\gamma \rangle$ by Remarks 4.8 and 4.9. An easy argument using parts (2) and (3) of Lemma 4.12 shows that $\gamma = (2, k+1, k+1)$. We conclude from Theorem 4.11 that Γ_A is of the form shown in Figure 4.

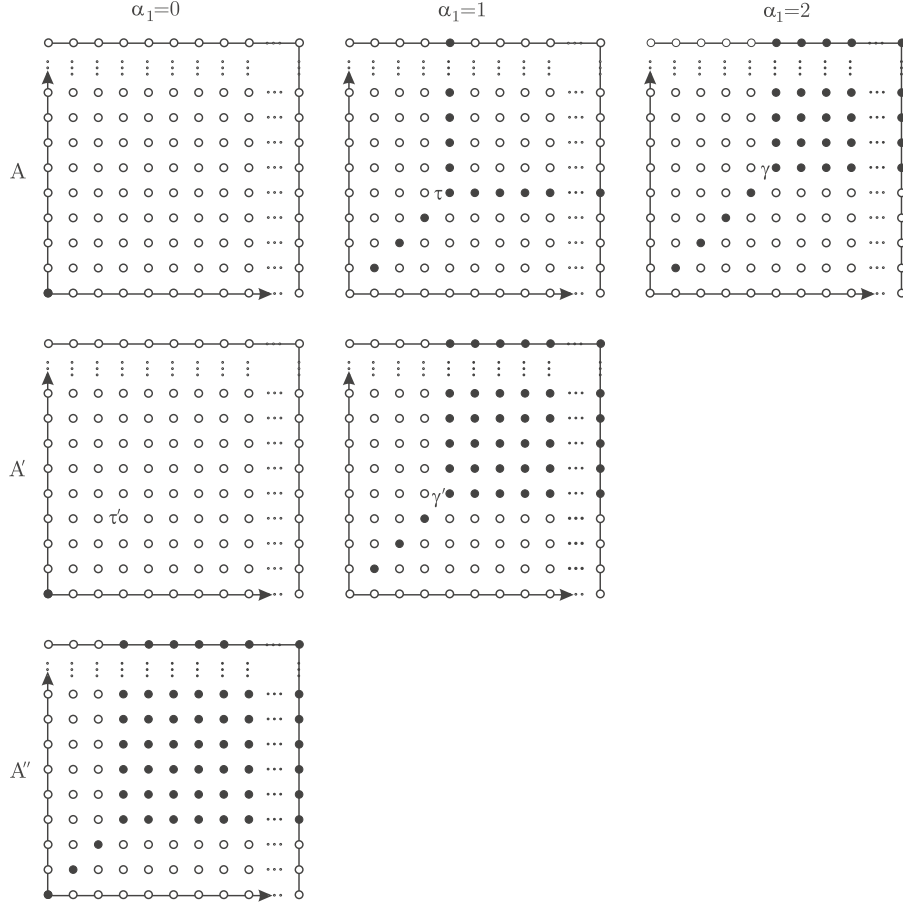


FIGURE 4. Normalization steps for a singularity of type D_{10} .

(A') Determining A' as before, we get $\mathcal{C}_{A'} = \langle t^{\gamma'} \rangle$, where $\gamma' = \tau = (1, k, k)$, and

$$A' = A + K \cdot z = K[[x, y, z]] \cong K[[X, Y, Z]]/I',$$

where $z = (0, t_2^k, -t_3^k)$ and

$$I' = \langle X, Z \rangle \cap \langle Y - Z, Z + X^k \rangle \cap \langle Y - Z, Z - Y^k \rangle.$$

Indeed, x, y, z satisfy the relations given by the generators of I' , and I' is a radical ideal with the right number of components of the right codimension. The coordinate change $(X, Y, Z) \mapsto (X, Y - Z, Y)$ then leads to the claimed ideal of minors (see part (1) of the proof).

(A'') Multiplying $(1, 0, 0)$ with the generators x, y, z of $\mathfrak{m}_{A'}$, we get either zero or $(1, 0, 0) \cdot y = (t_1, 0, 0) = y - z \in \mathfrak{m}_{A'}$. Hence, $(1, 0, 0) \in A'' = \text{End}_{A'}(\mathfrak{m}_{A'}) \subset \overline{A}$. It follows from Lemmas 4.16 and 4.14 that

$$A'' = A' + \overline{A} \cdot (1, 0, 0) + \overline{A} \cdot t^{\gamma''},$$

where $\gamma'' = \tau' = (0, k-1, k-1)$ (for the inclusion from left to right, use again that $A'' = \text{End}_{A'}(\mathfrak{m}_{A'})$). Using once more Lemma 4.14, and that $(1, 0, 0) \in A''$, we see that A'' admits a product decomposition

$$A'' = K[[t_1]] \times K[[x', y']],$$

where $x' = (t_2, t_3)$ and $y' = (t_2^{k-1}, -t_3^{k-1})$. Moreover,

$$K[[x', y']] \cong K[[X, Y]] / \langle Y^2 - X^{2k-2} \rangle$$

is of type A_{n-5} for $n \geq 6$, and it is smooth for $n = 4$. □

Proposition 4.19. *Suppose A is of type E_n , where $n = 6, 7, 8$. Then A' is a simple, non-Gorenstein space curve singularity of type $E_n(1)$, see [FK99, Tab. 2a]. Moreover, A'' is of type A_{n-6} for $n = 7, 8$, and it is smooth for $n = 6$. In particular, $n(A) = \lfloor \frac{n-1}{2} \rfloor$.*

Proof. (1) Let $n = 6$. We may assume $f = X^3 - Y^4$. Then $A \rightarrow K[[t]] = \overline{A}$, $x \mapsto t^4, y \mapsto t^3$ is the normalization, and we have

$$A = \langle 1, t^3, t^4 \rangle_K \oplus K[t] \cdot t^6.$$

Arguing as in the previous proofs, we get

$$A' = A + K \cdot z,$$

where $z = \tau = t^5$. Then

$$A' = K[[x, y, z]] \cong K[[X, Y, Z]] / I',$$

where I' is the ideal of minors of the matrix

$$\begin{pmatrix} Y & Z & X^2 \\ X & Y & Z \end{pmatrix}.$$

This means that A' is of type $E_6(1)$. Since $A' = K \cdot 1 \oplus K[t] \cdot t^3 = K \cdot 1 \oplus \mathfrak{m}_{A'}$, we obtain that $A'' = \text{End}_{A'}(\mathfrak{m}_{A'}) = K[t]$. See Figure 5.

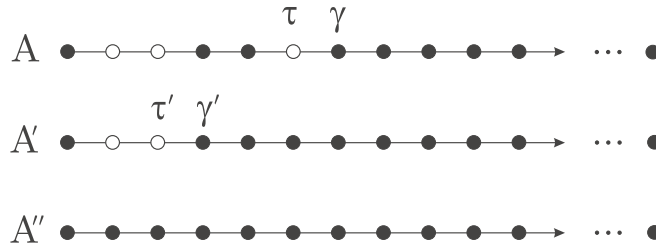


FIGURE 5. Normalization steps for a singularity of type E_6 .

(2) Let $n = 7$. We may assume $f = X(X^2 - Y^3)$. Then $A \rightarrow K[[t_1, t_2]] = \overline{A}$, $x \mapsto (0, t_2^3), y \mapsto (t_1, t_2^2)$ is the normalization.

Since $x^i y^j = (0, t_2^{3i+2j}) \in A$ for $i \geq 1$ and $j \geq 0$, and $(y^3 - x^2)y^i = (t_1^{3+i}, 0) \in A$ for $i \geq 0$, we have $(t_1^3, t_2^5) \in \mathcal{C}_A = \langle t^\gamma \rangle$ by Remarks 4.8 and 4.9. Then $\gamma = (3, 5)$ by Remark 4.9 since $(t_1^2, 0) \notin A$ and $(0, t_2^4) \notin A$.

where

$$I' = \langle XZ, Y^2Z - Z^2, Y^3 - X^2 - YZ \rangle.$$

After the coordinate change $(X, Y, Z) \mapsto (Y, -X, Z + X^2)$, this becomes the ideal of maximal minors of the matrix

$$\begin{pmatrix} Z + X^2 & Y & X \\ 0 & Z & Y \end{pmatrix}.$$

It follows that A' is of type $E_7(1)$. Since $A' = \langle 1, y \rangle_K \oplus \bar{A} \cdot (t_1^2, t_2^3)$, we must have $A'' = K \cdot 1 \oplus \bar{A} \cdot (t_1, t_2)$, so that A'' is of type A_1 .

- (3) Let $n = 8$. We may assume $f = X^3 - Y^5$. Then $A \rightarrow K[[t]] = \bar{A}$, $x \mapsto t^5$, $y \mapsto t^3$ is the normalization, and we have

$$A = \langle 1, t^3, t^5, t^6 \rangle_K \oplus K[t] \cdot t^8.$$

As before,

$$A' = A + K \cdot z,$$

where $z = \tau = t^7$. Then

$$A' = K[[x, y, z]] \cong K[[X, Y, Z]]/I',$$

where I' is the ideal of maximal minors of the matrix

$$\begin{pmatrix} X & Z & Y^3 \\ Y & X & Z \end{pmatrix}.$$

This means that A' is of type $E_8(1)$. Since $A' = \langle 1, t^3 \rangle_K \oplus K[t] \cdot t^5$, it follows that $A'' = K \cdot 1 \oplus K[t] \cdot t^2$. See Figure 7.

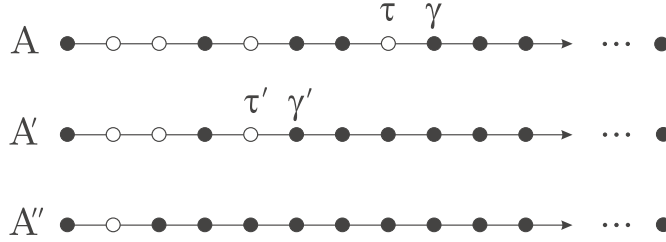


FIGURE 7. Normalization steps for a singularity of type E_8 .

□

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