

IRREGULARITY OF HYPERGEOMETRIC SYSTEMS VIA SLOPES ALONG COORDINATE SUBSPACES

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ABSTRACT. We study the irregularity sheaves attached to the A -hypergeometric D-module $M_A(\beta)$ introduced by Gel'fand et al., where $A \in \mathbb{Z}^{d \times n}$ is pointed of full rank and $\beta \in \mathbb{C}^d$. More precisely, we investigate the slopes of this module along coordinate subspaces.

In the process we describe the associated graded ring to a positive semi-group ring for a filtration defined by an arbitrary weight vector L on torus equivariant generators. To this end we introduce the (A, L) -umbrella, a simplicial complex determined by A and L , and identify its facets with the components of the associated graded ring.

We then establish a correspondence between the full (A, L) -umbrella and the components of the L -characteristic variety of $M_A(\beta)$. We compute in combinatorial terms the multiplicities of these components in the L -characteristic cycle of the associated Euler–Koszul complex, identifying them with certain intersection multiplicities.

We deduce from this that slopes of $M_A(\beta)$ are combinatorial, independent of β , and in one-to-one correspondence with jumps of the (A, L) -umbrella. This confirms a conjecture of Sturmfels and gives a converse of a theorem of Hotta: $M_A(\beta)$ is regular if and only if A defines a projective variety.

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1. INTRODUCTION AND OVERVIEW

The solutions f of an ordinary linear differential equation $P \bullet f(x) = 0$ where

$$P = p_m(x)\partial_x^m + p_{m-1}(x)\partial_x^{m-1} + \cdots + p_1(x)\partial_x + p_0(x)$$

form a \mathbb{C} -vector bundle of dimension m away from the zero locus of p_m . At $p_m(x_0) = 0$, two types of singularities may occur: at a regular singular point x_0 , the (multivalued) solutions have polynomial growth for $x \rightarrow x_0$ while in all other cases x_0 is called irregular. By Fuchs’ Theorem, P is regular at the origin if and only if the Newton polygon $N(P)$ of P is a quadrant [Inc44]. The slopes (or critical indices) of P are just the slopes of $N(P)$; they represent a refined notion of irregularity [Lau85] and indicate the growth of solutions near the critical point. Regularity at x_0 is equivalent to equality of formal and convergent solutions at x_0 .

The concept of regularity in higher dimension is considerably more involved. Denote by \mathcal{O}_X the structure sheaf of the complex manifold X and by $\widehat{\mathcal{O}_{X|Y}}$ the completion of \mathcal{O}_X along Y . For any coherent \mathcal{D}_X -module \mathcal{M} , Z. Mebkhout introduced the irregularity complex $\mathcal{I}rr_Y(\mathcal{M}) = \mathbf{R}\mathcal{H}om_{\mathcal{D}_X}(\mathcal{M}, \widehat{\mathcal{O}_{X|Y}}/\mathcal{O}_{X|Y})$, $\mathcal{O}_{X|Y}$

being the restriction of \mathcal{O}_X to Y [Meb89]. This intrinsically analytic notion is inspired by earlier work of B. Malgrange on regularity in the univariate case [Mal74], but quite difficult to use.

On the other hand, Y. Laurent [Lau87] gave a generalization for the concept of a slope to the multivariate case based on more algebraic methods. With X, Y as above, let V be the Kashiwara–Malgrange filtration along Y , let F be the order filtration, and put $L = pF + qV$. Then the rational number $p/q > 0$ is a slope of \mathcal{M} along Y if the L -characteristic variety $\text{Ch}^L(\mathcal{M}) = \text{supp}(\text{gr}^L(\mathcal{M})) \subseteq T^*X$ jumps (is not locally constant) at p/q .

The theorem of Fuchs generalizes to the multivariate case: by the analytic-algebraic comparison theorem for slopes [LM99, Thm. 2.4.2], \mathcal{M} has no slopes along Y precisely if $\mathcal{I}rr_Y(\mathcal{M})$ is exact. In fact, the slopes agree with the jumps of the Gevrey filtration on the irregularity complex which provide a measure of growth for the solutions of \mathcal{M} near Y .

All slopes of \mathcal{M} along Y are rational and there are only a finite number of them, [Lau87]. A. Assi, F. Castro, and M. Granger [ACJG96] developed a Gröbner basis algorithm to compute slopes of algebraic D -modules using the algebraic counterparts of F, V , and L on the Weyl algebra D . In the process they proved a comparison theorem: slopes of modules over the Weyl algebra can be computed without leaving the algebraic category, where rationality and finiteness follow from the existence of the Gröbner fan, [ACJG00]. Explicit formulæ for slopes of D -modules are very rare. The purpose of this article is to describe the slopes of A -hypergeometric D -modules.

I.M. Gel'fand, M.I. Graev, M.M. Kapranov and A.V. Zelevinskiĭ [GGZ87, GZK89] defined a class of D -modules that includes as particular cases the differential systems satisfied by the classical hypergeometric functions of Gauß, Appell, and others. These A -hypergeometric (or GKZ) systems are special cases of the equivariant D -modules of R. Hotta and M. Kashiwara [HK84, Hot98]. A $d \times n$ integer matrix A defines an action of a d -torus $\mathbb{T} := (\mathbb{C}^*)^d$ on $T_0^*\mathbb{C}^n = \mathbb{C}^n$. Our general hypothesis is that $\mathbb{N}A$ is a positive semigroup with $\mathbb{Z}A = \mathbb{Z}^d$. The closure of the orbit through $(1, \dots, 1)$ is defined by the toric ideal $I_A \subseteq \mathbb{C}[\partial] =: R$ where $\partial := \partial_1, \dots, \partial_n$ and $S_A := R/I_A = \mathbb{C}[\mathbb{N}A]$ is the associated semigroup ring. The Euler vector fields $E = (E_1, \dots, E_d)$ are the push-forwards to this orbit of the Lie algebra generators $t_1\partial_1, \dots, t_d\partial_d$ of \mathbb{T} . The A -hypergeometric system $M_A(\beta)$, depending on the Lie algebra character $\beta \in \mathbb{C}^d$, is the D -module defined by I_A and the Euler operators $E - \beta$. It arises in various situations in algebraic geometry such as in the theory of toric residues [CDS01], the study of hyperplane arrangements [OT01], and in the Picard–Fuchs equations governing the variation of Hodge structures for Calabi–Yau toric hypersurfaces [CK99].

By a theorem of R. Hotta [Hot98], homogeneous A -hypergeometric systems are regular and hence have no slopes. In dimension one and in codimension one, slopes of $M_A(\beta)$ were studied by F. Castro and N. Takayama [CJT03] and M. Hartillo [HH03, HH05]. Cohen–Macaulayness of the toric rings in question makes these cases comparatively tractable. In our general situation, a key tool is a (generalization of a) computation by A. Adolphson [Ado94] identifying candidate components of the F -characteristic variety with the set of faces not containing 0 of the convex hull Δ_A^F of 0 and the columns $\mathbf{a}_1, \dots, \mathbf{a}_n$ of A . In Section 2, L is the filtration on R defined by

an arbitrary weight vector on $\partial_1, \dots, \partial_n$. We introduce in Definition 2.7 the (A, L) -polyhedron Δ_A^L as the convex hull in projective space of 0 and all $\mathbf{a}_i^L := \mathbf{a}_i / \deg^L(\partial_i)$ for $i = 1, \dots, n$. Its faces not containing zero form the (A, L) -umbrella Φ_A^L , a combinatorial object independent of β that encodes all information regarding the L -characteristic variety of $M_A(\beta)$.

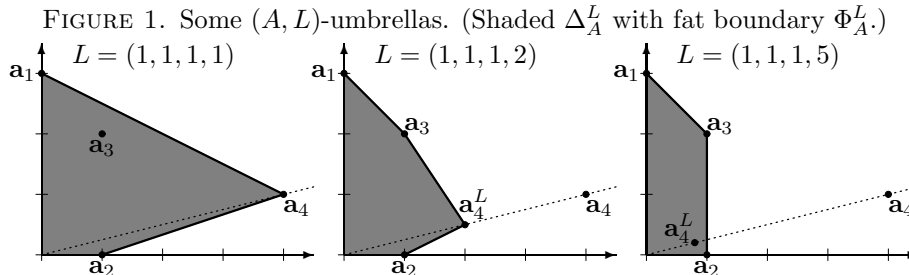


Figure 1 shows the (A, L) -umbrella in the example $A = \begin{pmatrix} 0 & 1 & 1 & 4 \\ 3 & 0 & 2 & 1 \end{pmatrix}$ for the family of filtrations $L = (1, 1, 1, t)$ for three parameters resulting in combinatorially different umbrellas. In Theorem 2.15, we identify Φ_A^L with both the \mathbb{Z}^d -graded spectrum of the L -graded toric ring $S_A^L := \text{gr}^L(S_A)$ and the collection of torus orbits in $\text{Spec}(S_A^L)$. In Theorem 2.21, we characterize Cohen–Macaulayness of S_A^L by the corresponding property of its subrings generated by facet cones.

In Section 3, we extend L to the Weyl algebra $D = \mathbb{C}[x]\langle \partial \rangle$ in the variables $x := x_1, \dots, x_n$ in such a way that E is L -homogeneous and $W := \text{gr}^L(D)$ is a polynomial ring. The family of filtrations in Figure 1 may be viewed as the restriction of $L = F + qV$ to R for the V -filtration along x_4 for $q = 0, 1, 4$. In Proposition 3.8 we show that all components of the L -characteristic variety of $M_A(\beta)$ correspond to faces $\tau \in \Phi_A^L$: each is the closure \bar{C}_A^τ of the conormal space C_A^τ to a torus orbit in $\text{Spec}(S_A^L)$. The facet components correspond to orbits in the smooth, and hence Cohen–Macaulay, locus of S_A^L on which the L -symbols of the Euler vector fields form a regular sequence. From this we conclude in Proposition 3.10 that the facet components actually occur and their multiplicity is given by an index formula independent of the parameter β . This shows in particular that p/q is a slope of $M_A(\beta)$ at the origin whenever Φ_A^{pF+qV} jumps at p/q . We thus obtain a converse to Hotta’s theorem in Corollary 3.16: regular A -hypergeometric systems are homogeneous.

Since orbits to non-facets may be outside the Cohen–Macaulay locus of S_A^L , we consider in Section 4 the full Euler–Koszul complex $K_{A, \bullet}(S_A; \beta)$ from [MMW05]. In order to apply methods of homological algebra, we discuss the basics of good L -filtrations on R - and D -modules. Using the spectral sequence for the L -filtration on $K_{A, \bullet}(S_A; \beta)$, we identify its L -characteristic cycle with the intersection cycle between S_A^L and the L -graded Euler ideal $\text{gr}^L(\langle E \rangle)$. In Theorem 4.11, we use Serre’s Intersection Theorem [Ser65] to show that the L -characteristic variety of $K_{A, \bullet}(S_A; \beta)$ contains all candidate components. To show that this also holds for $M_A(\beta) = H_0(K_{A, \bullet}(S_A; \beta))$ we use in Theorem 4.16 results from [MMW05] and an induction argument on toric modules. In particular (Corollary 4.17), the components of the L -characteristic variety of $M_A(\beta)$ are in one-to-one correspondence

to the faces in the (A, L) -umbrella Φ_A^L . It follows (Corollary 4.18) that the slopes of $M_A(\beta)$ along coordinate subspaces at the origin correspond exactly to jump parameters p/q of Φ_A^{F+qV} , confirming and extending a conjecture by B. Sturmfels.

The holonomic rank $\text{rk}(M) := \dim_{\mathbb{C}(x)}(M \otimes_{\mathbb{C}[x]} \mathbb{C}(x))$ of a D -module M is the dimension of the \mathbb{C} -vector space of its solution space near a regular point. It is a classical result [GZK89, Ado94] that for generic β the rank of $M_A(\beta)$ equals the volume of the convex hull Δ_A^F of 0 and $\mathbf{a}_1, \dots, \mathbf{a}_n$ where the volume of the unit simplex is normalized to 1. The exceptional set $\mathcal{E}(A)$ of A is the set of rank-jumping parameters: $\beta \in \mathcal{E}(A)$ precisely if the rank of $M_A(\beta)$ exceeds the volume. By [MMW05], $\mathcal{E}(A)$ is a finite subspace arrangement. In Theorem 4.21 we give a general index/volume formula for the multiplicity $\mu_A^{L,\tau}$ of \bar{C}_A^τ , $\tau \in \Phi_A^L$, in the L -characteristic cycle of $K_{A,\bullet}(S_A; \beta)$. We show then that for non-rank-jumping parameter β , or if τ is a facet, the number $\mu_A^{L,\tau}$ equals the multiplicity $\mu_{A,0}^{L,\tau}(\beta)$ of \bar{C}_A^τ in the L -characteristic cycle of $M_A(\beta)$. From [MMW05] it is known that the rank $\mu_{A,0}^{F,\emptyset}(\beta)$ of $M_A(\beta)$ is upper semi-continuous in the parameter β . Theorem 4.28 generalizes a weaker statement to all L and $\tau \in \Phi_A^L$: the multiplicity $\mu_{A,0}^{L,\tau}(\beta)$, $\tau \in \Phi_A^L$, is always minimal at generic β . We conjecture that $\mu_{A,0}^{L,\tau}(\beta)$ is upper semi-continuous in β .

In the last section, we generalize results of the previous sections to a natural extension of $M_A(\beta)$ to the projective closure $(\mathbb{P}_{\mathbb{C}}^1)^n$.

2. FILTRATIONS ON THE TORIC RING

2.1. Torus action and toric ring.

Notation 2.1. By \mathbb{Q}_+ we mean the non-negative rational numbers and we include 0 in \mathbb{N} .

Let $A = (a_{i,j}) \in \mathbb{Z}^{d \times n}$ be an integer matrix of rank d whose columns $\mathbf{a}_1, \dots, \mathbf{a}_n \in \mathbb{Z}^d$ are nonzero. We assume that $\mathbb{N}A$ is a *positive* [BH93, §6.1] (or *pointed* [MMW05, §1]) semigroup with $\mathbb{Z}A = \mathbb{Z}^d$. We write $\tau \subseteq A$ if τ is a subset of the column set of A and consider it both as a submatrix of A and a subset of the set of column indices $\{1, \dots, n\}$. Then the *dimension* $\dim(\tau)$ is $\dim_{\mathbb{Q}}(\mathbb{Q}\tau) - 1$. For a vector or collection with index set $\{1, \dots, n\}$, a lower index τ denotes the sub-vector or sub-collection with indices in τ . We abbreviate $\bar{\tau} := \{1, \dots, n\} \setminus \tau$. For any set τ , its cardinality is denoted $|\tau|$.

We shall frequently denote by \bar{C} the Zariski closure of a set C .

For $\mathbf{u} \in \mathbb{Z}^n$ define \mathbf{u}_+ by $(\mathbf{u}_+)_j = \max(0, \mathbf{u}_j)$ and put $\mathbf{u}_- = \mathbf{u}_+ - \mathbf{u}$. For $\mathbf{u}, \mathbf{v} \in \mathbb{N}^n$ write $\min(\mathbf{u}, \mathbf{v})$ for the vector whose j -th entry is $\min(\mathbf{u}_j, \mathbf{v}_j)$. For any vector \mathbf{u} we mean by “ $\mathbf{u} > 0$ ” that \mathbf{u} is componentwise positive: $u_i > 0$ for all i .

The base space in this note is $X := \text{Spec}(\mathbb{C}[x]) = \mathbb{C}^n$ where $x := x_1, \dots, x_n$. Let $R := \mathbb{C}[\partial]$ be the polynomial ring in n variables $\partial := \partial_1, \dots, \partial_n$. Identifying ∂_i with the partial derivation $\partial/\partial x_i$, $\text{Spec}(R)$ becomes the conormal space T_0^*X of X at 0 . The d -torus $\mathbb{T} := (\mathbb{C}^*)^d = \text{Spec}(\mathbb{C}[t_1^{\pm 1}, \dots, t_d^{\pm 1}])$ with coordinates $t := t_1, \dots, t_d$ acts on $\text{Spec}(R)$

$$(2.1.1) \quad (t, \xi) \longmapsto t \cdot \xi := (t^{\mathbf{a}_1} \xi_1, \dots, t^{\mathbf{a}_n} \xi_n).$$

This induces a \mathbb{Z}^d -grading on R by $\deg(\partial_i) = \mathbf{a}_i$. For a \mathbb{Z}^d -graded R -module N , we denote by $\deg(N)$ the set of its \mathbb{Z}^d -degrees.

Definition 2.2. We denote the orbit $\mathbb{T} \cdot \xi$ through $\xi \in T_0^*X$ by $\text{Orb}(\xi)$. Let τ be a subset of columns of A . We define $\mathbf{1}_A^\tau \in \{0, 1\}^n$ by

$$(\mathbf{1}_A^\tau)_j := \begin{cases} 1 & \text{if } \mathbf{a}_j \in \tau, \\ 0 & \text{if } \mathbf{a}_j \notin \tau, \end{cases}$$

and denote by O_A^τ the orbit of $\mathbf{1}_A^\tau$.

The *toric ideal* $I_\tau \subseteq R_\tau := \mathbb{C}[\partial_\tau]$ of τ is the \mathbb{Z}^d -graded prime ideal generated by $\square_{\mathbf{u}} = \partial^{\mathbf{u}^+} - \partial^{\mathbf{u}^-}$ where $\mathbf{u} \in \mathbb{Z}^{|\tau|}$ such that $\tau \cdot \mathbf{u} = 0$. We set $I_A^\tau := RI_\tau + J_\tau$ where J_τ is the R -ideal generated by $\{\partial_i \mid i \notin \tau\}$. Based on the following lemma, the *semigroup ring* of τ is

$$S_\tau := \mathbb{C}[\mathbb{N}\tau] = \bigoplus_{\mathbf{a} \in \mathbb{N}\tau} \mathbb{C} \cdot t^{\mathbf{a}} = \sum_{\mathbf{u} \in \mathbb{N}^{|\tau|}} \mathbb{C} \cdot (\partial_\tau^{\mathbf{u}} \bmod I_\tau) \subseteq S_A,$$

The normalization of S_A is the Cohen–Macaulay ring $\tilde{S}_A = \bigoplus_{\mathbf{a} \in (\mathbb{Q}_+ A \cap \mathbb{Z}^d)} \mathbb{C} \cdot t^{\mathbf{a}}$.

Lemma 2.3. For $\tau \subseteq A$, $I_A^\tau = \{f \in R \mid f(O_A^\tau) = 0\}$ and $S_\tau = R_\tau/I_\tau = R/I_A^\tau$. \square

2.2. L -filtration on the toric ring. Let $L = (L_{\partial_1}, \dots, L_{\partial_n}) \in \mathbb{Q}^n$ be a *weight vector*. It induces an increasing filtration L of \mathbb{C} -vector spaces on R via $[\partial^{\mathbf{u}} \in L_k R] \Leftrightarrow [L \cdot \mathbf{u} \leq k]$. Note that L has a rational discrete index set. If $f \in L_k R \setminus L_{<k} R$ then $k =: \deg^L(f)$ is the L -degree of f . Let $\sigma^L : R \rightarrow \text{gr}^L(R)$ be the L -symbol map defined by $\sigma^L(f) = f \bmod L_{<k} R$ if $\deg^L(f) = k$. An element $f \in R$ is L -homogeneous if $f = \sum_{L \cdot \mathbf{u} = k} f_{\mathbf{u}} \partial^{\mathbf{u}}$ for some $k \in \mathbb{Q}$ where $f_{\mathbf{u}} \in \mathbb{C}$. By abuse of notation, we identify R and $\text{gr}^L(R)$ via the \mathbb{C} -linear isomorphism induced by $f \mapsto \sigma^L(f)$ for L -homogeneous f .

For any $\tau \subseteq A$, L induces a filtration on S_τ by

$$L_k S_\tau := \sum_{\deg^L(\partial_\tau^{\mathbf{u}}) \leq k} \mathbb{C} \cdot (\partial_\tau^{\mathbf{u}} \bmod I_\tau).$$

With I_τ^L denoting the \mathbb{Z}^d -graded ideal $\text{gr}^L(I_\tau)$, we abbreviate

$$S_\tau^L := \text{gr}^L(S_\tau) \cong R/I_\tau^L.$$

The following is a mild extension of [Stu96, Cor. 4.4], see also [Ado94, Lem. 4.11].

Lemma 2.4. One has the identity $I_A^L = R \langle \sigma^L(\square_{\mathbf{u}}) \mid \mathbf{u} \in \mathbb{Z}^n, A \cdot \mathbf{u} = 0 \rangle$.

Proof. Generators of I_A^L are the L -symbols of a reduced Gröbner basis of I_A and can be computed by Buchberger’s algorithm. But each S-pair and reduction step of the algorithm preserves the generators $\{\square_{\mathbf{u}} \mid \mathbf{u} \in \mathbb{Z}^n \cap \ker(A)\}$: If $\mathbf{u}' + \mathbf{u}_+ = \mathbf{v}' + \mathbf{v}_+$ where $\mathbf{u}, \mathbf{v} \in \mathbb{Z}^n \cap \ker(A)$ and $\mathbf{u}', \mathbf{v}' \in \mathbb{N}^n$ then

$$A(\mathbf{v}' + \mathbf{v}_-) = A(\mathbf{v}' + \mathbf{v}_+) = A(\mathbf{u}' + \mathbf{u}_+) = A(\mathbf{u}' + \mathbf{u}_-).$$

and

$$\partial^{\mathbf{u}'} \square_{\mathbf{u}} - \partial^{\mathbf{v}'} \square_{\mathbf{v}} = \partial^{\mathbf{v}'+\mathbf{v}_-} - \partial^{\mathbf{u}'+\mathbf{u}_-} = \partial^{\min(\mathbf{u}'+\mathbf{u}_-, \mathbf{v}'+\mathbf{v}_-)} \square_{\mathbf{v}'+\mathbf{v}_- - \mathbf{u}' - \mathbf{u}_-}.$$

Note that this argument is valid also for $L \not\geq 0$ where Buchberger’s algorithm with (de-)homogenization is used. \square

In this section we study the geometry of S_A^L in terms of A and L . Since the Gröbner fan of any R -ideal is defined over \mathbb{Q} , the study of real weights can be reduced to the case $L \in \mathbb{Q}^n$. See [MS05, Sec. 7.4] for a discussion in the case where $L > 0$ defines a term order.

2.3. The (A, L) -umbrella. We consider the embedding of the affine space $\mathbb{Q}^d \supseteq A$ into the rational projective d -space

$$\mathbb{P}_{\mathbb{Q}}^d = \mathbb{P}_{\mathbb{Q}}(\mathbb{Q}^d \times \mathbb{Q})$$

via the map $q \mapsto (q : 1)$. Denote $\infty := \mathbb{P}_{\mathbb{Q}}^d \setminus \mathbb{Q}^d$ the hyperplane at infinity.

In $\mathbb{P}_{\mathbb{Q}}^d$, any two distinct points $\mathbf{a}, \mathbf{b} \in \mathbb{P}_{\mathbb{Q}}^d$ are joined by two line segments. If H is a hyperplane in $\mathbb{P}_{\mathbb{Q}}^d$ containing neither \mathbf{a} nor \mathbf{b} then there is a unique line segment joining the points and not meeting H . This is exactly the convex hull of \mathbf{a} and \mathbf{b} in the affine space $\mathbb{P}_{\mathbb{Q}}^d \setminus H$.

Definition 2.5. Let $H \subseteq \mathbb{P}_{\mathbb{Q}}^d$ be a hyperplane and let $U_H := \mathbb{P}_{\mathbb{Q}}^d \setminus H$ be its complement. For $B \subseteq U_H$, the *convex hull* of B relative to H is the set $\text{conv}_H(B)$ defined as the convex hull of B in the affine space U_H .

Note that $\text{conv}_H(B)$, for varying H , changes exactly when H is moved through a point of B . Within \mathbb{Q}^d , elements of convex hulls are linear combinations with non-negative coefficients that add to unity. Convex hulls relative to $H \neq \infty$, with coordinates from $\mathbb{Q}^d \subseteq \mathbb{P}_{\mathbb{Q}}^d$, obey slightly different rules. Let $h \in \text{Hom}_{\mathbb{Q}}(\mathbb{Q}^d, \mathbb{Q})$ be a linear form such that H is the closure of $h^{-1}(0)$ in $\mathbb{P}_{\mathbb{Q}}^d$. The line through 0 and $\mathbf{a} \in \mathbb{Q}^d \setminus \{0\}$ meets ∞ in a point that we denote $\mathbf{a}/0$. For $q \in \mathbb{Q}$ let $\text{sign}(q)$ be the usual signum function:

$$\text{sign}(q) = \begin{cases} -1 & \text{if } q < 0; \\ 0 & \text{if } q = 0; \\ 1 & \text{if } q > 0. \end{cases}$$

Lemma 2.6. Let $B = \{\mathbf{b}_1, \dots, \mathbf{b}_n\} \subseteq U_H$ with $B \cap \infty = \{\mathbf{b}_{m+1}, \dots, \mathbf{b}_n\}$ and pick $\{\mathbf{b}'_{m+1}, \dots, \mathbf{b}'_n\} \subseteq \mathbb{Q}^d \cap U_H$ such that $\mathbf{b}_j = \mathbf{b}'_j/0$ for $m < j \leq n$. If $\mathbf{b} \in \text{conv}_H(B) \cap \mathbb{Q}^d$ then in coordinates of \mathbb{Q}^d there is an equation $\mathbf{b} = \sum_{j=1}^m \varepsilon_j \mathbf{b}_j + \sum_{j=m+1}^n \varepsilon'_j \mathbf{b}'_j$ where $\sum_{j=1}^m \varepsilon_j = 1$, $\text{sign}(\varepsilon_j h(\mathbf{b}_j)) = \text{sign}(h(\mathbf{b}))$ for all j with $\varepsilon_j \neq 0$, and $\text{sign}(\varepsilon'_j h(\mathbf{b}'_j)) = \text{sign}(h(\mathbf{b}))$ for all j with $\varepsilon'_j \neq 0$.

Proof. Let B_+ (resp. B_-) be the subsets of $B \setminus \infty$ on which h evaluates positively (resp. negatively), and put $B_{\infty} = B \cap \infty$, $B' = \{\mathbf{b}'_{m+1}, \dots, \mathbf{b}'_n\}$. A general element $\mathbf{b} \in \text{conv}_H(B)$ is the convex combination of three points: $\mathbf{b}_+ \in \text{conv}_H(B_+)$, $\mathbf{b}_- \in \text{conv}_H(B_-)$, and $\mathbf{b}_{\infty} \in \text{conv}_H(B_{\infty})$.

Clearly, $\mathbf{b}_+ = \sum_{\mathbf{b}_j \in B_+} \varepsilon_j \mathbf{b}_j$ where $\varepsilon_j \geq 0$ and $\sum_{\mathbf{b}_j \in B_+} \varepsilon_j = 1$. A similar statement holds for \mathbf{b}_- . Now $\text{conv}_H(\mathbf{b}_+, \mathbf{b}_-) \cap \mathbb{Q}^d$ is the union of rays $\{\lambda_+ \mathbf{b}_+ + \lambda_- \mathbf{b}_- \mid \lambda_+ + \lambda_- = 1, \lambda_+ \lambda_- \leq 0\}$. Thus, if $\mathbf{b}_0 \in \text{conv}_H(\mathbf{b}_+, \mathbf{b}_-)$ then $\text{sign}(\lambda_+ h(\mathbf{b}_+)) = \text{sign}(\mathbf{b}_0) = \text{sign}(\lambda_- h(\mathbf{b}_-))$.

It suffices to show the lemma if $h(B') > 0$. Then \mathbf{b}_{∞} is of the form $\mathbf{b}'/0$ with $\mathbf{b}' \in \text{conv}_H(B')$, and points of $\text{conv}_H(B')$ are of the form $\sum_{\mathbf{b}_j \in B'} \varepsilon_j \mathbf{b}_j$ with $\sum_{\mathbf{b}_j \in B'} \varepsilon_j = 1$ and all $\varepsilon_j \geq 0$. We are thus reduced to considering $\text{conv}_H(\{\mathbf{b}_0, \mathbf{b}_{\infty}\})$ with $\mathbf{b}_{\infty} = \mathbf{b}'/0$ and $h(\mathbf{b}') > 0$. If $h(\mathbf{b}_0) > 0$ then $\text{conv}_H(\{\mathbf{b}_0, \mathbf{b}_{\infty}\})$ is the ray $\{\mathbf{b}_0 + \lambda' \mathbf{b}' \mid \lambda' \geq 0\}$, while if $h(\mathbf{b}_0) < 0$ then it is the ray $\{\mathbf{b}_0 + \lambda' \mathbf{b}' \mid \lambda' \leq 0\}$. The condition on λ' can, if $\lambda' \neq 0$, be packaged as $\text{sign}(\lambda' h(\mathbf{b}')) = \text{sign}(h(\mathbf{b}_0))$. The lemma follows. \square

We view the columns $\mathbf{a}_1, \dots, \mathbf{a}_n \in \mathbb{Z}^d$ of A as points in $\mathbb{Q}^d = \mathbb{P}_{\mathbb{Q}}^d \setminus \infty$. By assumption, $\mathbb{N}A$ is positive and hence $h \in \text{Hom}_{\mathbb{Q}}(\mathbb{Q}^d, \mathbb{Q})$ can be chosen such that $h(\mathbf{a}_j) > 0$ for all j . For any $\lambda \in \mathbb{Q}$, set $H_{\lambda} := h^{-1}(-\lambda)$ and $U_{\lambda} := \mathbb{P}_{\mathbb{Q}}^d \setminus H_{\lambda}$.

Definition 2.7. Choose $h \in \text{Hom}_{\mathbb{Q}}(\mathbb{Q}^d, \mathbb{Q})$ such that $h(\mathbf{a}_j) > 0$ for all j , and let ε be such that $0 < \varepsilon < |h(\mathbf{a}_j)/L_{\partial_j}|$ whenever $L_{\partial_j} \neq 0$. We set $\mathbf{a}_j^L := \mathbf{a}_j/L_{\partial_j}$ and call $\Delta_A^L := \text{conv}_{H_\varepsilon}(\{0, \mathbf{a}_1^L, \dots, \mathbf{a}_n^L\}) \subseteq \mathbb{P}_{\mathbb{Q}}^d$ the (A, L) -polyhedron.

Let the (A, L) -umbrella be the set Φ_A^L of faces of Δ_A^L which do not contain 0. In particular, Φ_A^L contains the empty face. Whenever it suits us, we identify $\tau \in \Phi_A^L$ with $\{j \mid \mathbf{a}_j^L \in \tau\}$, or with $\{\mathbf{a}_j \mid \mathbf{a}_j^L \in \tau\}$, or with the corresponding submatrix of A . By $\Phi_A^{L,k} \subseteq \Phi_A^L$, we denote the subset of faces of dimension k .

By $\Gamma_A^L := \bigcup \Phi_A^L$ (resp. $\Gamma_A^{L,k} := \bigcup \Phi_A^{L,k}$), we denote the underlying point set of Φ_A^L (resp. $\Phi_A^{L,k}$). Note that Γ_A^L is a piecewise linear manifold with boundary, homeomorphic to the $(d-1)$ -disk. If $\mathbf{a} \in \Delta_A^L \setminus \{0\}$ then the line through 0 and \mathbf{a} meets Γ_A^L in $\Gamma_A^L(\mathbf{a})$.

The matrix A is called L -homogeneous if all \mathbf{a}_j^L lie on a common hyperplane of $\mathbb{P}_{\mathbb{Q}}^d$. Every A is $\mathbf{0}$ -homogeneous and we call $\Phi_A^{\mathbf{0}}$ the A -umbrella. If A is L -homogeneous then $I_A^L = I_A$ under the identification of $\text{gr}^L(R)$ with R . Note that $\Phi_A^{\mathbf{0}}$ can be identified with the lattice of non-empty faces of the polyhedral cone $\mathbb{Q}_+ A$ via the face lattice of a cross-section $\mathbb{Q}_+ A \cap h^{-1}(1)$.

FIGURE 2. More (A, L) -umbrellas. (Shaded Δ_A^L with fat boundary Φ_A^L .)

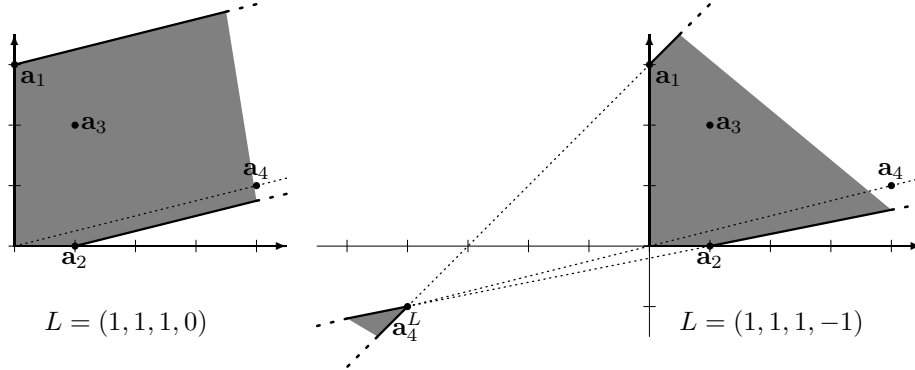


Figure 2 shows the intersection with \mathbb{Q}^d of the (A, L) -umbrella for $L = (1, 1, 1, t)$ and $t \leq 0$ in the example from the introduction, $A = \begin{pmatrix} 0 & 1 & 1 & 4 \\ 3 & 0 & 2 & 1 \end{pmatrix}$.

2.4. Monomials in the graded toric ideal. The following two results generalize Lemma 3.1 and 3.2 in [Ado94].

Lemma 2.8. *If $\mathbf{a}_k^L \notin \Gamma_A^L$ then $\partial_k \in \sqrt{I_A^L}$.*

Proof. We write v_j for L_{∂_j} . There are four cases to consider, depending on whether v_k is or is not zero, and whether $\Gamma_A^L(\mathbf{a}_k)$ is at infinity or not.

Case 1: $v_k \neq 0$, $\Gamma_A^L(\mathbf{a}_k) \notin \infty$. There is $0 \neq t \in \mathbb{Q}$ with $\Gamma_A^L(\mathbf{a}_k^L) = t\mathbf{a}_k^L \in \Gamma_A^L$ and hence either $\text{sign}(t/v_k) = 1$ and $t > 1$, or $\text{sign}(t/v_k) = -1$ and $t < 1$. Then $t\mathbf{a}_k^L$ is a convex combination in Γ_A^L of certain \mathbf{a}_i^L . By Lemma 2.6, this means in

\mathbb{Q}^d -coordinates:

$$t\mathbf{a}_k^L = \sum_{v_i \neq 0} \varepsilon_i \mathbf{a}_i^L + \sum_{v_i = 0} \varepsilon_i \mathbf{a}_i$$

for some $\varepsilon_i \in \mathbb{Q}$ where $\sum_{v_i \neq 0} \varepsilon_i = 1$, $\text{sign}(\varepsilon_i/v_i) = \text{sign}(t/v_k)$ if $\varepsilon_i v_i \neq 0$, and $\text{sign}(\varepsilon_i) = \text{sign}(t/v_k)$ if $v_i = 0$ but $\varepsilon_i \neq 0$. By scaling t and the ε_i by a suitable integer $s > 0$, we may assume that $st/v_k \in \mathbb{Z}$, $s\varepsilon_i/v_i \in \mathbb{Z}$ if $v_i \neq 0$, and $s\varepsilon_i \in \mathbb{Z}$ if $v_i = 0$. Then

$$\square = \partial_k^{|st/v_k|} - \prod_{v_i \neq 0} \partial_i^{|s\varepsilon_i/v_i|} \prod_{v_i = 0} \partial_i^{|s\varepsilon_i|} \in I_A,$$

$$\deg^L(\partial_k^{|st/v_k|}) = \text{sign}(v_k)|st| > \text{sign}(t/v_k) \sum_{v_i \neq 0} s\varepsilon_i = \deg^L\left(\prod_{v_i \neq 0} \partial_i^{|s\varepsilon_i/v_i|} \prod_{v_i = 0} \partial_i^{|s\varepsilon_i|}\right)$$

implies that $\partial_k^{|st/v_k|} = \sigma^L(\square) \in \text{gr}^L(I_A)$ and hence $\partial_k \in \sqrt{\text{gr}^L(I_A)}$.

Case 2: $v_k = 0$, $\Gamma_A^L(\mathbf{a}_k) \notin \infty$. Then $t\mathbf{a}_k = \Gamma_A^L(\mathbf{a}_k^L)$ for some $0 > t \in \mathbb{Q}$. For $t'/v'_k = t > 1/v'_k \in \mathbb{Q}$, $\Gamma_A^L(\mathbf{a}_k/v'_k) = \Gamma_A^L(\mathbf{a}_k^L) = t'(\mathbf{a}_k/v'_k)$ and the steps of Case 1 give an operator

$$\square = \partial_k^{|st'/v'_k|} - \prod_{v_i \neq 0} \partial_i^{|s\varepsilon_i/v_i|} \prod_{v_i = 0} \partial_i^{|s\varepsilon_i|} \in I_A$$

for which $\partial_k^{|st'/v'_k|}$ would be the L -symbol even if ∂_k had weight $v'_k < 0$.

Case 3: $v_k \neq 0$, $\Gamma_A^L(\mathbf{a}_k) \in \infty$. Consider the polyhedron $\Delta_A^L \cap \infty$. It is the convex hull of the points \mathbf{a}_i^L with $v_i = 0$, and of the intersections with ∞ of line segments from \mathbf{a}_i^L to \mathbf{a}_j^L with $v_j < 0 < v_i$. So if $\Gamma_A^L(\mathbf{a}_k) \in \infty$ and $v_k \neq 0$ then $v_k > 0$ and

$$\mathbf{a}_k^L = \sum_{v_i = 0} \varepsilon_i \mathbf{a}_i + \sum_{v_i \neq 0 \neq v_j} \varepsilon_{i,j} (\mathbf{a}_i^L - \mathbf{a}_j^L),$$

with $\varepsilon_i \geq 0$ whenever $v_i = 0$ and $\text{sign}(\varepsilon_{i,j}/v_i) > 0 > \text{sign}(\varepsilon_{i,j}/v_j)$ whenever $\varepsilon_{i,j} v_i \neq 0 \neq \varepsilon_{i,j} v_j$. Hence, for a suitable s as in Case 1,

$$\partial_k^{s/v_k} - \prod_{v_i = 0} \partial_i^{s\varepsilon_i} \prod_{v_i \neq 0 \neq v_j} \partial_i^{|s\varepsilon_{i,j}/v_i|} \partial_j^{|s\varepsilon_{i,j}/v_j|} \in I_A.$$

As $\varepsilon_{i,j} = -\varepsilon_{j,i}$, the L -degree of the left term is positive while that of the right term is zero.

Case 4: $v_k = 0$, $\Gamma_A^L(\mathbf{a}_k) \in \infty$. This is impossible in view of the hypothesis $\mathbf{a}_k^L \notin \Gamma_A^L$. □

Lemma 2.9. *If $\mathbf{a}_k^L, \mathbf{a}_l^L \in \Gamma_A^L$ do not lie in a common $\tau \in \Phi_A^{L,d-1}$ then $\partial_k \partial_l \in \sqrt{I_A^L}$.*

Proof. We write again v_j for L_{∂_j} and consider separate cases.

Case 1: $v_k \neq 0 \neq v_l$. By hypothesis, there is a convex combination \mathbf{a} of \mathbf{a}_k^L and \mathbf{a}_l^L in Δ_A^L which is not in $\Gamma_A^L \cup \infty$. By Lemma 2.6, $\mathbf{a} = \varepsilon_k \mathbf{a}_k^L + \varepsilon_l \mathbf{a}_l^L$ where $\varepsilon_k + \varepsilon_l = 1$ and $\text{sign}(\varepsilon_k/v_k) = \text{sign}(\varepsilon_l/v_l)$. Suppose first that $\Gamma_A^L(\mathbf{a}) \notin \infty$. Hence there is $t \in \mathbb{Q}$ such that $t\mathbf{a} = \Gamma_A^L(\mathbf{a})$. Following the argument in Case 1 of Lemma 2.8, $t\mathbf{a} = \sum_{v_i \neq 0} \varepsilon_i \mathbf{a}_i^L + \sum_{v_i = 0} \varepsilon_i \mathbf{a}_i$ for some $\varepsilon_i \in \mathbb{Q}$ where $\sum_{v_i \neq 0} \varepsilon_i = 1$, $\text{sign}(\varepsilon_i/v_i) =$

$\text{sign}(th(\mathbf{a}))$ if $\varepsilon_i v_i \neq 0$, and $\text{sign}(\varepsilon_i) = \text{sign}(th(\mathbf{a}))$ if $v_i = 0$ but $\varepsilon_i \neq 0$. For a suitable integer $s > 0$ as in Case 1 of Lemma 2.8,

$$\square = \partial_k^{st\varepsilon_k/v_k} \partial_l^{st\varepsilon_l/v_l} - \prod_{v_i \neq 0} \partial_i^{s\varepsilon_i/v_i} \prod_{v_i=0} \partial_i^{s\varepsilon_i} \in I_A$$

with $\sigma^L(\square) = \partial_k^{st\varepsilon_k/v_k} \partial_l^{st\varepsilon_l/v_l}$.

If $\Gamma_A^L(\mathbf{a}) \in \infty$, one proceeds as in Case 3 of Lemma 2.8, using \mathbf{a} instead of \mathbf{a}_k^L .

Case 2: $v_k \neq 0 = v_l$. Then, in \mathbb{Q}^d -coordinates, the line segment from \mathbf{a}_k^L to \mathbf{a}_l^L consists of \mathbf{a}_l^L and the points $\mathbf{a} = \mathbf{a}_k^L + \varepsilon_l \mathbf{a}_l \in \Delta_A^L \setminus (\Gamma_A^L \cup \infty)$ where $\text{sign}(\varepsilon_l) = \text{sign}(v_k)$. If $\Gamma_A^L(\mathbf{a}) \notin \infty$ then one can proceed as in Case 1 of this lemma. On the other hand, if $\Gamma_A^L(\mathbf{a}) \in \infty$ then similarly to Case 3 of Lemma 2.8,

$$\mathbf{a}_k^L + \varepsilon_l \mathbf{a}_l = \sum_{v_i=0} \varepsilon_i \mathbf{a}_i + \sum_{v_i \neq 0 \neq v_j} \varepsilon_{i,j} (\mathbf{a}_i^L - \mathbf{a}_j^L)$$

with $\text{sign}(\varepsilon_i) = \text{sign}(\varepsilon_l) = \text{sign}(v_k)$ whenever $\varepsilon_i \neq 0$ and $\text{sign}(\varepsilon_{i,j}/v_i) > 0 > \text{sign}(\varepsilon_{i,j}/v_j)$ whenever $\varepsilon_{i,j} v_i \neq 0 \neq \varepsilon_{i,j} v_j$. Clearing fractions with $s > 0$, one obtains the operator

$$\partial_k^{s/v_k} \partial_l^{s\varepsilon_l} - \prod_{v_i=0} \partial_i^{s\varepsilon_i} \prod_{v_i \neq 0 \neq v_j} \partial_i^{s\varepsilon_{i,j}/v_i} \partial_j^{s\varepsilon_{i,j}/v_j} \in I_A$$

where the left term has positive L -degree and the right term has L -degree zero.

Case 3: $v_k = 0 = v_l$. The points on the line segment from \mathbf{a}_k^L to \mathbf{a}_l^L are, in \mathbb{Q}^d -coordinates, the points $\mathbf{a}/0$ with $\mathbf{a} = \varepsilon_k \mathbf{a}_k + \varepsilon_l \mathbf{a}_l$ where $\varepsilon_k, \varepsilon_l \geq 0$ and $\varepsilon_k > 0$ or $\varepsilon_l > 0$. By hypothesis, for $\varepsilon_k, \varepsilon_l > 0$ the point $\mathbf{a}/0$ is in the interior of Δ_A^L and, in particular, $\Gamma_A^L(\mathbf{a}) \notin \infty$. Then $t\mathbf{a} = \Gamma_A^L(\mathbf{a})$ for some $0 > t \in \mathbb{Q}$. By Lemma 2.6,

$$t\mathbf{a} = \sum_{v_i \neq 0} \varepsilon_i \mathbf{a}_i^L + \sum_{v_i=0} \varepsilon_i \mathbf{a}_i$$

where $\sum_{v_i \neq 0} \varepsilon_i = 1$, $\varepsilon_i/v_i < 0$ whenever $\varepsilon_i v_i \neq 0$, and $\varepsilon_i < 0$ whenever $v_i = 0$ but $\varepsilon_i \neq 0$. Clearing fractions with $s > 0$, the L -symbol of the resulting operator

$$\partial_k^{-st\varepsilon_k} \partial_l^{-st\varepsilon_l} - \prod_{v_i \neq 0} \partial_i^{-s\varepsilon_i/v_i} \prod_{v_i=0} \partial_i^{-s\varepsilon_i} \in I_A$$

is $\partial_k^{-t\varepsilon_k} \partial_l^{-t\varepsilon_l}$ since the right term has L -degree $\sum_{v_i \neq 0} (-s\varepsilon_i/v_i)v_i = -s < 0$. \square

2.5. Homogeneity in the graded toric ideal.

Definition 2.10. Let τ be a set of columns of A . For $\mathbf{u} \in \mathbb{Z}^n$, we write $\text{supp}(\mathbf{u}) \subseteq \tau$ if $\mathbf{u}_i \neq 0$ implies $\mathbf{a}_i \in \tau$. For $f \in R$, we write $\text{supp}(f) \subseteq \tau$ if $\text{supp}(\mathbf{u}) \subseteq \tau$ for all monomials $\partial^{\mathbf{u}}$ of f .

Lemma 2.11. Let $\tau \in \Phi_A^L$ and pick $\mathbf{u} \in \mathbb{Z}^n$ such that $A \cdot \mathbf{u} = 0$.

- (1) If $\text{supp}(\square_{\mathbf{u}}) \subseteq \tau$ then $\sigma^L(\square_{\mathbf{u}}) = \square_{\mathbf{u}}$. In particular, the toric ideal I_{τ} is L -homogeneous.
- (2) If $\text{supp}(\partial^{\mathbf{u}^{\pm}}) \subseteq \tau$ and $\text{supp}(\partial^{\mathbf{u}^{\mp}}) \not\subseteq \tau$ then $\sigma^L(\square_{\mathbf{u}}) = \mp \partial^{\mathbf{u}^{\mp}}$.

Proof. We abbreviate $\mathbf{v} := L\partial$.

Consider first the case where the facet $\tau \in \Phi_A^{L,d-1}$ lies entirely in ∞ . Then $\mathbf{a}_i^L \in \tau$ implies $v_i = 0$ and hence $\text{supp}(\square_{\mathbf{u}}) \subseteq \tau$ implies that $\square_{\mathbf{u}}$ is L -homogeneous of degree zero. Suppose $\text{supp}(\mathbf{u}_+) \subseteq \tau$ but $\text{supp}(\mathbf{u}_-) \not\subseteq \tau$. As $\tau \subseteq \infty$ is a facet of

Δ_A^L and by Definition 2.7, the interior of Δ_A^L meets neither ∞ nor H_ε . Hence Δ_A^L is completely contained in the $\mathbb{P}_{\mathbb{Q}}^d$ -closure of one of the regions $\{t \in \mathbb{Q}^d \mid h(t) \geq -\varepsilon\}$ and $\{t \in \mathbb{Q}^d \mid h(t) \leq -\varepsilon\}$. Since $0 \in \Delta_A^L$, it must be the former. Moreover, by definition of ε , $h(\mathbf{a}_j^L) \notin [-\varepsilon, 0]$ for all $v_j \neq 0$, whence $v_j \geq 0$ in all cases. Thus $\deg^L(\partial^{\mathbf{u}^+}) = 0$ and $\deg^L(\partial^{\mathbf{u}^-}) > 0$, hence $\sigma^L(\square_{\mathbf{u}}) = -\partial^{\mathbf{u}^-}$. The case $\text{supp}(\mathbf{u}_-) \subseteq \tau \not\subseteq \text{supp}(\mathbf{u}_+)$ is similar.

Now let $\tau \in \Phi_A^L$ be not contained in ∞ , or let $\tau \in \Phi_A^L \setminus \Phi_A^{L,d-1}$ be a non-facet face contained in ∞ . Then there is a linear form $h_\tau \in \text{Hom}_{\mathbb{Q}}(\mathbb{Q}^d, \mathbb{Q})$ such that the closure of $h_\tau^{-1}(1)$ in $\mathbb{P}_{\mathbb{Q}}^d$ meets Δ_A^L precisely in τ , and for which $\text{sign}(v_i)h_\tau(\mathbf{a}_i^L) \leq \text{sign}(v_i)$ whenever $v_i \neq 0$.

For all $\mathbf{a}_i^L \in \tau$ with $v_i = 0$ the line through 0 and \mathbf{a}_i meets τ in $\mathbf{a}_i^L \in \infty$. This means that \mathbf{a}_i is parallel to $h_\tau^{-1}(1)$ and so $h_\tau(\mathbf{a}_i) = 0$. Thus, $A \cdot \mathbf{u} = 0$ implies that

$$0 = h_\tau(A \cdot \mathbf{u}) = h_\tau\left(\sum_{i=1}^n u_i \mathbf{a}_i\right) = \sum_{u_i > 0 \neq v_i} u_i v_i h_\tau(\mathbf{a}_i^L) - \sum_{u_i < 0 \neq v_i} (-u_i) v_i h_\tau(\mathbf{a}_i^L).$$

If $\text{supp}(\square_{\mathbf{u}}) \subseteq \tau$ and $u_i v_i \neq 0$ then $h_\tau(\mathbf{a}_i^L) = 1$ and hence $\deg^L(\partial^{\mathbf{u}^+}) = \deg^L(\partial^{\mathbf{u}^-})$.

Now suppose $\text{supp}(\partial^{\mathbf{u}^+}) \subseteq \tau \not\subseteq \text{supp}(\partial^{\mathbf{u}^-})$. Hence $h_\tau(\mathbf{a}_i^L) = 1$ if $u_i > 0 \neq v_i$, $\text{sign}(v_i)h_\tau(\mathbf{a}_i^L) \leq \text{sign}(v_i)$ if $u_i < 0 \neq v_i$, and $\text{sign}(v_i)h_\tau(\mathbf{a}_i^L) < \text{sign}(v_i)$ for at least one $u_i < 0$. Thus,

$$\begin{aligned} \deg^L(\partial^{\mathbf{u}^+}) &= \sum_{u_i > 0} v_i u_i = \sum_{u_i > 0 \neq v_i} v_i u_i h_\tau(\mathbf{a}_i^L) \\ &= \sum_{u_i < 0 \neq v_i} v_i (-u_i) h_\tau(\mathbf{a}_i^L) < \sum_{u_i < 0 \neq v_i} v_i (-u_i) = \deg^L(\partial^{\mathbf{u}^-}). \end{aligned}$$

The case $\text{supp}(\mathbf{u}_-) \subseteq \tau \not\subseteq \text{supp}(\mathbf{u}_+)$ is similar. \square

2.6. Minimal associated prime ideals. We now identify the components of the L -graded toric ring S_A^L .

Definition 2.12. For $\mathbf{u} \in \mathbb{Z}^n$ let

$$\tau_{\mathbf{u}}^L = \bigcap_{\text{supp}(\mathbf{u}) \subseteq \tau \in \Phi_A^L} \tau$$

be the smallest element of Φ_A^L containing $\text{supp}(\mathbf{u})$ and put $\tau_{\mathbf{u}}^L = A$ if there is none. Note that $[\partial^{\mathbf{u}} \in J_\tau] \Rightarrow [\tau_{\mathbf{u}}^L \not\subseteq \tau]$ for $\tau \subseteq A$ and $\mathbf{u} \in \mathbb{N}^n$.

Recall that I_A , I_A^r , and J_τ are \mathbb{Z}^d -graded prime ideals.

Lemma 2.13. Let $\tilde{I}_A^L \subseteq R$ be generated by all elements of the following types:

- (1) ∂_k where $\mathbf{a}_k^L \notin \Gamma_A^L$;
- (2) $\partial_k \partial_l$ where $\mathbf{a}_k^L, \mathbf{a}_l^L \in \Gamma_A^L$ do not lie in a common facet $\tau \in \Phi_A^{L,d-1}$;
- (3) $\square_{\mathbf{u}}$ where $\mathbf{u} \in \mathbb{Z}^n$, $A \cdot \mathbf{u} = 0$, and $\tau_{\mathbf{u}}^L \neq A$.

Then $\tilde{I}_A^L = \bigcap_{\tau \in \Phi_A^{L,d-1}} I_A^r$.

Proof. Let $I = \bigcap_{\tau \in \Phi_A^{L,d-1}} I_A^r$. We show first that $\tilde{I}_A^L \subseteq I$. Let I_0 denote the ideal of R generated by the elements from (1) and (2). Then clearly $I_0 \subseteq J_\tau \subseteq I_A^r$ for any $\tau \in \Phi_A^{L,d-1}$. Pick $\square_{\mathbf{u}}$ as in (3). By Part (1) of Lemma 2.11, $\sigma^L(\square_{\mathbf{u}}) = \square_{\mathbf{u}}$. Let $\tau \in \Phi_A^{L,d-1}$ and suppose $\square_{\mathbf{u}} \notin J_\tau$. Then (without loss of generality) $\text{supp}(\mathbf{u}_+) \subseteq \tau$.

By Part (2) of Lemma 2.11, since $\sigma^L(\square_{\mathbf{u}}) = \square_{\mathbf{u}}$, $\text{supp}(\mathbf{u}_-) \subseteq \tau$ as well. Hence $\square_{\mathbf{u}} \in I_\tau \subseteq I_A^\tau$ and so $\tilde{I}_A^L \subseteq I$.

For the converse inclusion suppose now $m \in I$. We write $m = m_0 + \sum_{\tau \in \Phi_A^L} m_\tau$ where $m_0 \in I_0$, and

$$m_\tau = \sum_{\tau_{\mathbf{u}}^L = \tau} c_{\mathbf{u}} \partial^{\mathbf{u}} \in \mathbb{C}[\partial_\tau]$$

collects the monomials in m minimally supported in τ . Since $I_0 \subseteq \tilde{I}_A^L \subseteq I$ we may assume that $m_0 = 0$. Now pick any $\tau \in \Phi_A^L$. Then $m = \hat{m}_\tau + m_{\overline{\tau}}$ where $\hat{m}_\tau = \sum_{\tau' \subseteq \tau} m_{\tau'}$ are the terms in m supported in τ . Since $m_{\overline{\tau}} \in J_\tau \subseteq I_A^\tau$ and $m \in I \subseteq I_A^\tau$, $\hat{m}_\tau = m - m_{\overline{\tau}} \in I_A^\tau \cap \mathbb{C}[\partial_\tau] = I_\tau \subseteq \tilde{I}_A^L$ in view of (3). Since every \hat{m}_τ is in \tilde{I}_A^L , so is every m_τ , and hence $m \in \tilde{I}_A^L$ as well. \square

Lemma 2.14. $\sqrt{I_A^L} = \tilde{I}_A^L$, the radical ideal from Lemma 2.13.

Proof. By Lemmas 2.8 and 2.9, the elements in 2.13.(1) and 2.13.(2) are in $\sqrt{I_A^L}$.

By Part (1) of Lemma 2.11, all elements from 2.13.(3) are in I_A^L . Hence $\tilde{I}_A^L \subseteq \sqrt{I_A^L}$.

By Lemma 2.4 and Lemma 2.13, it suffices to show conversely that, for any $\mathbf{u} \in \mathbb{Z}^n$ with $A \cdot \mathbf{u} = 0$, $\sigma^L(\square_{\mathbf{u}}) \in I_A^\tau$ for all $\tau \in \Phi_A^{L, d-1}$. Pick such \mathbf{u} and let $\tau \in \Phi_A^{L, d-1}$. If $\tau_{\mathbf{u}}^L \subseteq \tau$ then by Part (1) of Lemma 2.11, $\sigma^L(\square_{\mathbf{u}}) = \square_{\mathbf{u}} \in I_\tau \subseteq I_A^\tau$. If $\tau_{\mathbf{u}_+}^L \subseteq \tau$ but $\tau_{\mathbf{u}_-}^L \not\subseteq \tau$ then by Part (2) of Lemma 2.11, $\sigma^L(\square_{\mathbf{u}}) = -\partial^{\mathbf{u}_-} \in J_\tau \subseteq I_A^\tau$. Similarly, $\sigma^L(\square_{\mathbf{u}}) \in I_A^\tau$ if $\tau_{\mathbf{u}_-}^L \subseteq \tau$ but $\tau_{\mathbf{u}_+}^L \not\subseteq \tau$. Finally, if $\tau_{\mathbf{u}_+}^L \not\subseteq \tau \not\subseteq \tau_{\mathbf{u}_-}^L$ then $\sigma^L(\square_{\mathbf{u}}) \in J_\tau \subseteq I_A^\tau$. \square

The following consequence of Lemmas 2.3, 2.13, and 2.14 generalizes [Ado94, Lem. 3.2].

Theorem 2.15. *The set of \mathbb{Z}^d -graded prime ideals of R containing I_A^L equals $\{I_A^\tau \mid \tau \in \Phi_A^L\}$ and hence the (A, L) -umbrella encodes the geometry of S_A^L :*

$$\text{Spec}(S_A^L) = \text{Var}(I_A^L) = \bigcup_{\tau \in \Phi_A^{L, d-1}} \bar{O}_A^\tau = \bigsqcup_{\tau \in \Phi_A^L} O_A^\tau.$$

Adjacencies of orbit strata correspond to inclusions in the (A, L) -umbrella:

$$[O_A^{\tau'} \subseteq \bar{O}_A^\tau] \Leftrightarrow [\tau' \subseteq \tau]. \quad \square$$

In particular, Theorem 2.15 identifies the \mathbb{Z}^d -graded prime ideals containing $I_A = I_A^{\mathbf{0}}$ with the elements of $\Phi_A^{\mathbf{0}}$ from Definition 2.7.

Remark 2.16. Geometrically, $\text{Spec}(S_A^L)$ reflects certain asymptotics of $\text{Spec}(S_A)$. We make this precise in the case where $L_{\partial_j} > 0$ for all j . We may assume that the components of L are coprime positive integers. Then $X = \mathbb{C}^n$ can be considered as the smooth chart defined by $\partial_0 \neq 0$ in the weighted projective space $\mathbb{P}_{(1, L)}^n$ where $(1, L) := (1, L_{\partial_1}, \dots, L_{\partial_n})$, and $\text{Spec}(S_A) \subseteq X$. The closure Z of $\text{Spec}(S_A)$ in $\mathbb{P}_{(1, L)}^n$ is defined by the L -homogenization $I_A^L(\partial_0)$ of I_A with respect to ∂_0 . The ideal of the intersection of Z with the weighted projective space $\mathbb{P}_L^{n-1} = \text{Var}(\partial_0) \subseteq \mathbb{P}_{(1, L)}^n$ is $I_A^L(\partial_0)|_{\partial_0=0} = I_A^L$. Thus $Z \cap \mathbb{P}_L^{n-1} = \text{Proj}(S_A^L) \subseteq \mathbb{P}_L^{n-1}$.

2.7. Index formula for multiplicities. By Theorem 2.15, there is a composition chain of \mathbb{Z}^d -graded R -modules

$$(2.7.1) \quad 0 = N_0 \subsetneq N_1 \subsetneq \cdots \subsetneq N_{l-1} \subsetneq N_l = S_A^L$$

with $N_i/N_{i-1} \cong S_{\tau_i}(-\mathbf{u}_i)$ for some $\tau_i \in \Phi_A^L$ and $\mathbf{u}_i \in \mathbb{Z}^d$. As \mathbb{Z}^d -graded vector spaces $S_A^L = S_A$ and the \mathbb{Z}^d -graded Hilbert function of both rings has values in $\{0, 1\}$. Thus, the composition chain (2.7.1) induces a partition of \mathbb{Z}^d -degrees

$$(2.7.2) \quad \deg(S_A^L) = \deg(S_A) = \mathbb{N}A = \bigsqcup_{i=1}^l (\mathbf{u}_i + \mathbb{N}\tau_i) = \bigsqcup_{i=1}^l \deg(S_{\tau_i}(-\mathbf{u}_i)).$$

Definition 2.17. If $\tau \in \Phi_A^{L, d-1}$ is a facet then the number $\nu_A^{L, \tau}$ of indices i in the chain (2.7.1), and hence in the partition (2.7.2), with $\tau_i = \tau$ is the *multiplicity* of S_A^L along \bar{O}_A^τ . Note that $\nu_A^{L, \tau}$ is the length of the localization of S_A^L at I_A^τ and hence independent of the particular composition chain.

Proposition 2.18. For all $\tau \in \Phi_A^{L, d-1}$, $\nu_A^{L, \tau} = [\mathbb{Z}^d : \mathbb{Z}\tau]$. In particular, the degree of I_A^L equals $\sum_{\tau \in \Phi_A^{L, d-1}} \nu_A^{L, \tau} = \sum_{\tau \in \Phi_A^{L, d-1}} [\mathbb{Z}^d : \mathbb{Z}\tau]$.

Proof. Fix $\tau \in \Phi_A^{L, d-1}$. For disjoint $\mathbf{u} + \mathbb{N}\tau$ and $\mathbf{v} + \mathbb{N}\tau$, $\mathbf{u} - \mathbf{v} \notin \mathbb{Z}\tau$ since $\mathbb{N}\tau$ contains a shifted copy of its normalization $\mathbb{Q}_+\tau \cap \mathbb{Z}\tau$. This means that $\nu_A^{L, \tau} \leq [\mathbb{Z}^d : \mathbb{Z}\tau]$ and it remains to show that $\bigsqcup_{\tau_i = \tau} (\mathbf{u}_i + \mathbb{N}\tau_i)$ meets every coset of $\mathbb{Z}^d/\mathbb{Z}\tau$. Pick $\mathbf{u} \in \mathbb{N}\tau$ outside $\mathbb{Q}_+\tau'$ for all $\tau' \in \Phi_A^L$ with $\tau' \subsetneq \tau$. Since $\mathbb{N}A$ contains a shifted copy of its normalization $\mathbb{Q}_+A \cap \mathbb{Z}^d$, for $k \gg 0$

$$k\mathbf{u} + (\mathbb{Q}_+\tau \cap \mathbb{Z}^d) \subseteq k\mathbf{u} + (\mathbb{Q}_+A \cap \mathbb{Z}^d) \subseteq \mathbb{N}A = \deg(S_A^L) = \bigsqcup_{i=1}^l \deg(S_{\tau_i}(-\mathbf{u}_i)).$$

By Lemma 2.19 below, for $k \gg 0$

$$k\mathbf{u} + (\mathbb{Q}_+\tau \cap \mathbb{Z}^d) \subseteq \bigsqcup_{\tau_i = \tau} (\mathbf{u}_i + \mathbb{N}\tau_i)$$

and the left hand side meets every coset of $\mathbb{Z}^d/\mathbb{Z}\tau$. This yields $\nu_A^{L, \tau} \geq [\mathbb{Z}^d : \mathbb{Z}\tau]$. \square

Lemma 2.19. Let $\tau \in \Phi_A^{L, d-1}$ and suppose $\mathbf{u} \in \mathbb{N}\tau \setminus \mathbb{Q}_+\tau'$ for all $\tau' \in \Phi_A^L$ with $\tau' \subsetneq \tau$. Fix $\mathbf{u}' \in \mathbb{Z}^d$ and $\tau' \in \Phi_A^L$ such that $\tau' \neq \tau$. Then for $k \gg 0$

$$(k\mathbf{u} + \mathbb{Q}_+\tau) \cap (\mathbf{u}' + \mathbb{Q}_+\tau') = \emptyset.$$

Proof. Suppose that $k\mathbf{u} + \mathbf{v} = \mathbf{u}' + \mathbf{v}'$ with $\mathbf{v} \in \mathbb{Q}_+\tau$ and $\mathbf{v}' \in \mathbb{Q}_+\tau'$. Pick a linear form $0 \neq h_{\tau, \tau'} \in \text{Hom}_{\mathbb{Q}}(\mathbb{Q}^d, \mathbb{Q})$ with $h_{\tau, \tau'}(\tau) \geq 0$, $h_{\tau, \tau'}(\tau') \leq 0$. Then, by hypothesis, $h_{\tau, \tau'}(\mathbf{u}) > 0$. But $kh_{\tau, \tau'}(\mathbf{u}) + h_{\tau, \tau'}(\mathbf{v}) = h(\mathbf{u}') + h_{\tau, \tau'}(\mathbf{v}')$ implies that $kh_{\tau, \tau'}(\mathbf{u}) \leq h(\mathbf{u}')$ which is impossible for $k \gg 0$. \square

2.8. Newton filtration and Cohen–Macaulayness. Until the end of this section, we fix a weight vector L with $L_{\partial_i} > 0$ for all i . Let $V = \bigoplus_{\mathbf{a} \in \mathbb{N}A} V_{\mathbf{a}}$ be a \mathbb{Z}^d -graded vector space. Then the *Newton filtration* $N = N_A^L$ on V with respect to Δ_A^L is defined by

$$N_i V = \bigoplus_{\mathbf{a} \in i \cdot \Delta_A^L} V_{\mathbf{a}}.$$

Note that $\text{gr}^N(V) = V$ as \mathbb{Z}^d -graded vector spaces. For $\tau \in \Phi_A^L$ or $\tau = A$, we denote $V(\tau) = \bigoplus_{\mathbf{a} \in \mathbb{Q}_+\tau} V_{\mathbf{a}}$. Generalizing [Kou76, Prop. 2.6], there is a complex

$$(2.8.1) \quad 0 \longrightarrow V_{d-1} \longrightarrow V_{d-2} \longrightarrow \cdots \longrightarrow V_0 \longrightarrow 0$$

where the \mathbb{Z}^d -graded vector space V_i is the direct sum of all $V(\tau)$ for which $\tau \in \Phi_A^{L,i}$ is not contained in the boundary of Γ_A^L , cf. Definition 2.7. The cohomology of this complex is concentrated in homological degree $d-1$ and equal to V .

Now assume that V is a \mathbb{Z}^d -graded \mathbb{C} -algebra. Then $\text{gr}^N(V)(\tau) = V(\tau)$ as \mathbb{C} -algebras for $\tau \in \Phi_A^L$. For $\tau, \tau' \in \Phi_A^L$ (or $\tau = A$) with $\tau \supseteq \tau'$, the maps

$$\gamma_{\tau, \tau'} : \text{gr}^N(V)(\tau) \longrightarrow \text{gr}^N(V)(\tau')$$

in the complex (2.8.1) for $\text{gr}^N(V)$ are natural projections of \mathbb{C} -algebras. In particular, (2.8.1) is a complex of $\text{gr}^N(V)$ -modules.

Lemma 2.20. *For all $\tau \in \Phi_A^L$ and all $k > 0$, $\mathbb{C}[\mathbb{N} \cdot k \cdot (\tau \cap \Phi_A^{L,0})] \subseteq S_A^L(\tau)$ is a module-finite ring extension for all $\tau \in \Phi_A^L$.*

Proof. One can construct a \mathbb{Z}^d -graded composition chain of S_A^L as in (2.7.1) such that the resulting partition (2.7.2) refines the partition $\mathbb{Q}_+A = \mathbb{Q}_+\tau \sqcup (\mathbb{Q}_+A \setminus \mathbb{Q}_+\tau)$, and hence $\deg(S_{\tau_i}(-\mathbf{u}_i))$ meets $\mathbb{Q}_+\tau$ only if $\tau_i \subseteq \tau$. Namely, given any chain as in (2.7.1) we refine it at any index $i \in \{1, \dots, l\}$ for which $\deg(S_{\tau_i}(-\mathbf{u}_i)) = \mathbf{u}_i + \mathbb{N}\tau_i$ meets both $\mathbb{Q}_+\tau$ and $\mathbb{Q}_+A \setminus \mathbb{Q}_+\tau$. Let $\ell \in \text{Hom}_{\mathbb{Q}}(\mathbb{Q}^d, \mathbb{Q})$ be a separating linear form, $\ell(\tau) \geq 0$ and $\ell(\tau_i) \leq 0$. If $\tau_i \subseteq \tau$ (and hence $\mathbf{u}_i \notin \mathbb{Q}_+\tau$), we chose $\mathbf{u}'_i \in (\mathbf{u}_i + \mathbb{N}\tau_i) \cap \mathbb{Q}_+\tau$. The submodule $S_{\tau_i}(-\mathbf{u}'_i)$ of $S_{\tau_i}(-\mathbf{u}_i)$ has its \mathbb{Z}^d -degrees entirely in $\mathbb{Q}_+\tau$, and the cokernel of the inclusion has smaller dimension. If conversely $\tau_i \not\subseteq \tau$, choose $\ell \in \text{Hom}_{\mathbb{Q}}(\mathbb{Q}^d, \mathbb{Q})$ above such that $\ell(\tau_i) \neq 0$ and pick $\mathbf{u}'_i \in \mathbf{u}_i + \mathbb{N}\tau_i$ with $\ell(\mathbf{u}'_i) < 0$. Then $S_{\tau_i}(-\mathbf{u}'_i) \subseteq S_{\tau_i}(-\mathbf{u}_i)$ and has degrees completely outside of $\mathbb{Q}_+\tau$.

Iterating this procedure we arrive at a composition chain as claimed. Then, however, the statement of the lemma is obvious: for each $i \in \{1, \dots, l\}$ with $\deg(S_{\tau_i}(-\mathbf{u}_i)) \cap \mathbb{Q}_+\tau \neq \emptyset$, $\tau_i \subseteq \tau$ and $(S_{\tau_i}(-\mathbf{u}_i))(\tau) \cong S_{\tau_i}$ is even module-finite over the subring $\mathbb{C}[\mathbb{N} \cdot k \cdot (\tau_i \cap \Phi_A^{L,0})]$ of $\mathbb{C}[\mathbb{N} \cdot k \cdot (\tau \cap \Phi_A^{L,0})]$. \square

Theorem 2.21. *If $S_A^L(\tau)$ is Cohen–Macaulay for all $\tau \in \Phi_A^L$ not contained in the topological boundary of Γ_A^L then S_A^L is Cohen–Macaulay.*

Proof. There is an integer $k > 0$ such that $k \cdot \Gamma_A^L$ has vertex set $k \cdot \Gamma_A^{L,0} \subseteq \mathbb{N}A$. Within the space of sequences f_1, \dots, f_d in S_A^L for which each f_i is a sum of terms whose \mathbb{Z}^d -degrees are in $k \cdot \Gamma_A^{L,0}$, choose one sequence that is generic.

Each f_i is homogeneous with respect to $N = N_A^L$ and can hence be identified with its N -symbol in $\text{gr}^N(S_A^L)$. By Lemma 2.20, the vector space spanned by $\gamma_{A, \tau}(f_1), \dots, \gamma_{A, \tau}(f_d) \in \mathbb{C}[\mathbb{N}\tau]$ contains a system of parameters on $S_A^L(\tau)$ for all $\tau \in \Phi_A^L$. This system of parameters is a regular sequence by the Cohen–Macaulay hypothesis. By the (spectral sequence) argument in [Kou76, §2.12], the Koszul complex induced by f_1, \dots, f_d on the complex (2.8.1) with $V = S_A^L$ is a resolution. It follows that the Koszul complex induced by f_1, \dots, f_d on $\text{gr}^N(S_A^L)$, and hence on S_A^L , is a resolution as well. Therefore, f_1, \dots, f_d is a regular sequence on S_A^L and the claim follows. \square

3. CHARACTERISTIC VARIETY OF THE HYPERGEOMETRIC SYSTEM

3.1. Characteristic varieties and slopes. The *Weyl algebra* $D = \mathbb{C}[x]\langle \partial \rangle$ in n variables $x = x_1, \dots, x_n$ is the ring of \mathbb{C} -linear differential operators on $X = \mathbb{C}^n$ and contains R as a commutative subring. The \mathbb{Z}^d -grading on R extends to D by setting $-\deg(x_i) = \mathbf{a}_i = \deg(\partial_i)$.

With $L_x = (L_{x_1}, \dots, L_{x_n})$ and $L_\partial = (L_{\partial_1}, \dots, L_{\partial_n})$, $L = (L_x, L_\partial) \in \mathbb{Q}^{2d}$ is a *weight vector on D* if $L_x + L_\partial \geq 0$. Fix any such weight vector L ; it defines an increasing filtration L on D by $[x^{\mathbf{u}}\partial^{\mathbf{v}} \in L_k D] \Leftrightarrow [L \cdot (\mathbf{u}, \mathbf{v}) \leq k]$. Since the Gröbner fan of any D -ideal is defined over \mathbb{Q} (cf. [ACJG00]), the study of real weights can be reduced to the present rational case. If $P \in L_k D \setminus L_{<k} D$ then $k =: \deg^L(P)$ is the L -degree of P . The multiplicative, but not additive, L -symbol map

$$\sigma^L : D \longrightarrow \text{gr}^L(D) =: W$$

is defined by $\sigma^L(P) = P \bmod L_{<k} R$ if $\deg^L(P) = k$. An element of the form $P = \sum_{L \cdot (\mathbf{u}, \mathbf{v}) = k} P_{\mathbf{u}, \mathbf{v}} x^{\mathbf{u}} \partial^{\mathbf{v}} \in D$ with $P_{\mathbf{u}, \mathbf{v}} \in \mathbb{C}$ is L -homogeneous. By abuse of notation, we identify L -homogeneous elements in D with their image under σ^L in W . We restrict ourselves to the case $L_x + L_\partial > 0$ in which case $W \cong \mathbb{C}[x, \partial]$ can be considered as the ring of polynomial functions on the cotangent space T^*X .

The definition of the characteristic variety of a D -module is based on the concept of good filtrations discussed in more detail in Section 4 and [Sch85, II.§1.1-3]. Let F be a filtration on a ring T . Then G is called a *good F -filtration* on a T -module N , cf. [Sch85, II. Def. 1.1.1], if there are generators n_1, \dots, n_m of N and $\mathbf{u} \in \mathbb{Z}^m$ such that for all k one has

$$G_k N = \sum_{i=1}^m F_{k+u_i} T \cdot n_i.$$

Note that good F -filtrations on N exist if and only if N is T -finite. From the definition follows that all good F -filtrations on N are equivalent in the sense that for all $k, l \in \mathbb{N}$ there are $k_l, l_k \in \mathbb{N}$ with $G_k N \subseteq G'_{l_k} N$ and $G'_l \subseteq G_{k_l}$.

Definition 3.1. The L -characteristic variety $\text{Ch}^L(M)$ of a finite D -module M on X is the support of $\text{gr}^L(M)$ in T^*X for some good L -filtration on M .

The independence of $\text{Ch}^L(M)$ of the choice of the good L -filtration on M follows from [Sch85, II. Prop. 1.3.1.a]. Y. Laurent proved in the micro-local setting that $\text{Ch}^L(M)$ is involutive and Lagrangian for holonomic M . The following algebraic statement is a special case of a result by G.G. Smith [Smi01].

Theorem 3.2. *The dimension of any component of $\text{Ch}^L(M)$ is at least n and if M is holonomic then each component has dimension n . \square*

Important special cases of filtrations on D are the *order filtration* $F = (F_x, F_\partial) = (\mathbf{0}, \mathbf{1})$ and the *V -filtration* $V = (V_x, V_\partial)$ along the coordinate variety

$$(3.1.1) \quad Y := \text{Var}(x_{\mathfrak{Y}}) \subseteq X, \quad \mathfrak{Y} \subseteq \{1, \dots, n\},$$

defined by $-V_{x_i} = 1 = V_{\partial_i}$ for $i \in \mathfrak{Y}$ and $V_{x_i} = 0 = V_{\partial_i}$ for $i \notin \mathfrak{Y}$. The notion of slopes along Y , see Definition 3.4 below, involves the family of intermediate filtrations L between F and V defined by the linear combination of weight vectors

$$(3.1.2) \quad L = pF + qV, \quad p/q \in \mathbb{Q}_{>0} \cup \{\infty\}.$$

Note that $L_x + L_\partial > 0$ since $p > 0$. If $p'/q' = p/q$ then the filtrations L and L' are identical, up to a dilation in the index: $L_{p'k} = L'_{pk}$. By abuse of notation, we shall frequently identify the filtration L with the number p/q . For $Y \subseteq X$ closed and reduced, let T_Y^*X be (the closure of) the conormal bundle of (the smooth points of) Y in X . With notation as in (3.1.1) and (3.1.2), the ring $W = \text{gr}^L(D)$ can be considered as the ring of polynomial functions on the cotangent space $T^*T_Y^*X$ of T_Y^*X . For $i \in \mathfrak{A}$, $-x_i$ can be interpreted as the partial derivative with respect to the variable ∂_i . This sets up an explicit isomorphism between $T^*T_Y^*X$ and T^*X .

Remark 3.3. For $L = pF + qV$, $\text{Ch}^L(M)$ is a global algebraic version of the analytic characteristic variety of type $r = p/q + 1$ defined by Y. Laurent [Lau87, §3.2]. Our algebraic L -filtration corresponds to the filtration on the sheaf of analytic differential operators on X along Y induced by the microlocal filtration $F_{\Lambda, r}$ along $\Lambda = T_Y^*X$ from [Lau87, Def. 3.2.1]. A flatness argument in [ACJG96, Lem. 1.1.2] shows that the analytification functor commutes with the grading by these two corresponding filtrations. Thus, the components of our $\text{Ch}^L(M)$ which meet the preimage of Y in T^*X correspond to the components of Y. Laurent's characteristic variety of type r . This motivated the following algebraic version of Y. Laurent's critical indices or *slopes* [Lau87, §3.4].

Definition 3.4. For $L = pF + qV$, we mean by $f(L')$ jumps at $L' = L$ that the set-valued function

$$\mathbb{Q} \ni p'/q' \longmapsto L' = p'F + q'V \mapsto f(L')$$

is not locally constant at p/q . A *slope* of a finite D -module M at $y \in Y$ along Y is a value $L = p/q$ such that the set of components of $\text{Ch}^L(M)$ which meet T_y^*X jumps at $L' = L$.

It follows from the existence of the Gröbner fan that a fixed D -module has only finitely many and only rational slopes along all coordinate varieties, cf. [ACJG00]. Y. Laurent proved finiteness and rationality in the micro-local setting along general varieties [Lau87, Thm. 3.4.1].

3.2. Hypergeometric system and candidate components. We now define our main object of interest, the hypergeometric D -module $M_A(\beta)$, introduced in [GGZ87, GZK89].

Definition 3.5. The *Euler vector fields* $E = (E_1, \dots, E_d)$ of A are defined by $E_i := \sum_j a_{i,j} x_j \partial_j$ for $i = 1, \dots, d$. The *A-hypergeometric* (or *GKZ*) *system* defined by A and a complex parameter vector $\beta \in \mathbb{C}^d$ is the D -module

$$M_A(\beta) := D/D\langle I_A, E - \beta \rangle$$

on X defined by the toric ideal I_A and the *Euler operators* $E - \beta$.

The A -hypergeometric system is always holonomic, cf. [Ado94]. Our goal is to understand its L -characteristic varieties and slopes under the assumption that

$$(3.2.1) \quad L_x + L_\partial = (c, \dots, c) =: \mathbf{c} \text{ for some rational } c > 0.$$

This guarantees that W is a polynomial ring and E is L -homogeneous of positive L -degree. Note that $M_A(\beta)$ is L -homogeneous if A is L -homogeneous as defined in Definition 2.7. The following statement is a consequence of Buchberger's algorithm.

Lemma 3.6. *One has the identity $\mathrm{gr}^L(DI_A) = WI_A^L$.* \square

The vector fields $t_1\partial_{t_1}, \dots, t_d\partial_{t_d}$ span the tangent space at any point of the torus $\mathbb{T} = (\mathbb{C}^*)^d$. Hence, for any $\xi \in T_0^*X$, the tangent space of the orbit $\mathrm{Orb}(\xi)$ is spanned by the push-forwards $E_i^T := \sum_j a_{i,j}\partial_j(-x_j)$ of $t_i\partial_{t_i}$ under the map $\mathbb{T} \rightarrow T_0^*X$, $t \mapsto t \cdot \xi$ from (2.1.1). Since E_i is L -homogeneous of positive degree with $\sigma^L(E_i) = -\sigma^L(E_i^T)$, the equations $\sigma^L(E - \beta) = 0$ impose the conormal condition to the orbit $\mathrm{Orb}(\xi) \subseteq T_0^*X$ in $T^*T_0^*X = T^*X$. In the sequel we abuse notation by writing $\sigma^L(E) = E$.

Definition 3.7. For a subset τ of $\{1, \dots, n\}$, we denote by $C_A^\tau \subseteq T^*X$ the conormal space to the orbit $O_A^\tau \subseteq T_0^*X$ from Definition 2.2. We denote by $P_C \subseteq W$ the defining ideal of any irreducible variety $C \subseteq T^*X$ and abbreviate $P_{C_A^\tau}$ by P_τ .

Proposition 3.8. *The L -characteristic variety of $M_A(\beta)$ is*

$$\mathrm{Ch}^L(M_A(\beta)) = \bigcup_{\tau \in \phi_A^L(\beta)} \bar{C}_A^\tau = \bigsqcup_{\tau \in \phi_A^L(\beta)} C_A^\tau$$

for some subset $\phi_A^L(\beta) \subseteq \Phi_A^L$.

Proof. By definition, $\mathrm{Ch}^L(M_A(\beta)) \subseteq \mathrm{Var}(W\langle I_A^L, E \rangle)$. By Theorem 2.15 and the preceding arguments,

$$(3.2.2) \quad \mathrm{Var}(W\langle I_A^L, E \rangle) = \bigsqcup_{\tau \in \Phi_A^L} C_A^\tau = \bigcup_{\tau \in \Phi_A^L} \bar{C}_A^\tau.$$

Then Theorem 3.2 assures that $\mathrm{Ch}^L(M_A(\beta))$ is purely n -dimensional and hence a union of closures of conormals C_A^τ for certain $\tau \in \Phi_A^L$. \square

3.3. Existence of facet components. In this subsection, we use an elementary localization argument to give an index formula for the multiplicity of components in the characteristic cycle $\mathrm{CC}^L(M_A(\beta))$ (cf. Definition 4.4) corresponding to facets in the (A, L) -umbrella. It shows in particular that these multiplicities are independent of β and positive, and that all facet components occur in $\mathrm{Ch}^L(M_A(\beta))$. We deduce from this a converse to R. Hotta's Theorem: regular A -hypergeometric systems are homogeneous in the usual sense.

Definition 3.9. We denote by $\mu_{A,0}^{L,C}(\beta)$ the multiplicity of $\mathrm{gr}^L(M_A(\beta))$ along C . (The reason for the appearance of the subscript "0" will become apparent in the next section). This is the length of the W_{P_C} -module $\mathrm{gr}^L(M_A(\beta)) \otimes_W W_{P_C}$. We write $\mu_{A,0}^{L,\tau}(\beta)$ if $C = \bar{C}_A^\tau$.

Theorem 3.10. *For all $\tau \in \Phi_A^{L,d-1}$ and all $\beta \in \mathbb{C}^d$,*

$$\mu_{A,0}^{L,\tau}(\beta) = \nu_A^{L,\tau} = [\mathbb{Z}^d : \mathbb{Z}\tau]$$

and, in particular, $\Phi_A^{L,d-1} \subseteq \phi_A^L(\beta)$.

Proof. Let $\tau \in \Phi_A^{L,d-1}$ and relabel columns such that $\mathbf{a}_1^L, \dots, \mathbf{a}_d^L \in \tau$ and such that $\mathbf{a}_1, \dots, \mathbf{a}_d$ are linearly independent. We have to show that $C_A^\tau \subseteq \mathrm{Ch}^L(M_A(\beta))$. First, we verify that E is a regular sequence on $W[\partial_\tau^{-1}]/W[\partial_\tau^{-1}]I_A^L$. After Gauss reduction on E and multiplying E_i by ∂_i^{-1} , $E_i \equiv x_i$ modulo terms independent of x_j for all $j \leq d$. After a change of the coordinates x_1, \dots, x_d in $W[\partial_\tau^{-1}]$, leaving I_A^L

invariant, $E_i = x_i$. As I_A^L does not involve the variables x_1, \dots, x_d , E is a regular sequence modulo $W[\partial_\tau^{-1}]I_A^L$ as claimed.

Since $W[\partial_\tau^{-1}]I_A^L = \text{gr}^L(D[\partial_\tau^{-1}]I_A)$ by Lemma 3.11 below, $\text{gr}^L(D[\partial_\tau^{-1}]\langle I_A, E - \beta \rangle) = \text{gr}^L(D[\partial_\tau^{-1}]I_A) + \text{gr}^L(D[\partial_\tau^{-1}]\langle E \rangle)$ by the argument in [SST00, Thm. 4.3.5]. Again by Lemma 3.11,

$$\text{gr}^L(M_A(\beta))[\partial_\tau^{-1}] = W[\partial_\tau^{-1}]/W[\partial_\tau^{-1}]\langle I_A^L, E \rangle.$$

Since $\partial_j \notin P_\tau$ for all $j \in \tau$, this yields

$$\text{gr}^L(M_A(\beta))_{P_\tau} = W_{P_\tau}/W_{P_\tau}\langle I_A^L, E \rangle \cong (R_{I_A^L}/I_A^L)(x_{d+1}, \dots, x_n)$$

with the ring isomorphism defined by the above coordinate change and hence

$$\mu_{A,0}^{L,\tau}(\beta) = \ell(\text{gr}^L(M_A(\beta))_{P_\tau}) = \ell(R_{I_A^L}/I_A^L) = \nu_A^{L,\tau}.$$

Then Proposition 2.18 finishes the proof. \square

Lemma 3.11. *For $I \subseteq D$ and $\tau \subseteq \{1, \dots, n\}$, $\text{gr}^L(D[\partial_\tau^{-1}]I) = \text{gr}^L(D[\partial_\tau^{-1}])\text{gr}^L(I)$. In particular, $\text{gr}^L((D/I)[\partial_\tau^{-1}]) = (\text{gr}^L(D/I))[\partial_\tau^{-1}]$.*

Proof. The inclusion $\text{gr}^L(D[\partial_\tau^{-1}]I) \supseteq \text{gr}^L(D[\partial_\tau^{-1}])\text{gr}^L(I)$ holds trivially. For $Q \in D[\partial_\tau^{-1}]$, $\partial_\tau^{\mathbf{u}}Q \in D$ for some $\mathbf{u} \in \mathbb{N}^{|\tau|}$. If $\{P_i\}$ is a finite set of generators for I then the opposite inclusion follows from

$$\sigma^L\left(\sum_i Q_i P_i\right) = \partial_\tau^{-\mathbf{u}}\sigma^L\left(\sum_i \partial_\tau^{\mathbf{u}}Q_i P_i\right) \in \text{gr}^L(D[\partial_\tau^{-1}])\text{gr}^L(I)$$

with a common \mathbf{u} for the finitely many $Q_i \in D[\partial_\tau^{-1}]$. In particular,

$$\begin{aligned} \text{gr}^L((D/I)[\partial_\tau^{-1}]) &= \text{gr}^L(D[\partial_\tau^{-1}]/D[\partial_\tau^{-1}]I) = \text{gr}^L(D[\partial_\tau^{-1}])/\text{gr}^L(D[\partial_\tau^{-1}])\text{gr}^L(I) \\ &= W[\partial_\tau^{-1}]/W[\partial_\tau^{-1}]\text{gr}^L(I) = (W/\text{gr}^L(I))[\partial_\tau^{-1}] = (\text{gr}^L(D/I))[\partial_\tau^{-1}]. \end{aligned} \quad \square$$

Example 3.12. Let A be as in Figure 1; we consider $L = F + tV$ where V is the V -filtration along $\text{Var}(x_4)$, induced by the weight vector $(0, 0, 0, -1, 0, 0, 0, 1)$. We consider specifically $t \in \{0, 1, 4\}$. As these weights are generic, for all $\tau \subseteq \{1, \dots, 4\}$ the conormal closures \bar{C}_A^τ are coordinate subspaces $\text{Var}(x_\tau, \partial_\tau)$. The facets in Φ_A^L for these values of t are shown in Figure 1.

For $t = 0$ one finds $\mu_{A,0}^{L,\{1,4\}}(\beta) = 12$ and $\mu_{A,0}^{L,\{2,4\}}(\beta) = 1$ with sum 13 = $\text{vol}_{\mathbb{Z}^d}(\Delta_A^F)$. (For relevant notation and more information, see Definition 4.19 and the continuation in Example 4.24). For $t = 1$ one finds three facets, with $\mu_{A,0}^{L,\{1,3\}}(\beta) = 3$, $\mu_{A,0}^{L,\{3,4\}}(\beta) = 7$ and $\mu_{A,0}^{L,\{2,4\}}(\beta) = 1$. On the other hand, $t = 4$ yields two facets only, with $\mu_{A,0}^{L,\{1,3\}}(\beta) = 3$ and $\mu_{A,0}^{L,\{2,3\}}(\beta) = 2$. The respective sums, $3 + 7 + 1$ and $3 + 2$, are the degrees of the L -graded toric ideals, cf. Proposition 2.18.

Lemma 3.13. $\Phi_A^{L',d-1}$ jumps at $L' = L$ if and only if $\Phi_A^{L'}$ jumps at $L' = L$.

Proof. The ‘‘only if’’ part in the statement follows trivially from $\Phi_A^{L',d-1} \subseteq \Phi_A^{L'}$. Assume that $\Phi_A^{L',d-1}$ is locally constant at L and let $\tau \in \Phi_A^L$. Then there is a $\tau' \in \Phi_A^{L',d-1}$ such that τ is a face of τ' . By assumption, $\tau' \in \Phi_A^{L',d-1}$ for L' close to L . Since τ' is not contained in a hyperplane through origin and since the $\mathbf{a}_j^{L'}$ depend on L' only by scaling, τ remains a face of τ' for L' close to L . In particular, $\tau \in \Phi_A^{L'}$ for L' close to L and hence $\Phi_A^{L'}$ is locally constant at L . \square

Theorem 3.10 and Lemma 3.13 yield the existence of slopes of $M_A(\beta)$ corresponding to jumps of the (A, L) -umbrella with L as in (3.1.2). The following result on the candidate components \bar{C}_A^τ of $\text{Ch}^L(M_A(\beta))$ allows us to show that all these slopes occur at the origin.

Lemma 3.14. *For all $\tau \in \Phi_A^L$, the candidate component \bar{C}_A^τ meets T_0^*X .*

Proof. Since the ideal $W\langle I_A^L, E \rangle$ is homogeneous in the x -variables, this holds also for its associated prime ideals P_τ where $\tau \in \Phi_A^L$. Thus $\bar{C}_A^\tau = \text{Var}(P_\tau)$ meets T_0^*X as claimed. \square

Corollary 3.15. *For $L = pF + qV$ as in (3.1.2), if Φ_A^L jumps at $L = L'$ then $L' = p'/q'$ is a slope of $M_A(\beta)$ along Y at $0 \in Y$.* \square

In particular, we obtain the following converse to a Theorem by R. Hotta [Hot98].

Corollary 3.16. *Regular A -hypergeometric systems are homogeneous with respect to the order filtration F .*

Proof. The polytope $\text{conv}(A)$ is an intersection of closed half spaces and $0 \notin \text{conv}(A)$ by positivity of NA . Thus there is a facet τ of $\text{conv}(A)$ and a linear form $\ell \in \text{Hom}_{\mathbb{Q}}(\mathbb{Q}^d, \mathbb{Q})$ such that $\ell(\mathbf{a}_i) \geq 1$ with equality equivalent to $i \in \tau$. If $\tau \in \Phi_A^{F, d-1}$ then τ is a facet of $\Delta_A^F = \text{conv}(\{0, \mathbf{a}_1, \dots, \mathbf{a}_n\})$. But then $0 \in \Delta_A^F$ and $\ell(0) = 0 < 1$ implies that $\ell(\mathbf{a}_i) \leq 1$ and hence $\ell(\mathbf{a}_i) = 1$ for $i = 1, \dots, n$. So in this case A is F -homogeneous. Assume now that $\tau \notin \Phi_A^{F, d-1}$ and let $L = F + \frac{q}{p}V$ where $\mathfrak{V} = \bar{\tau}$ in (3.1.1). For $i \in \tau$, $\mathbf{a}_i^L = \mathbf{a}_i$ and hence $\ell(\mathbf{a}_i^L) = 1$. But for $i \notin \tau$ and $p/q \rightarrow 0$, $\mathbf{a}_i^L \rightarrow 0$ and hence eventually $\ell(\mathbf{a}_i^L) < 1$. Thus, $\tau \in \Phi_A^{L, d-1}$ for $p/q \rightarrow 0$ while $\tau \notin \Phi_A^{F, d-1}$ and hence $L \mapsto \Phi_A^L$ can not be constant. Then, by Corollary 3.15, $M_A(\beta)$ has slopes. By [LM99, Thm. 2.4.2], $M_A(\beta)$ is hence irregular. \square

We shall see later in Corollary 4.18 that Corollary 3.15 actually gives a complete list of all slopes along coordinate varieties at the origin.

Remark 3.17. For generic β , the equivalence of the regularity of $M_A(\beta)$ with the homogeneity of A was already obtained in [SST00, Thm. 2.4.11].

4. CHARACTERISTIC CYCLE OF THE EULER–KOSZUL COMPLEX

In this section we assume that $L_x + L_\partial = \mathbf{c}$ is a constant positive vector as in (3.2.1). By way of a dilation we further may, and do, assume that the index set of the L -filtration is \mathbb{Z} .

By Proposition 3.8, all components of the L -characteristic variety of $M_A(\beta)$ are of the form \bar{C}_A^τ where $\tau \in \Phi_A^L$. In this section we prepare the way for Corollary 4.17 which ascertains the presence of every such candidate component. The approach is to consider $M_A(\beta)$ as the 0-th homology of a Koszul type complex by operators $E - \beta$ on the D -module $D \otimes_R N$ for the \mathbb{Z}^d -graded R -module $N = S_A$. For modules N having a composition series with quotients of type S_θ , $\theta \in \Phi_A^0$, we can apply results of the previous sections combined with homological algebra. For basic results on filtered rings and modules we refer to [Sch85, II.§1.1-3].

4.1. Good filtrations and toric modules. In order to combine homological methods with good L -filtrations on D -modules, we need the L -filtration on D to be Noetherian, which means the following.

The *Rees ring* of a ring T with a \mathbb{Z} -indexed filtration F is the graded ring

$$\text{Rees}^F(T) = \bigoplus_{i \in \mathbb{Z}} F_i(T)t^i \subseteq T[t, t^{-1}]$$

and F is called *Noetherian* if $\text{Rees}^F(T)$ is Noetherian.

Lemma 4.1. *The filtrations L on D and L_∂ on R are Noetherian.*

Proof. Let us write $(\mathbf{u}, \mathbf{v}) := (L_x, L_\partial)$, then $\mathbf{u} + \mathbf{v} = \mathbf{c} > 0$ by hypothesis. The elements of $\text{Rees}^L(D)$ are of the form $\sum_{k \in \mathbb{Z}} \sum_{\mathbf{u} \cdot \mathbf{u}' + \mathbf{v} \cdot \mathbf{v}' \leq k} c_{\mathbf{u}', \mathbf{v}', k} x^{\mathbf{u}'} \partial^{\mathbf{v}' } t^k$. Apply the change of variables $x_i \mapsto x_i t^{-u_i}$, $\partial_i \mapsto \partial_i t^{u_i}$ and set $m = k + \mathbf{u} \cdot (\mathbf{v}' - \mathbf{u}')$. Then the elements are transformed to $\sum_{m \in \mathbb{Z}} \sum_{(\mathbf{v} + \mathbf{u}) \cdot \mathbf{v}' \leq m} x^{\mathbf{u}'} \partial^{\mathbf{v}' } t^m$. Thus, $\text{Rees}^L(D) \cong \text{Rees}^{F'}(D)$ where $F' := (0, \mathbf{v} + \mathbf{u}) = (0, \mathbf{c})$, which is Noetherian by [Sch85, II. Prop. 1.1.8] since $F'_0 D = \mathbb{C}[x]$ is Noetherian. Similarly, a change of variables $\partial_i \mapsto \partial_i t^{1-v_i}$ shows that $\text{Rees}^{L_\partial}(R) \cong \text{Rees}^{F_\partial}(R)$ is Noetherian. \square

The *Rees module* of a T -module N with an F -filtration G is the graded $\text{Rees}^F(T)$ -module

$$\text{Rees}^G(N) = \bigoplus_{i \in \mathbb{Z}} G_i(N)t^i \subseteq N[t, t^{-1}].$$

For Noetherian F , G is a good F -filtration on N in the sense of Subsection 3.1 if and only if $\text{Rees}^G(N)$ is Noetherian over $\text{Rees}^F(T)$, cf. [Sch85, II. Prop. 1.1.7]. The advantage of this new definition is the following. Consider a short exact sequence of T -modules

$$(4.1.1) \quad 0 \longrightarrow N' \longrightarrow N \longrightarrow N'' \longrightarrow 0,$$

a good F -filtration G on N , and the induced F -filtrations G' on N' and G'' on N'' . Then the associated Rees sequence

$$0 \longrightarrow \text{Rees}^{G'}(N') \longrightarrow \text{Rees}^G(N) \longrightarrow \text{Rees}^{G''}(N'') \longrightarrow 0$$

is exact and hence G' and G'' are good according to the new definition. By definition of G' and G'' , the maps in the sequence (4.1.1) are strict and hence

$$0 \longrightarrow \text{gr}^{G'}(N') \longrightarrow \text{gr}^G(N) \longrightarrow \text{gr}^{G''}(N'') \longrightarrow 0$$

is exact.

For D -modules of the type $M = D \otimes_R N$ where N is an R -module, a good L_∂ -filtration on N induces a good L -filtration on M by

$$(4.1.2) \quad L_k(D \otimes_R N) = \sum_{i+j=k} (L_x)_i \mathbb{C}[x] \otimes_{\mathbb{C}} (L_\partial)_j N.$$

Here, $(L_x)_i \mathbb{C}[x]$ denotes the level- i piece of the filtration L_x on $\mathbb{C}[x]$. For many purposes, we can replace a given good L -filtration on M by that in (4.1.2) and then Lemma 3.6 generalizes as follows.

Lemma 4.2. *For any R -module N , $\text{gr}^L(D \otimes_R N) = W \otimes_R \text{gr}^{L_\partial}(N)$ as W -modules. \square*

The following definition, cf. [MMW05, Def. 4.5], describes a class of R -modules that arise naturally in the study of A -hypergeometric systems. Recall that, by Theorem 2.15, the \mathbb{Z}^d -graded prime ideals containing I_A are of the form I_A^θ for $\theta \in \Phi_A^0$.

Definition 4.3. A \mathbb{Z}^d -graded R -module N is *toric* if it has a *toric filtration*

$$0 = N_0 \subsetneq N_1 \subsetneq \cdots \subsetneq N_{l-1} \subsetneq N_l = N$$

which means that for all i , $N_i/N_{i-1} = S_{\theta_i}(\mathbf{u}_i)$ for some $\theta_i \in \Phi_A^0$ and some $\mathbf{u}_i \in \mathbb{Z}^d$. The minimal such l is called the *toric length* of N . A *toric morphism* is a \mathbb{Z}^d -graded R -linear map between toric modules.

4.2. Characteristic cycle and Euler–Koszul homology. By Theorem 3.2, the L -characteristic variety $\text{Ch}^L(M)$ is purely n -dimensional for holonomic M .

Definition 4.4. The L -characteristic cycle of a holonomic D -module M is the formal sum of n -dimensional irreducible varieties in T^*X

$$\text{CC}^L(M) := \sum_C \mu^{L,C}(M) \cdot C, \quad \mu^{L,C}(M) := \ell(\text{gr}^L(M)_{P_C}),$$

for some good L -filtration on M .

A bounded complex of D -modules K_\bullet is called *homologically holonomic* if all its homology modules $H_i(K_\bullet)$ are holonomic. The L -characteristic cycle of a homologically holonomic complex of finite D -modules K_\bullet is

$$\text{CC}^L(K_\bullet) := \sum_{i \in \mathbb{Z}} (-1)^i \text{CC}^L(H_i(K_\bullet)) = \sum_C \mu^{L,C}(K_\bullet) \cdot C,$$

where

$$\mu^{L,C}(K_\bullet) := \sum_{i \in \mathbb{Z}} (-1)^i \mu_i^{L,C}(K_\bullet), \quad \mu_i^{L,C}(K_\bullet) := \ell(\text{gr}^L(H_i(K_\bullet))_{P_C}).$$

For a holonomic D -module M considered as a complex with trivial differential concentrated in homological degree zero, the two definitions of $\text{CC}^L(M)$ coincide.

The independence of $\text{CC}^L(M)$ of the particular choice of the good L -filtration follows from [Sch85, II. Prop. 1.3.1.a]. Essentially by definition, the L -characteristic variety $\text{Ch}^L(M)$ of a holonomic M in Definition 3.1 is the union

$$\text{Ch}^L(M) = \bigcup_{\mu^{L,C}(M) > 0} C.$$

For $L = pF + qV$, $\text{CC}^L(M)$ is the algebraic counterpart of the *characteristic cycle of type $r = p/q + 1$* defined by Y. Laurent [Lau87] as explained in Remark 3.3. Lemma 4.1 and [Sch85, II. Prop. 1.3.1.b] yield the following statement.

Lemma 4.5. CC^L is additive. □

Our main technical tool for the study of $\text{CC}(M_A(\beta))$ is the Euler–Koszul functor from [MMW05]. For a \mathbb{Z}^d -graded left D -module M and a \mathbb{Z}^d -homogeneous $y \in M$,

$$(4.2.1) \quad (E_i - \beta_i) \circ y := (E_i - \beta_i + \deg_i(y))y$$

defines d commuting D -linear endomorphisms $E - \beta$ of M .

Definition 4.6. Let N be a \mathbb{Z}^d -graded R -module and $\beta \in \mathbb{C}^d$. The *Euler–Koszul complex* $K_{A,\bullet}(N; \beta)$ is the Koszul complex of the endomorphisms $E - \beta$ on the left D -module $D \otimes_R N$. Its i -th homology $H_{A,i}(N; \beta) := H_i(K_{A,\bullet}(N; \beta))$ is the i -th *Euler–Koszul homology* of N . By [MMW05, Thm. 5.1], $K_{A,\bullet}(N; \beta)$ is homologically holonomic for toric N . Note that $H_{A,0}(S_A; \beta) = M_A(\beta)$.

To a good L_∂ -filtration on a \mathbb{Z}^d -graded R -module N we associate a good L -filtration on $K_{A,\bullet}(N; \beta)$ by setting for $i_1 < \dots < i_k$ the L -degree of the (i_1, \dots, i_k) -th unit vector in $K_{A,k}(R; \beta) = D^{\binom{n}{k}}$ equal to $\sum_{t=1}^k \deg^L(E_{i_t})$, and using Equation (4.1.2). Then Lemma 4.2 implies that

$$(4.2.2) \quad \mathrm{gr}^L(K_{A,\bullet}(N; \beta)) = W \otimes_R K_\bullet(\mathrm{gr}^L(N); E),$$

by which, abusing notation, we mean the usual Koszul complex induced by the L -symbols of E on $W \otimes_R \mathrm{gr}^L(N)$.

Recall Definition 3.9.

Definition 4.7. For a toric module N , we call

$$\mathrm{CC}^L(N) := \mathrm{CC}^L(K_{A,\bullet}(N; \beta)) = \sum_{i=0}^d (-1)^i \mathrm{CC}^L(H_{A,i}(N; \beta))$$

the *L -characteristic Euler–Koszul cycle* of N . We denote the i -th *Euler–Koszul L -multiplicity* and the *Euler–Koszul L -characteristic* of N along C by

$$\mu_{A,i}^{L,C}(N; \beta) := \mu_i^{L,C}(K_{A,\bullet}(N; \beta)), \quad \mu_A^{L,C}(N) := \mu^{L,C}(K_{A,\bullet}(N; \beta)).$$

If $N = S_A$ we skip the argument N , while if $C = \bar{C}_A^\tau$ we write τ rather than \bar{C}_A^τ as upper index.

The independence of β suggested by the notation $\mu_A^{L,C}(N)$ will be established in Theorem 4.11 below. Lemma 4.5 and the long exact Euler–Koszul homology sequence imply the following statement.

Lemma 4.8. $\mu_A^{L,C}$ is additive. □

4.3. Euler–Koszul characteristic and intersection multiplicity. We now interpret $\mu_A^{L,\tau}$ as an intersection multiplicity, cf. [Ser65]. That will lead to an explicit formula in Theorem 4.21.

Definition 4.9. For two W -modules M and M' and a variety $C \subseteq \mathrm{Spec}(W)$ not strictly contained in an irreducible component of $\mathrm{supp}(M \otimes_W M')$, the *intersection multiplicity of M and M' along C* is the alternating sum

$$\chi^C(M, M') := \sum_i (-1)^i \ell \left(\mathrm{Tor}_i^{W_{P_C}}(M_{P_C}, M'_{P_C}) \right)$$

of lengths of W_{P_C} -modules. If $C = \bar{C}_A^\tau$ we write τ rather than \bar{C}_A^τ as upper index.

Lemmas 4.1 and 4.2 and [Sch85, II. Prop. 1.3.1.b] yield the following statements.

Lemma 4.10. Fix a W -module M and a variety $C \subseteq \mathrm{Spec}(W)$. Whenever defined, the quantity $\chi^C(M, W \otimes_R \mathrm{gr}^L(N))$ is independent of the choice of a good L -filtration on the R -module N and additive in N . □

Theorem 4.11. *For any toric module N and all $\beta \in \mathbb{C}^d$,*

$$\mu_A^{L,C}(N) = \chi^C(W/W\langle E \rangle, W \otimes_R \text{gr}^L(N))$$

and is hence independent of β . For any $i \in \mathbb{N}$, $\mu_{A,i}^{L,C}(N; \beta) > 0$ only if $C = \bar{C}_A^\tau$ for some $\tau \in \Phi_A^L$, while $\mu_A^{L,C}(N) > 0$ if and only if $\dim(N) = d$ and $C = \bar{C}_A^\tau$ for some $\tau \in \Phi_A^L$.

Proof. We consider $K_\bullet = K_{A,\bullet}(N; \beta)$ and fix a variety C for which $\mu_A^{L,C}(N)$ is defined. By Corollary A.2 and Equation (4.2.2), there is a sequence of homologically graded W_{P_C} -modules K^1, \dots, K^r with W_{P_C} -linear endomorphisms $d^i: K^i \rightarrow K^{i+1}$ such that

$$K^1 = W_{P_C} \otimes_R K_\bullet(\text{gr}^L(N); E),$$

$K^{i+1} = H(K^i, d^i)$ for $i = 1, \dots, r-1$, and $K^r = \text{gr}^L(H_{A,\bullet}(N; \beta))_{P_C}$ as homologically graded modules. Thus for all $\beta \in \mathbb{C}^d$, $\mu_A^{L,C}(N) = \chi(K^r) = \dots = \chi(K^1)$ is independent of β by additivity of $\chi(K) := \sum_i (-1)^i \ell(H_i(K, d))$. Since E is a regular sequence in W , Koszul homology against E computes Tor over both W and W_{P_C} and hence

$$\begin{aligned} \chi(K^1) &= \sum_{i=1}^d (-1)^i \ell \left(\text{Tor}_i^{W_{P_C}}(W_{P_C}/W_{P_C}\langle E \rangle, W_{P_C} \otimes_R \text{gr}^L(N)) \right) \\ &= \chi^C(W/W\langle E \rangle, W \otimes_R \text{gr}^L(N)). \end{aligned}$$

By Lemmas 4.8 and 4.10 we may assume now that $N = S_\theta$ for some $\theta \in \Phi_A^0$ with the standard good L_θ -filtration induced by the single generator $1 \in S_\theta$. Then the complex K^1 above is the Koszul complex of E on $W_{P_C} \otimes_R S_\theta^L$. By [BH93, Prop. 1.6.5], its homology is annihilated by both $\text{gr}^L(I_A^\theta) = R \cdot I_\theta^L + J_\theta \supseteq I_A^L$ and E . Then Equation (3.2.2) shows that $\mu_{A,i}^{L,C}(S_\theta; \beta) > 0$ for some i implies that $C = \bar{C}_A^\tau$ is an irreducible component of $\text{Spec}(W/W\langle E \rangle \otimes_R S_A^L)$ with $\theta \supseteq \tau \in \Phi_A^L$. By Theorem 2.15,

$$\text{codim}(W/W\langle E \rangle) + \text{codim}(W \otimes_R S_\theta^L) = d + n - \dim(S_\theta) \geq n = \text{codim}(\bar{C}_A^\tau).$$

So Serre's Intersection Theorem [Ser65, V.B.3.1] shows that $\mu_A^{L,\tau}(S_\theta) > 0$ if and only if $\dim(S_\theta) = d$, which is to say $\theta = A$. \square

In the special case $L = F$, Theorem 4.11 yields the holonomicity of Euler–Koszul homology of toric modules proved in [MMW05, Thm. 5.1].

In order to discuss $\text{CC}^L(M_A(\beta))$ we need more information about $\mu_{A,0}^{L,\tau}(\beta)$: Theorem 4.11 does not state that $\mu_{A,0}^{L,\tau}(\beta) > 0$. We show next that for generic β this inequality does indeed hold. In Subsections 4.4 and 4.6 we will show that the genericity assumption is not necessary and in fact identify a global combinatorial lower bound for $\mu_{A,0}^{L,\tau}(\beta)$, similar to classical statements about the holonomic rank of $M_A(\beta)$ in [GZK89].

Corollary 4.12. *For generic (more precisely, not rank-jumping) $\beta \in \mathbb{C}^d$*

$$\mu_{A,0}^{L,\tau}(\beta) = \mu_A^{L,\tau} = \chi^\tau(W/W\langle E \rangle, W \otimes_R S_A^L) > 0$$

for all $\tau \in \Phi_A^L$. In particular, $\text{Ch}^L(M_A(\beta)) = \bigcup_{\tau \in \Phi_A^L} \bar{C}_A^\tau$.

Proof. By assumption on β and [MMW05, Thm. 6.6], the Euler–Koszul complex $K_{A,\bullet}(S_A; \beta)$ is a resolution of $M_A(\beta)$. Hence $\mathrm{CC}^L(K_{A,\bullet}(S_A; \beta)) = \mathrm{CC}^L(M_A(\beta))$ and $\mu_{A,0}^{L,\tau}(\beta) = \mu_A^{L,\tau} > 0$ by Theorem 4.11. So the last claim follows from Proposition 3.8. \square

Recall that \tilde{S}_A is the normalization of S_A . In Section 4.4, we show that $\mu_{A,0}^{L,\tau}(\beta) > 0$ for all β . This is done by reducing to \tilde{S}_A and applying the following statement.

Corollary 4.13. *For all $\tau \in \Phi_A^L$,*

$$\mu_{A,0}^{L,\tau}(\tilde{S}_A; \beta) = \mu_A^{L,\tau}(\tilde{S}_A) = \mu_A^{L,\tau} = \chi^\tau(W/W\langle E \rangle, W \otimes_R S_A^L) > 0.$$

Proof. Since \tilde{S}_A is a Cohen–Macaulay module of full dimension over S_A , its Euler–Koszul complex is a resolution of $H_{A,0}(\tilde{S}_A; \beta)$ by [MMW05, Thm. 6.6]. Hence $\mu_{A,i}^{L,\tau}(\tilde{S}_A; \beta) = 0$ for all $i > 0$ which proves the first equality. There is a short exact sequence

$$(4.3.1) \quad 0 \longrightarrow S_A \longrightarrow \tilde{S}_A \longrightarrow Q \longrightarrow 0$$

where $\dim(Q) < d$ and the rest of the claim follows from Theorem 4.11 and Corollary 4.12. \square

4.4. Rigidity of Euler–Koszul multiplicities. In this section we show in Theorem 4.16 that the strict positivity of $\mu_{A,0}^{L,\tau}(\beta)$ in Corollary 4.12 holds without the genericity assumption. As a consequence, we obtain a complete description of the L -characteristic variety and the slopes of the A -hypergeometric system along coordinate varieties at the origin.

Definition 4.14. Let τ be a subset of the column set of A . We set $W_\tau = \mathbb{C}[x_\tau, \partial_\tau]$ so that $W/WI_A^\tau = \mathbb{C}[x_{\bar{\tau}}] \otimes_{\mathbb{C}} (W_\tau \otimes_{R_\tau} S_\tau)$.

Let $g \in \mathrm{Gl}(\mathbb{Z}, d)$ and put $A' = gA$ and $\beta' = g\beta$. Then note that $S_A = S_{A'}$ and $E' - \beta' = g(E - \beta)$. Hence, the Euler–Koszul complexes $K_{A,\bullet}(S_A; \beta)$ and $K_{A',\bullet}(S_{A'}; \beta')$ are homotopy equivalent. In particular, when investigating homotopy invariant properties of $K_{A,\bullet}(S_A; \beta)$ one may replace A by gA .

Lemma 4.15. *Let $\theta \in \Phi_A^0$ and assume that the top $\dim(\theta)$ rows A' of θ are a lattice basis for the \mathbb{Z} -row span of θ . Then, for all $\tau \in \Phi_A^L$,*

$$\mu_{A,i}^{L,\tau}(S_\theta; \beta) = \begin{cases} \mu_{A',i}^{L',\tau'}(S_{\theta'}; \beta') & \text{if } \tau \subseteq \theta; \\ 0 & \text{if } \tau \not\subseteq \theta; \end{cases}$$

where $L' = (L_{x_{A'}}, L_{\partial_{A'}})$, $\tau = \tau'$, $\theta' = A'$, and $\beta' := (\beta_1, \dots, \beta_{\dim(\theta)})$.

Proof. By [MMW05, Lem. 4.8], $H_{A,i}(S_\theta; \beta) = \mathbb{C}[x_{\bar{\theta}}] \otimes_{\mathbb{C}} H_{A',i}(S_{\theta'}; \beta')$ and hence

$$\mathrm{gr}^L(H_{A,i}(S_\theta; \beta)) = \mathbb{C}[x_{\bar{\theta}}] \otimes_{\mathbb{C}} \mathrm{gr}^{L'}(H_{A',i}(S_{\theta'}; \beta')).$$

For $j \in \tau \setminus \theta$, ∂_j is both annihilator and unit on $\mathrm{gr}^L(H_{A,i}(S_\theta; \beta))_{P_\tau}$ and hence $\tau \subseteq \theta$ if $\mu_{A,i}^{L,\tau}(S_\theta; \beta) \neq 0$. Assume now that $\tau \subseteq \theta$. Then

$$W_{P_\tau} \otimes_W \mathrm{gr}^L(H_{A,i}(S_\theta; \beta)) = \mathbb{C}(x_{\bar{\theta}}) \otimes_{\mathbb{C}} \left((W_{A'})_{P_\tau} \otimes_{W_{A'}} \mathrm{gr}^{L'}(H_{A',i}(S_{\theta'}; \beta')) \right)$$

where the symbol P_τ is used for the two ideals in W_A and $W_{A'}$ from Definition 3.7 induced by $\tau \in \Phi_A^L$ and $\tau \in \Phi_{A'}^{L'}$. In particular, the length of $W_{P_\tau} \otimes_W \mathrm{gr}^L(H_{A,i}(S_\theta; \beta))$

over W_{P_τ} is the same as the length of $(W_{A'})_{P_\tau} \otimes_{W_{A'}} \text{gr}^{L'}(H_{A',i}(S_{\theta'}; \beta'))$ over $(W_{A'})_{P_\tau}$. \square

Theorem 4.16. *For any toric module N and for every parameter $\beta \in \mathbb{C}^d$, if $\mu_{A,i}^{L,\tau}(N; \beta) > 0$ for some $i > 0$ then $\mu_{A,0}^{L,\tau}(N; \beta) > 0$.*

Proof. Suppose there is a counterexample (A, β, L, τ, N) to the theorem. This means that $\mu_{A,i}^{L,\tau}(N; \beta) \neq 0$ for some $i > 0$ while $\mu_{A,0}^{L,\tau}(N; \beta) = 0$. We choose a minimal counterexample in the sense that

1. A has minimal number of columns,
2. for a fixed such A , $\dim(N)$ is minimal, and
3. no quotient of N is part of such a counterexample.

This last choice can be made since toric modules are Noetherian. Let $\theta \in \Phi_A^0$ such that I_A^θ is an associated prime of N . Then there is a short exact sequence

$$0 \longrightarrow S_\theta(-\mathbf{a}) \longrightarrow N \longrightarrow N' \longrightarrow 0$$

where $\mathbf{a} \in \mathbb{Z}^d$ is the degree of the image of $1 \in S_\theta$ within N . The long exact Euler–Koszul homology sequence shows that $H_{A,0}(N'; \beta)$ is a quotient of $H_{A,0}(N; \beta)$. Recall that by Lemma 4.1 the L -characteristic cycle is additive. Since by hypothesis $\mu_{A,0}^{L,\tau}(N; \beta) = 0$, we conclude that $\mu_{A,0}^{L,\tau}(N'; \beta) = 0$ as well.

By assumption, (A, β, L, τ, N') is not a counterexample and so $\mu_{A,i}^{L,\tau}(N'; \beta) = 0$ for all i . It follows that the $\mu_{A,i}^{L,\tau}$ -vanishing patterns of N and of $S_\theta(-\mathbf{a})$ coincide. In particular, $S_\theta(-\mathbf{a})$ is a counterexample. Since $\dim(S_\theta) \leq \dim(N)$, and since S_θ is a domain, $S_\theta(-\mathbf{a})$ is actually a minimal counterexample. Since $H_{A,i}(S_\theta(-\mathbf{a}); \beta) = H_{A,i}(S_\theta; \beta - \mathbf{a})$ up to a \mathbb{Z}^d -shift we may assume that $N = S_\theta$. But if $\theta \neq A$ then Lemma 4.15 yields a counterexample whose matrix A' has strictly fewer columns than A . Therefore $\theta = A$ by minimality of A and (A, β, L, τ, S_A) is a minimal counterexample.

By Corollary 4.13, $\mu_{A,0}^{L,\tau}(\tilde{S}_A; \beta) > 0$ where \tilde{S}_A is the normalization of S_A . For appropriate $\mathbf{a} \in \mathbb{Z}^d$, there is a short exact sequence

$$0 \longrightarrow \tilde{S}_A(-\mathbf{a}) \longrightarrow S_A \longrightarrow Q \longrightarrow 0$$

where Q is not a counterexample as $\dim(Q) < d$. By additivity of the L -characteristic cycle, $\mu_{A,0}^{L,\tau}(S_A; \beta) = 0$ implies $\mu_{A,0}^{L,\tau}(Q; \beta) = 0$ and so, as Q is not a counterexample, $\mu_{A,i}^{L,\tau}(Q; \beta) = 0$ for all i . This yields $\mu_{A,i}^{L,\tau}(S_A; \beta) = \mu_{A,i}^{L,\tau}(\tilde{S}_A; \beta)$ for all i which is a contradiction for $i = 0$. \square

Finally, from Theorem 4.16, Corollary 4.12 and Theorem 4.11, we obtain the L -characteristic variety of the A -hypergeometric system and, as a consequence, the slopes of the A -hypergeometric system along coordinate varieties at the origin.

Corollary 4.17. *For all $\tau \in \Phi_A^L$ and $\beta \in \mathbb{C}^d$, $\mu_{A,0}^{L,\tau}(\beta) > 0$. In consequence, the L -characteristic variety of $M_A(\beta)$ consists for every β precisely of all conormal closures of the torus orbits indexed by the faces in the (A, L) -umbrella:*

$$\text{Ch}^L(M_A(\beta)) = \bigcup_{\tau \in \Phi_A^L} \bar{C}_A^\tau. \quad \square$$

Corollary 4.18. *For $L = pF + qV$ as in (3.1.2), $L' = p'/q'$ is a slope of $M_A(\beta)$ along Y at $0 \in Y$ if and only if Φ_A^L jumps at $L = L'$. \square*

4.5. Euler–Koszul characteristic and volume. In this section we develop an explicit combinatorial formula for the Euler–Koszul characteristic $\mu_A^{L,\tau}$ for all $\tau \in \Phi_A^L$, Theorem 4.21 below. For generic β , this formula determines the characteristic cycle of $M_A(\beta)$ as stated in Corollary 4.17 below, but in general it provides only a lower bound.

Note that $\text{rk}(M_A(\beta)) = \mu_{A,0}^{F,\emptyset}(\beta)$, and that $\mu_{A,0}^{L,\tau}(\beta) = \mu_A^{L,\tau}$ if β is not a rank-jumping parameter by [MMW05, Thm. 6.6]. It is a classical result that $\text{rk}(M_A(\beta)) = \text{vol}_{\mathbb{Z}^d}(\Delta_A^F)$ for generic β [GZK89, Ado94]. Theorem 4.21 below contains this result as a special case when $\tau = \emptyset$ and $L = F$. Our generalization of “generic rank equals volume” reads

$$\mu_A^{L,\emptyset} = \text{vol}_{\mathbb{Z}^d} \left(\bigcup_{\tau \in \Phi_A^{L,d-1}} (\Delta_0^F \setminus \text{conv}(\tau)) \right).$$

This paragraph outlines our strategy towards Theorem 4.21. By Theorem 4.11, $\mu_A^{L,\tau}$ is an intersection multiplicity and hence additive. Applying $\mu_A^{L,\tau}$ to the composition chain (2.7.1) of S_A^L yields

$$\begin{aligned} (4.5.1) \quad \mu_A^{L,\tau} &= \chi^\tau(W/\langle E \rangle, S_A^L) = \sum_{i=1}^l \chi^\tau(W/\langle E \rangle, S_{\tau_i}) \\ &= \sum_{\tau \subseteq \tau' \in \Phi_A^{L,d-1}} \nu_A^{L,\tau'} \cdot \chi^\tau(W/\langle E \rangle, S_{\tau'}) \\ &= \sum_{\tau \subseteq \tau' \in \Phi_A^{L,d-1}} [\mathbb{Z}^d : \mathbb{Z}\tau'] \cdot \mu_A^{L,\tau}(S_{\tau'}). \end{aligned}$$

For the third equality note that $(R/I_A^{\tau'})_{I_A^\tau} = 0$ unless $\tau \subseteq \tau'$, and then use $\dim(S_{\tau'}) = \dim(\tau') + 1$ in Theorem 4.11. Proposition 2.18 gives the last equality. By Equation (4.5.1) it is sufficient to know the multiplicities $\mu_A^{L,\tau}(S_{\tau'})$ for $\tau \subseteq \tau' \in \Phi_A^{L,d-1}$. To determine these we first reduce to $\tau' = A$ and then to $\tau = \emptyset$ while controlling the appropriate intersection multiplicity. Finally, we give a combinatorial description in that case. Some of our arguments are similar to [GZK89, §2.3–4] and [Ado94, §5].

Let $\Phi_A^L \ni \tau \subseteq \tau' \in \Phi_A^{L,d-1}$ and abbreviate $\mu := \mu_A^{L,\tau}(S_{\tau'}) = \chi^\tau(W/\langle E \rangle, S_{\tau'})$. Since E is a regular sequence in W , μ can be computed from the Koszul complex of E on $W_{P_\tau} \otimes_R S_{\tau'}$. Since $\partial_{\tau'}$ is zero on $S_{\tau'}$ one can erase the $\partial_{\tau'}$ -terms in E . Now the (canonical basis of the) ambient lattice $\mathbb{Z}^d = \mathbb{Z}A$ can be replaced by (a lattice basis of) $\mathbb{Z}\tau'$; this leaves the quantities W , $\langle E \rangle$, I_τ and $S_{\tau'}$ invariant. We have thus reduced the problem to the case $\tau' = A$.

We now reduce to the case $\tau = \emptyset$. By the Fourier transform we may use ∂ as base variables and identify $-x_j \in W$ with the (symbol of the) partial derivation along ∂_j . Since χ^τ is determined at a generic point of C_A^τ , we may replace W by $W' = W[\partial_\tau^{-1}]$ whose spectrum T^*U is the cotangent space of $U = \text{Spec}(R[\partial_\tau^{-1}])$.

We now modify the pair (τ', τ) into a more convenient one, while keeping track of μ . A row operation $(\hat{\tau}', \hat{\tau}) = g(\tau', \tau)$ defined by $g \in \text{Gl}(\mathbb{Z}, d)$ amounts to a basis change in the d -torus \mathbb{T} , so $\mu = \mu_{\hat{\tau}'}^{L,\hat{\tau}}(S_{\hat{\tau}'})$. On the other hand, for a triple $j \in \tau'$, $i \in \tau$, $m \in \mathbb{Z}$, the elementary column operation $\hat{\mathbf{a}}_j = \mathbf{a}_j + m\mathbf{a}_i$ transforming τ' into $\hat{\tau}'$ corresponds to the automorphism $\hat{\partial}_j = \partial_j \partial_i^m$ of U . The induced map on W'

obviously transforms $I_{\tau'}$ into $I_{\hat{\tau}'}$. Note that $x_i \partial_i = \hat{x}_i \hat{\partial}_i + m \hat{x}_j \hat{\partial}_j$ and $x_k \partial_k = \hat{x}_k \hat{\partial}_k$ for $k \neq i$. Thus, $\mu = \mu_{\hat{\tau}'}^{L, \hat{\tau}'}(S_{\hat{\tau}'})$. In other words, all elementary column operations $\mathbf{a}_j \mapsto \mathbf{a}_j + m \mathbf{a}_i$ with $i \in \tau$, all column switches $\mathbf{a}_i \leftrightarrow \mathbf{a}_j$ with $i, j \in \tau$ or $i, j \notin \tau$, and all \mathbb{Z} -invertible row operations leave μ invariant.

After a suitable sequence of such transformations, we may assume that

$$\tau = \begin{pmatrix} I & 0 \\ 0 & 0 \end{pmatrix}, \quad I = \begin{pmatrix} k_1 & & 0 \\ & \ddots & \\ 0 & & k_e \end{pmatrix}$$

where $k_i \in \mathbb{N} \setminus \{0\}$. Note that the product $k_1 \cdots k_e$ is the index $[(\mathbb{Z}\tau' \cap \mathbb{Q}\tau) : \mathbb{Z}\tau]$ of $\mathbb{Z}\tau$ in $\mathbb{Z}\tau' \cap \mathbb{Q}\tau$.

We next consider the k_1 -fold covering space $\kappa_1 : U \rightarrow U$ induced by $\hat{\partial}_1^{k_1} = \partial_1$. Let $(\hat{\tau}, \hat{\tau}')$ be the matrices obtained from (τ, τ') by dividing the first column by k_1 and let \hat{E} be the Euler vector fields of $\hat{\tau}$. As one checks, $\hat{x}_1 \hat{\partial}_1 = k_1 x_1 \partial_1$ and so $\kappa_1(\text{Var}(\hat{E})) = \text{Var}(E)$, $\kappa_1(\text{Var}(I_{\hat{\tau}'})) = \text{Var}(I_{\tau'})$, and $\kappa_1(C_A^{\hat{\tau}}) = C_A^{\tau}$. Moreover, the degree of κ_1 is equal to k_1 on C_A^{τ} , and equal to 1 on both $\text{Var}(E)$ and $\text{Var}(I_{\tau'})$. By Example 8.2.5 in [Ful98],

$$\mu = k_1 \cdot \chi^{\hat{\tau}}(W/\langle \hat{E} \rangle, S_{\hat{\tau}'}) .$$

Analogous transformations for k_2, \dots, k_e followed by suitable elementary column operations as above yield

$$\hat{\tau} = \begin{pmatrix} I & 0 \\ 0 & 0 \end{pmatrix}, \quad \hat{\tau}' = \begin{pmatrix} I & 0 & 0 \\ 0 & 0 & \check{\tau} \end{pmatrix}, \quad I = \begin{pmatrix} 1 & & 0 \\ & \ddots & \\ 0 & & 1 \end{pmatrix},$$

where now

$$\mu = k_1 \cdots k_e \cdot \chi^{\hat{\tau}}(W/E_{\hat{\tau}'}, S_{\hat{\tau}'}) = [(\mathbb{Z}\tau' \cap \mathbb{Q}\tau) : \mathbb{Z}\tau] \cdot \chi^{\hat{\tau}}(W/E_{\hat{\tau}'}, S_{\hat{\tau}'}) .$$

Note that $\check{\tau}$ is pointed since τ is a face of τ' . Let \check{E} be the Euler vector fields of $\check{\tau}$. Then on T^*U , $\langle E \rangle = \langle \hat{x}_1 \hat{\partial}_1, \dots, \hat{x}_e \hat{\partial}_e, \check{E} \rangle = \langle \hat{x}_1, \dots, \hat{x}_e, \check{E} \rangle$ as well as $I_{\hat{\tau}'} = \langle \hat{\partial}_{e+1}, \dots, \hat{\partial}_{|\tau|} \rangle + I_{\check{\tau}}$. This establishes the announced reduction to the case $\tau = \emptyset$,

$$\mu = [(\mathbb{Z}\tau' \cap \mathbb{Q}\tau) : \mathbb{Z}\tau] \cdot \chi^{\emptyset}(W/\langle \check{E} \rangle, S_{\check{\tau}}) = [(\mathbb{Z}\tau' \cap \mathbb{Q}\tau) : \mathbb{Z}\tau] \cdot \mu_{\check{\tau}}^{L, \emptyset}(S_{\check{\tau}}) .$$

The constructed lattice $\mathbb{Z}\check{\tau} \subseteq \mathbb{Z}\tau' \subseteq \mathbb{Z}^d$ corresponds to a splitting of the natural projection

$$\pi_{\tau, \tau'} : \mathbb{Z}\tau' \longrightarrow \mathbb{Z}\tau' / (\mathbb{Z}\tau' \cap \mathbb{Q}\tau)$$

which identifies the positive semigroups $\mathbb{N}\check{\tau}$ and $\pi_{\tau, \tau'}(\mathbb{N}\tau')$.

Definition 4.19. In a lattice Λ , the volume function vol_{Λ} is normalized so that the unit simplex of Λ has volume 1. We abbreviate $\text{vol}_{\tau, \tau'} := \text{vol}_{\pi_{\tau, \tau'}(\mathbb{Z}\tau')}$.

We continue under the assumption that $\tau = \emptyset$, $\check{\tau} = \tau' = A$, and compute the intersection multiplicity of $\text{Var}(E)$ with $\text{Var}(I_A)$ along $C_A^{\emptyset} = X$ inside $T^*U = \text{Spec}(W')$,

$$\check{\mu} := \mu_{\check{\tau}}^{L, \emptyset}(S_{\check{\tau}}) = \chi^{\emptyset}(W/\langle \check{E} \rangle, S_{\check{\tau}}) .$$

Since for $y \in (\mathbb{C}^*)^{|\check{\tau}|}$ the linear polynomials $x_1 - y_1, \dots, x_{|\check{\tau}|} - y_{|\check{\tau}|}$ form a regular sequence on $W/\langle E \rangle$, on $S_{\check{\tau}}$ and on W/P_{\emptyset} , we may replace W by R , E_i by $\bar{E}_i :=$

$\sum_j a_{i,j} y_j \partial_j$, and C_A^\emptyset by $\{0\} = \text{Var}(\partial_{\tilde{\tau}})$. By the sequence (4.3.1), we may further replace $S_{\tilde{\tau}}$ by its normalization $\tilde{S}_{\tilde{\tau}}$. Thus, we have reduced to computing

$$\check{\mu} = \chi^{\{0\}}(R/\langle \bar{E} \rangle, \tilde{S}_{\tilde{\tau}}).$$

For generic y , the function on $\text{Spec}(\tilde{S}_{\tilde{\tau}})$

$$f := \sum_{j=1}^{|\tilde{\tau}|} y_j \partial_j = \sum_{j=1}^{|\tilde{\tau}|} y_j t^{\mathbf{a}_j}$$

is by [Kou76, Thm. 6.1] and its proof Newton non-degenerate at the origin. We can interpret \bar{E} as functions on $\text{Spec}(\tilde{S}_{\tilde{\tau}})$,

$$\bar{E}_i = t_i \frac{\partial f}{\partial t_i}.$$

Let $\mathfrak{m}_{\tilde{\tau}} \subseteq S_{\tilde{\tau}}$ and $\tilde{\mathfrak{m}}_{\tilde{\tau}} \subseteq \tilde{S}_{\tilde{\tau}}$ be the maximal ideals at 0. In the special case where $\tilde{S}_{\tilde{\tau}}$ a polynomial ring, [Kou76, Thm. 2.8] states that the Koszul complex of \bar{E} on $\tilde{S}_{\tilde{\tau}}$ is acyclic in positive dimension when completed at $\tilde{\mathfrak{m}}_{\tilde{\tau}}$. The $\mathfrak{m}_{\tilde{\tau}}$ - and $\tilde{\mathfrak{m}}_{\tilde{\tau}}$ -adic topologies on $\tilde{S}_{\tilde{\tau}}$ are equivalent and completion at $\tilde{\mathfrak{m}}_{\tilde{\tau}}$ can be replaced by completion at $\mathfrak{m}_{\tilde{\tau}}$. By faithful flatness of completion this implies that the Koszul complex of \bar{E} on $(\tilde{S}_{\tilde{\tau}})_{\mathfrak{m}_{\tilde{\tau}}}$ is acyclic in positive dimension and hence

$$\check{\mu} = \dim_{\mathbb{C}}((\tilde{S}_{\tilde{\tau}})_{\mathfrak{m}_{\tilde{\tau}}}/\langle \bar{E} \rangle).$$

Since this dimension is finite, localization can be replaced by completion at $\mathfrak{m}_{\tilde{\tau}}$ or equivalently at $\tilde{\mathfrak{m}}_{\tilde{\tau}}$. Then [Kou76, Thm. A.I] states that this dimension equals

$$\check{\mu} = \text{vol}_{\mathbb{Z}\tilde{\tau}}(\Gamma_-(f)), \quad \Gamma_-(f) = \text{conv}(\tilde{\tau} \cup \{0\}) \setminus \text{conv}(\tilde{\tau}).$$

One checks that Kouchnirenko's result applies to a general normal semigroup ring $\tilde{S}_{\tilde{\tau}}$ for which $\tilde{\tau}$ is pointed, cf. [Kou76, §2.12]. The following definition serves to interpret this volume in terms of the original matrix A , and mirrors the one given in [GZK89, §2.1].

Definition 4.20. For $\Phi_A^L \ni \tau \subseteq \tau' \in \Phi_A^{L,d-1}$, define the polyhedra

$$P_{\tau,\tau'} := \text{conv}(\pi_{\tau,\tau'}(\tau' \cup \{0\})), \quad Q_{\tau,\tau'} := \text{conv}(\pi_{\tau,\tau'}(\tau' \setminus \tau)).$$

We are now ready to give the promised multiplicity formula.

Theorem 4.21. For all $\tau \in \Phi_A^L$, the multiplicity of \bar{C}_A^τ in $\text{CC}^L(K_{A,\bullet}(S_A; \beta))$ is

$$\mu_A^{L,\tau} = \sum_{\tau \subseteq \tau' \in \Phi_A^{L,d-1}} [\mathbb{Z}^d : \mathbb{Z}\tau'] \cdot [(\mathbb{Z}\tau' \cap \mathbb{Q}\tau) : \mathbb{Z}\tau] \cdot \text{vol}_{\tau,\tau'}(P_{\tau,\tau'} \setminus Q_{\tau,\tau'}). \quad \square$$

Corollary 4.22. If $\tau = \emptyset$, then the local degree of S_A^L at the origin equals

$$\mu_A^{L,\emptyset} = \text{vol}_{\mathbb{Z}^d} \left(\bigcup_{\tau' \in \Phi_A^{L,d-1}} (\Delta_{\tau'}^F \setminus \text{conv}(\tau')) \right) \leq \text{vol}_{\mathbb{Z}^d}(\Delta_A^F).$$

Proof. Let $\tau' \in \Phi_A^{L,d-1}$. For $\tau = \emptyset$, $\text{vol}_{\tau,\tau'} = \text{vol}_{\mathbb{Z}\tau'}$, $P_{\tau,\tau'} = \Delta_{\tau'}^F$, and $Q_{\tau,\tau'} = \text{conv}(\tau')$. Clearly, $[(\mathbb{Q}\tau \cap \mathbb{Z}\tau') : \mathbb{Z}\tau] = 1$ and $\text{vol}_{\mathbb{Z}\tau'}(\Delta_{\tau'}^F) = \text{vol}_{\mathbb{Z}^d}(\Delta_{\tau'}^F)/[\mathbb{Z}^d : \mathbb{Z}\tau']$. Then Theorem 4.21 implies the equality while the inequality is obvious. \square

Remark 4.23. If all facets τ' of the (A, L) -umbrella are F -homogeneous then the volumes $\text{vol}_{\mathbb{Z}^d}(\text{conv}(\tau'))$ are zero and one obtains

$$\mu_A^{L, \emptyset} = \text{vol}_{\mathbb{Z}^d} \left(\bigcup_{\tau' \in \Phi_A^{L, d-1}} \Delta_{\tau'}^F \right).$$

In particular, $\mu_A^{F, \emptyset} = \text{vol}_{\mathbb{Z}^d}(\Delta_A^F)$.

Example 4.24. We continue Example 3.12 and investigate characteristic cycles. The following table, whose rows are indexed by the three weight vectors $L = F + tV$ considered in Example 3.12, lists in its columns the nonzero multiplicities $\mu_{A,0}^{L,\tau}(\beta)$ for generic β . Genericity implies that $\mu_{A,i}^{L,\tau}(\beta) = 0$ for $i > 0$, so that $\mu_{A,0}^{L,\tau}(\beta) = \mu_{A,0}^{L,\tau}$.

$t \backslash \tau$	\emptyset	$\{1\}$	$\{2\}$	$\{3\}$	$\{4\}$	$\{1,3\}$	$\{1,4\}$	$\{2,3\}$	$\{2,4\}$	$\{3,4\}$
0	13	12	1		13		12		1	
1	11	3	1	10	8	3			1	7
4	5	3	2	5		3		2		

The matrix A permits two rank-jumping parameters, $\mathcal{E}(A) = \{(2, 1), (3, 1)\}$. In both cases, $H_{A,1}(S_A; \beta) \cong D/\langle \partial_1, \dots, \partial_n \rangle$ while $H_{A,2}(S_A; \beta) \cong 0$. As $\langle \partial \rangle = P_\emptyset$, with $\beta \in \mathcal{E}(A)$ one has $\mu_{A,1}^{L,\tau}(\beta) = 1$ if $\tau = \emptyset$, and $\mu_{A,1}^{L,\tau}(\beta) = 0$ otherwise. (This holds for all $L \in \mathbb{Q}^{2n}$ as long as $L_x + L_\partial > 0$ since $\text{gr}^L(\langle \partial_1, \dots, \partial_n \rangle)$ is independent of L). In accordance with Corollary 4.13, the multiplicities $\mu_{A,0}^{L,\tau}(\beta)$ for $\beta \in \mathcal{E}(A)$ are given by the data in the table above, with $\mu_{A,0}^{L,\emptyset}(\beta)$ incremented by one. This behavior is typical in dimension two as the following proposition shows.

Proposition 4.25. *If $d = 2$ then for every rank-jumping β and for all L with $L_x + L_\partial > 0$ one has $\mu_{A,1}^{L,\tau}(\beta) = 1$ if $\tau = \emptyset$ and $\mu_{A,1}^{L,\tau}(\beta) = 0$ otherwise.*

Proof. Recall that \tilde{S}_A is the Cohen–Macaulay S_A -module $\bigoplus_{\mathbf{a} \in \mathbb{Q}_+ A \cap \mathbb{Z}^d} t^{\mathbf{a}}$. We first construct the minimal toric submodule of \tilde{S}_A that contains S_A and satisfies Serre’s condition S_2 . For, apply the local cohomology functor $H_{\mathfrak{m}}^\bullet(-)$ to the sequence (4.3.1); we obtain $H_{\mathfrak{m}}^0(Q) = H_{\mathfrak{m}}^1(S_A)$. Let C be the (toric) preimage of $H_{\mathfrak{m}}^0(Q)$ under the projection $\tilde{S}_A \rightarrow Q$; as $C \subseteq \tilde{S}_A$ we have $H_{\mathfrak{m}}^0(C) = 0$. Applying $H_{\mathfrak{m}}^\bullet(-)$ to

$$(4.5.2) \quad 0 \longrightarrow S_A \longrightarrow C \longrightarrow H_{\mathfrak{m}}^0(Q) \longrightarrow 0$$

we find an exact piece $0 \rightarrow H_{\mathfrak{m}}^0(H_{\mathfrak{m}}^0(Q)) \rightarrow H_{\mathfrak{m}}^1(S_A) \rightarrow H_{\mathfrak{m}}^1(C) \rightarrow H_{\mathfrak{m}}^1(H_{\mathfrak{m}}^0(Q)) = 0$. As the second arrow is an isomorphism by construction, the two-dimensional module C has depth two and is hence Cohen–Macaulay.

Application of the Euler–Koszul functor to the short exact sequence (4.5.2) shows that $H_{A,1}(S_A; \beta) \cong H_{A,2}(H_{\mathfrak{m}}^0(Q); \beta)$. Since $H_{\mathfrak{m}}^0(Q)$ is a toric Artinian quotient of C , it has a toric filtration whose quotients are of the form $R/\langle \partial_1, \dots, \partial_n \rangle \cdot t^\beta$ where $\beta \in \mathcal{E}(A)$ and each such β occurs exactly once (cf. [MMW05, Thm. 6.6, Thm. 9.1]). Hence, $H_{A,1}(S_A; \beta) \cong D/\langle \partial_1, \dots, \partial_n \rangle$ if $\beta \in \mathcal{E}(A)$ and zero otherwise. It follows as in the example above that $\text{gr}^L(H_{A,1}(S_A; \beta)) \cong W/\langle \partial_1, \dots, \partial_n \rangle$ in the non-vanishing case. \square

Remark 4.26. (1) The module C constructed in the above proof is actually a ring, the *ideal transform* of S_A relative to \mathfrak{m} from [BS98, Thm. 2.2.4].

(2) It is suggestive, particularly in the light of Example 4.24, to view $\mu_A^{L,\emptyset}$ as “ L -rank”. Specifically, one might speculate whether the L -rank detects rank jumps in the sense that $\mu_{A,i}^{L,\emptyset}(\beta) = 0$ for all $i > 0$ implies that β is not rank-jumping. We think that this is plausible.

4.6. Generic Euler–Koszul homology. In [GZK89], I.M. Gel’fand, M.M. Kapranov and A.V. Zelevinskiĭ proved that in the projective case the rank of $M_A(\beta)$ is always at least equal to $\text{vol}_{\mathbb{Z}^d}(\Delta_A^F) = \mu_A^{F,\emptyset}$, cf. Remark 4.23; this was generalized by A. Adolphson in [Ado94]. In this section, we prove that every Euler–Koszul multiplicity $\mu_{A,0}^{L,\tau}(\beta)$ is bounded from below by the Euler–Koszul characteristic $\mu_A^{L,\tau}$ for all β . This yields some evidence for the following conjecture.

Conjecture 4.27. *For fixed A , L and τ , $\mu_{A,0}^{L,\tau}(\beta)$ is upper semi-continuous in β .*

A proof of this would generalize [MMW05, Thm. 2.6, Thm. 7.5] which correspond to $\tau = \emptyset$ and $L = F$. However, we do not know how to approach this question since it involves a flat deformation in combination with a specialization. The case where $\tau = \emptyset$ and $L = F$ is much easier since there one may skip the computation of the graded by the Cauchy–Kovalevskaya–Kashiwara Theorem [SST00, Thm. 1.4.14, Thm. 1.4.19].

Theorem 4.28. *For any $\tau \in \Phi_A^L$ and $\beta \in \mathbb{C}^d$, $\mu_{A,0}^{L,\tau}(\beta) \geq \mu_A^{L,\tau}$. Equality holds if β is generic (more precisely, not rank-jumping).*

Proof. The sequence (4.3.1) yields a four-step exact sequence

$$0 \longrightarrow H_{A,1}(Q; \beta) \longrightarrow H_{A,0}(S_A; \beta) \longrightarrow H_{A,0}(\tilde{S}_A; \beta) \longrightarrow H_{A,0}(Q; \beta) \longrightarrow 0.$$

By Corollary 4.13 and Lemma 4.29 below,

$$\mu_{A,0}^{L,\tau}(S_A; \beta) \geq \mu_{A,0}^{L,\tau}(\tilde{S}_A; \beta) = \mu_A^{L,\tau}.$$

The second claim is part of Corollary 4.12. □

Lemma 4.29. *For any toric module N with $\dim(N) < d$, $\tau \in \Phi_A^L$, and $\beta \in \mathbb{C}^d$,*

$$\mu_{A,0}^{L,\tau}(N; \beta) \leq \mu_{A,1}^{L,\tau}(N; \beta).$$

Proof. Let $(A', \beta') = (gA, g\beta)$ for some $g \in \text{Gl}(\mathbb{Z}, d)$. Then the Euler–Koszul complexes $K_{A,\bullet}(N; \beta)$ and $K_{A',\bullet}(N; \beta')$ are homotopic for any toric module N . Hence we may replace (A, β) by (A', β') and choose g generic. Fix the toric module N of dimension less than d . Then the Koszul complex $K_{A,\bullet}(N; \beta')$ of the endomorphisms $E' - \beta' := (E_1 - \beta_1, \dots, E_{d-1} - \beta_{d-1})$ on the \mathbb{Z}^d -graded left D -module $D \otimes_R N$, defined as in Definition 4.6, is homologically holonomic. To see this it is by the long Euler–Koszul homology sequence enough to consider a length one toric module $N = S_\theta$ for some $\theta \in \Phi_A^0$. Then genericity of g assures that $W\langle I_\theta^F, E' \rangle$ is an n -dimensional ideal and hence $\dim \text{Ch}^F(H_{A,i}(N; \beta')) \leq n$ by the spectral sequence

$$H_i(\text{gr}^F(K_{A,\bullet}(N; \beta'))) \implies \text{gr}^F(H_{A,i}(N; \beta')).$$

We interpret $K_{A,\bullet}(N; \beta)$ as the complex induced by $E_d - \beta_d$ on $K_{A,\bullet}(N; \beta')$ and abbreviate $H_i := H_{A,i}(N; \beta)$ and $H'_i := H_{A,i}(N; \beta')$. There is a double complex spectral sequence abutting to H_\bullet with E_1 -term H'_\bullet and differential d_1 induced by

$E_d - \beta_d$ which collapses at the E_2 -term. In particular, $H_0 = H'_0/(E_d - \beta_d)H'_0$ and there is a short exact sequence of holonomic D -modules

$$0 \longrightarrow \ker_{H'_0}(E_d - \beta_d) \longrightarrow H_1 \rightarrow H'_1/(E_d - \beta_d)H'_1 \longrightarrow 0.$$

Using the exact sequence

$$0 \longrightarrow \ker_{H'_0}(E_d - \beta_d) \longrightarrow H'_0 \xrightarrow{E_d - \beta_d} H'_0 \longrightarrow H'_0/(E_d - \beta_d)H'_0 \longrightarrow 0$$

it follows that

$$\begin{aligned} \mathrm{CC}^L(H_0) &= \mathrm{CC}^L(H'_0/(E_d - \beta_d)H'_0) \\ &= \mathrm{CC}^L(\ker_{H'_0}(E_d - \beta_d)) \\ &= \mathrm{CC}^L(H_1) - \mathrm{CC}^L(H'_1/(E_d - \beta_d)H'_1). \end{aligned}$$

We conclude that $\mathrm{CC}^L(H_1) - \mathrm{CC}^L(H_0) = \mathrm{CC}^L(H'_1/(E_d - \beta_d)H'_1)$ is non-negative which implies the claim. \square

5. PROJECTIVIZED HYPERGEOMETRIC SYSTEMS

Here we introduce and study a natural extension of the A -hypergeometric system $M_A(\beta)$ on $X = \mathbb{C}^n$ to a sheaf on $(\mathbb{P}_{\mathbb{C}}^1)^n$ where $\mathbb{P}_{\mathbb{C}}^1 = \mathbb{C} \sqcup \{\infty\}$ in each factor. We call the resulting extension $\mathcal{M}_A(\beta)$ of $M_A(\beta)$ the *projectivized A -hypergeometric system* defined by A and β . We use the results of the previous sections to discuss, in this sequence, $\mathcal{M}_A(\beta)$ from the points of view of the previous sections: (A, L) -umbrella, L -characteristic variety, L -characteristic cycle, and, as a special case, its slopes along all projective coordinate subspaces. Once again the (A, L) -umbrella will play a pivotal role in our investigation of $\mathcal{M}_A(\beta)$.

Notation 5.1. We use the canonical embedding

$$\mathbb{C}^1 = \mathrm{Spec}(\mathbb{C}[x_j]) \hookrightarrow \mathrm{Proj}(\mathbb{C}[y_j, y'_j]) = \mathbb{P}_{\mathbb{C}}^1 = \mathbb{C}^1 \sqcup \{\infty\}$$

induced by $x_j = y_j/y'_j$ to embed $X = \mathrm{Spec}(\mathbb{C}[x_1, \dots, x_n])$ into $(\mathbb{P}_{\mathbb{C}}^1)^n =: \bar{X}$ as the complement of the variety of the ideal sheaf generated by $\prod_{j=1}^n y'_j$. Recall from Notation 2.1, that the symbol $\bar{\tau}$ denotes the complement of a set τ in the set $\{1, \dots, n\}$. On the other hand, \bar{C} will always refer to the Zariski closure of a set C in $T^*\bar{X}$. We denote by $x'_j := y'_j/y_j$ the coordinate on $\mathbb{P}_{\mathbb{C}}^1 \setminus \{0\}$. A subset $\mathfrak{P} \subseteq \{1, \dots, n\}$ defines an affine patch $X_{\mathfrak{P}} \cong \mathbb{C}^n$ of \bar{X} by

$$X_{\mathfrak{P}} = \mathrm{Spec}(\mathbb{C}[\{x_j \mid j \notin \mathfrak{P}\} \cup \{x'_j \mid j \in \mathfrak{P}\}]).$$

In particular, $X = X_{\emptyset}$ in this notation. If \mathcal{F} is a sheaf on \bar{X} then we denote its sections over $X_{\mathfrak{P}}$ by $F_{(\mathfrak{P})} := \mathcal{F}(X_{\mathfrak{P}})$. On the other hand, if g is an \bar{X} -global section then we denote by $g_{(\mathfrak{P})}$ the restriction of g to $X_{\mathfrak{P}}$.

Denote by \mathcal{D} the sheaf of algebraic \mathbb{C} -linear differential operators on \bar{X} . Then with the above notation the ring $D_{(\mathfrak{P})} = \mathcal{D}(X_{\mathfrak{P}})$ is the Weyl algebra in the variables $\{x_j \mid j \notin \mathfrak{P}\} \cup \{x'_j \mid j \in \mathfrak{P}\}$. Let ∂'_j denote the derivation on $\mathbb{P}_{\mathbb{C}}^1$ relative to x'_j . The sheaf \mathcal{D} has a special subsheaf of rings \mathcal{R} , consisting entirely of the global differential operators in $R = \mathbb{C}[\partial]$. There are ring isomorphisms

$$(5.0.1) \quad \rho_{\mathfrak{P}}: R_{(\emptyset)} = \mathbb{C}[\partial] \longrightarrow \mathbb{C}[\partial_{(\mathfrak{P})}] = \mathbb{C}[\partial_{\bar{\mathfrak{P}}}] [\{x_j'^2 \partial'_j \mid j \in \mathfrak{P}\}] = R_{(\mathfrak{P})}$$

defined by $\partial_j \mapsto -x_j'^2 \partial_j'$ for $j \in \mathfrak{P}$. Any \mathbb{Z}^d -graded R -module induces a \mathbb{Z}^d -graded \mathcal{R} -module via Equation (5.0.1).

Note that x_j' naturally inherits the \mathbb{Z}^d -degree $\deg(x_j') := -\deg(x_j) = \mathbf{a}_j$. Since $x_j \partial_j = -x_j' \partial_j'$, we put $\deg(\partial_j') := \deg(x_j \partial_j) - \deg(x_j') = -\mathbf{a}_j$. This makes \mathcal{D} a \mathbb{Z}^d -graded sheaf of right \mathcal{R} -modules.

Similarly, let $L = (L_x, L_\partial)$ be a weight vector on $D = D_\emptyset$, so $L_\partial + L_x \geq 0$. We extend L to the variables x_j' and ∂_j' by setting $L_{x_j'} := -L_{x_j}$ and $L_{\partial_j'} := L_{x_j} + L_{\partial_j} - L_{x_j'}$. This extends the L -filtration on $D = D_\emptyset$ to a *global L -filtration* on the sheaf \mathcal{D} . Let $\mathcal{W} := \text{gr}^L(\mathcal{D})$ be the sheaf of associated graded algebras.

By definition, $L_\partial + L_x = L_{\partial'} + L_{x'}$. Suppose $L_x + L_\partial > 0$, then $W_{(\mathfrak{P})} = \text{gr}^L(D_{(\mathfrak{P})})$ is the polynomial ring $\mathbb{C}[x_{\mathfrak{P}}, \partial_{\mathfrak{P}}, x_{\mathfrak{P}}', \partial_{\mathfrak{P}}']$ and \mathcal{W} can be interpreted as the sheaf of regular functions on $T^* \bar{X}$.

Definitions 3.1 and 4.4 of the L -characteristic variety Ch^L and L -characteristic cycle Ch^L generalize to finite \mathcal{D} -modules \mathcal{M} and homologically holonomic complexes \mathcal{K}_\bullet of \mathcal{D} -modules. Both L -characteristic variety and cycle behave well under restriction to charts as L is global.

Since both $\partial_j, x_j \partial_j \in D = D_\emptyset$ extend to global differential operators on \bar{X} , the same is true for the generators $\square_{\mathbf{u}}$ with $A \cdot \mathbf{u} = 0$ and $E - \beta$ with $1 \leq i \leq n$ in Definition 3.5. We will denote these extended operators by the same symbols as the ones we used on X_\emptyset . This provides the motive for introducing a projectivized version of $M_A(\beta)$ as follows.

Definition 5.2. Define the \mathcal{R} -module $\mathcal{I}_A := \mathcal{R} \cdot I_A$ as the extension of I_A to \bar{X} . The *projectivized A -hypergeometric system* $\mathcal{M}_A(\beta)$ is the sheaf of \mathcal{D} -modules

$$\mathcal{M}_A(\beta) := \frac{\mathcal{D}}{\mathcal{D}\langle I_A, E - \beta \rangle}.$$

On the affine patch $X_{\mathfrak{P}}$, using notation introduced above, one has

$$M_{A,(\mathfrak{P})}(\beta) = D_{(\mathfrak{P})}/D_{(\mathfrak{P})}\langle I_{A,(\mathfrak{P})}, E_{(\mathfrak{P})} - \beta \rangle.$$

Note, how $I_{A,(\mathfrak{P})} = \rho_{\mathfrak{P}}(I_A)$ and $E_{(\mathfrak{P})} = \rho_{\mathfrak{P}}(E)$ behave under change of \mathfrak{P} : moving the index j into \mathfrak{P} results in the change $a_{i,j} x_j \partial_j \mapsto -a_{i,j} x_j' \partial_j'$ in $E_{(\mathfrak{P})}$, and in the substitution $\partial_j \mapsto -x_j'^2 \partial_j'$ in $I_{A,(\mathfrak{P})}$.

We begin our study of $\mathcal{M}_A(\beta)$ with a basic observation. Since the L -filtration is compatible with restriction to charts, we have $(\sigma^L(P))_{(\mathfrak{P})} = \sigma^L(P_{(\mathfrak{P})})$ for all global sections P of \mathcal{D} . Thus the local sections of the ideal sheaf

$$\mathcal{I}_A^L := \text{gr}^L(\mathcal{I}_A) = \mathcal{R}\langle \sigma^L(\square_{\mathbf{u}}) \mid A \cdot \mathbf{u} = 0 \rangle$$

are given by $I_{A,(\mathfrak{P})}^L = \text{gr}^L(I_{A,(\mathfrak{P})})$. More interestingly, Buchberger's algorithm yields the following analog to Lemma 2.13 regarding the L -graded ideal of $\mathcal{D} \cdot I_A$.

Lemma 5.3. *As ideal sheaves in \mathcal{W} ,*

$$\text{gr}^L(\mathcal{D} \cdot \mathcal{I}_A) = \mathcal{W} \cdot \mathcal{I}_A^L$$

so that for all \mathfrak{P} one has

$$\text{gr}^L(D_{(\mathfrak{P})} \cdot I_{A,(\mathfrak{P})}) = W_{(\mathfrak{P})} \cdot I_{A,(\mathfrak{P})}^L = W_{(\mathfrak{P})}\langle \sigma^L(\square_{\mathbf{u}})_{(\mathfrak{P})} \mid \mathbf{u} \in \mathbb{Z}^n, A \cdot \mathbf{u} = 0 \rangle. \quad \square$$

We now study the variety of $T^*\bar{X}$ defined by $\mathcal{W} \cdot \mathcal{I}_A^L$. Since \mathcal{I}_A^L is \mathbb{Z}^d -graded, the associated prime ideals of $\mathcal{W} \cdot \mathcal{I}_A^L$ will be \mathbb{Z}^d -graded as well and we hence search for \mathbb{Z}^d -graded sheaves of prime ideals $\mathcal{P} \subseteq \mathcal{W}$ containing $\mathcal{W} \cdot \mathcal{I}_A^L$. By Theorem 2.15 the \mathbb{Z}^d -graded prime ideals of $R_{(\emptyset)}$ containing \mathcal{I}_A^L are of the form I_A^τ with $\tau \in \Phi_A^L$. Since their sections are \bar{X} -global, every \mathcal{P} as above must contain one of the ideals $\mathcal{I}_A^\tau := \mathcal{R} \cdot I_A^\tau$ with $\tau \in \Phi_A^L$. Thus we shall look for the components of the variety

$$\Upsilon_A^\tau := \text{Var}(\mathcal{W} \cdot \mathcal{I}_A^\tau) \subseteq T^*\bar{X}.$$

The sections $I_{A,(\mathfrak{P})}^\tau$ of \mathcal{I}_A^τ over a chart $X_\mathfrak{P}$ are the elements of I_A^τ with ∂_j replaced by $-x_j'^2 \partial_j'$ for all $j \in \mathfrak{P}$. On T^*X_\emptyset , $I_{A,(\emptyset)}^\tau = I_A^\tau$ defines an irreducible variety whose (again irreducible) closure in $T^*\bar{X}$ we denote by $\Upsilon_A^{\tau, \emptyset}$.

Assume first that \mathfrak{P} is contained in τ . In this case, $W_{(\mathfrak{P})} \cdot I_{A,(\mathfrak{P})}^\tau$ is generated by $I_{\tau,(\mathfrak{P})}$ and $\partial_{\tau \cup \mathfrak{P}}$ which reduces the problem to understanding $W_{(\mathfrak{P})} \cdot I_{\tau,(\mathfrak{P})}$.

Lemma 5.4. *If $\mathfrak{P} \subseteq \tau$ then the ideal $W_{(\mathfrak{P})} \cdot I_{\tau,(\mathfrak{P})}$ is a prime ideal.*

Proof. By Equation (5.0.1), $I_{\tau,(\mathfrak{P})} = \rho_\mathfrak{P}(I_\tau)$ is prime in $R_{(\mathfrak{P})}$ and hence so is $I_{\tau,(\mathfrak{P})} \cdot R_{(\mathfrak{P})}[x'_\mathfrak{P}, x'_{\bar{\mathfrak{P}}}]$. Now $W_{(\mathfrak{P})} = R_{(\mathfrak{P})}[x'_\mathfrak{P}, x'_{\bar{\mathfrak{P}}}][\partial'_\mathfrak{P}]$ and $\partial_j' = -\rho_\mathfrak{P}(\partial_j)/x_j'^2$ for $j \in \mathfrak{P}$. Since for $j \in \mathfrak{P}$, x_j' is not in $R_{(\mathfrak{P})}[x'_\mathfrak{P}, x'_{\bar{\mathfrak{P}}}] \cdot I_{\tau,(\mathfrak{P})}$ (and, by primeness, not in any associated prime either), it follows that under the ring extension $R_\mathfrak{P} \hookrightarrow W_\mathfrak{P}$ the ideal $I_{\tau,(\mathfrak{P})}$ remains prime. \square

It follows that for $\mathfrak{P} \subseteq \tau$, Υ_A^τ is irreducible on $X_\mathfrak{P}$ and $W_{(\mathfrak{P})} \cdot I_{A,(\mathfrak{P})}^\tau$ is precisely the defining ideal of $\Upsilon_A^{\tau, \emptyset} \cap T^*X_\mathfrak{P}$.

We now view $\mathfrak{P} = (\mathfrak{P} \cap \tau) \sqcup (\mathfrak{P} \setminus \tau)$. By the above, $W_{(\mathfrak{P})} \cdot I_{A,(\mathfrak{P})}^\tau$ is the ideal of $W_{(\mathfrak{P})}$ generated by $I_{\tau,(\mathfrak{P})}$, $\partial_{\tau \cup \mathfrak{P}}$ and $\{x_j'^2 \partial_j' \mid j \in \mathfrak{P} \setminus \tau\}$. By Lemma 5.4, the radical of this ideal has a decomposition into prime ideals

$$\sqrt{I_{A,(\mathfrak{P})}^\tau} = \bigcap_{\mathfrak{T} \subseteq \mathfrak{P} \setminus \tau} W_{(\mathfrak{P})} \langle I_{\tau,(\mathfrak{P})}, \partial_{\tau \cup \mathfrak{P}}, x'_\mathfrak{T}, \partial'_{\mathfrak{P} \setminus (\mathfrak{T} \cup \tau)} \rangle.$$

We are prompted to make the following definitions generalizing Definitions 2.2 and 2.7.

Definition 5.5. Let

$$\bar{\Phi}_A^L := \{(\tau, \mathfrak{T}) \mid \tau \in \Phi_A^L, \mathfrak{T} \subseteq \{1, \dots, n\}, \mathfrak{T} \cap \tau = \emptyset\}$$

and $\bar{\Phi}_A^{L,k} := \{(\tau, \mathfrak{T}) \in \bar{\Phi}_A^L \mid \tau \in \Phi_A^{L,k}\}$. For $(\tau, \mathfrak{T}) \in \bar{\Phi}_A^L$, put

$$J_{\tau, \mathfrak{T}, (\mathfrak{P})} := R_{(\mathfrak{P})} \langle x'_\mathfrak{T}, \partial_{\tau \cup \mathfrak{T} \cup \mathfrak{P}}, \partial'_{\tau \cup \mathfrak{T} \cap \mathfrak{P}} \rangle$$

and define the prime ideal sheaf $\mathcal{I}_A^{\tau, \mathfrak{T}} \supseteq \mathcal{W} \cdot I_A^\tau \supseteq \mathcal{W} \cdot I_A^L$ on each chart $X_\mathfrak{P}$ by

$$I_{A,(\mathfrak{P})}^{\tau, \mathfrak{T}} := \begin{cases} W_{(\mathfrak{P})} \langle I_{\tau,(\mathfrak{P})}, J_{\tau, \mathfrak{T}, (\mathfrak{P})} \rangle, & \text{if } \mathfrak{P} \supseteq \mathfrak{T}, \\ W_{(\mathfrak{P})}, & \text{otherwise.} \end{cases}$$

The irreducible varieties $\Upsilon_A^{\tau, \mathfrak{T}} := \text{Var}(\mathcal{I}_A^{\tau, \mathfrak{T}}) \subseteq T^*\bar{X}$ define quasi-affine varieties

$$\dot{\Upsilon}_A^{\tau, \mathfrak{T}} := \text{Var}(I_{A,(\mathfrak{T})}^{\tau, \mathfrak{T}}) \setminus \bigcup_{\substack{(\tau', \mathfrak{T}') \in \bar{\Phi}_A^L \\ \tau' \subseteq \tau}} \text{Var}(I_{A,(\mathfrak{T}')}^{\tau', \mathfrak{T}'}) \subseteq T^*X_\mathfrak{T}.$$

Note that $\Upsilon_A^{\tau, \mathfrak{I}}$ (and hence $\hat{\Upsilon}_A^{\tau, \mathfrak{I}}$) has dimension $\dim(\tau) + n$ and meets the chart $T^*X_{\mathfrak{P}}$ exactly if $\mathfrak{P} \supseteq \mathfrak{I}$. Let finally $\nu_A^{L, \tau, \mathfrak{I}}$ be the multiplicity of \mathcal{I}_A^L along $\Upsilon_A^{\tau, \mathfrak{I}}$.

The quasi-affine varieties $\hat{\Upsilon}_A^{\tau, \mathfrak{I}}$ will play the role of the orbits in Section 2. In fact, the action of the d -torus \mathbb{T} on T_0^*X can be extended to \bar{X} via the extension of the coordinate ring R of T_0^*X : Denote

$$Z_{\mathfrak{P}} := \text{Var}(x'_{\mathfrak{P}}) \subseteq X_{\mathfrak{P}}.$$

For $p \in Z_{\mathfrak{P}}$, $\{\partial_j \mid j \notin \mathfrak{P}\} \cup \{\partial'_j \mid j \in \mathfrak{P}\}$ are coordinates on $T_p^*\bar{X}$. Similar to (2.1.1), $t \in \mathbb{T}$ applies componentwise to $\xi = (\xi, \xi') \in T_p^*\bar{X}$ by $t \cdot \xi_j = t^{\mathfrak{a}_j} \xi_j$ but $t \cdot \xi'_j = 0$ because $\partial_j = -x'_j{}^2 \partial'_j$ vanishes on $Z_{\mathfrak{P}}$. When restricted to $T_p^*Z_{\mathfrak{P}}$, this becomes a group action uniform in $p \in Z_{\mathfrak{P}}$. To generalize Definition 2.2, let $(\tau, \mathfrak{I}) \in \bar{\Phi}_A^L$, denote by $\mathbf{1}_A^{\tau, \mathfrak{I}}$ the point $\mathbf{1}_A$ in $T_0^*Z_{\mathfrak{I}}$, and set

$$O_A^{\tau, \mathfrak{I}} := \text{Orb}(\mathbf{1}_A^{\tau, \mathfrak{I}}) \subseteq T_0^*Z_{\mathfrak{I}}.$$

By Lemma 2.3, $O_A^{\tau, \mathfrak{I}} = \text{Var}(I_{A \setminus \mathfrak{I}}^{\tau, \mathfrak{I}})$ so that $O_A^{\tau, \mathfrak{I}} \times \mathbb{C}^{|\mathfrak{I}|}$ identifies with O_A^{τ} . The inclusion $Z_{\mathfrak{I}} = \text{Var}(x'_{\mathfrak{I}}) \subseteq X_{\mathfrak{I}}$ induces natural maps of cotangent spaces

$$T^*X_{\mathfrak{I}} \xleftarrow{\iota_{\mathfrak{I}}} T^*X_{\mathfrak{I}|Z_{\mathfrak{I}}} \xrightarrow{\pi_{\mathfrak{I}}} T^*Z_{\mathfrak{I}}.$$

and $\hat{\Upsilon}_A^{\tau, \mathfrak{I}} = \iota_{\mathfrak{I}}(\pi_{\mathfrak{I}}^{-1}(O_A^{\tau, \mathfrak{I}} \times Z_{\mathfrak{I}}))$. The partial Fourier transform

$$(5.0.2) \quad W_{(\mathfrak{I})} \longrightarrow W_{(\emptyset)}, \quad x'_{\mathfrak{I}} \longmapsto \partial_{\mathfrak{I}}, \quad \partial'_{\mathfrak{I}} \longmapsto -x_{\mathfrak{I}},$$

identifies $\hat{\Upsilon}_A^{\tau, \mathfrak{I}}$ with the product $O_A^{\tau} \times \mathbb{C}^n$

By the preceding discussion, Theorem 2.15 and Proposition 2.18 can be generalized as follows.

Theorem 5.6. *A decomposition of Υ_A^L into irreducible components is given by*

$$\Upsilon_A^L = \bigcup_{(\tau, \mathfrak{I}) \in \bar{\Phi}_A^{L, d-1}} \Upsilon_A^{\tau, \mathfrak{I}}.$$

More generally, there is a stratification

$$\Upsilon_A^L = \bigsqcup_{(\tau, \mathfrak{I}) \in \bar{\Phi}_A^L} \hat{\Upsilon}_A^{\tau, \mathfrak{I}}$$

such that $\hat{\Upsilon}_A^{\tau', \mathfrak{I}'}$ is in the closure of $\hat{\Upsilon}_A^{\tau, \mathfrak{I}}$ if and only if $\tau' \subseteq \tau$ and $\tau \setminus \tau' \subseteq \mathfrak{I}' \supseteq \mathfrak{I}$. For $(\tau, \mathfrak{I}) \in \bar{\Phi}_A^{L, d-1}$, the multiplicity of $\Upsilon_A^{\tau, \mathfrak{I}}$ in Υ_A^L is

$$\nu_A^{L, \tau, \mathfrak{I}} = 2^{|\mathfrak{I}|} \cdot \nu_A^{L, \tau} = 2^{|\mathfrak{I}|} \cdot [\mathbb{Z}^d : \mathbb{Z}\tau].$$

Proof. The irreducible decomposition is a consequence of Lemma 5.4. The existence of the stratification follows from Theorem 2.15 and the preceding discussion. For $j \in \tau \setminus \mathfrak{P}$, $\rho_{\mathfrak{P} \cup \{j\}}(\partial_j) = -x'_j{}^2 \partial'_j$ and the adjacencies in Theorem 2.15 imply that

$$\sqrt{\langle I_{A, (\mathfrak{I} \cup \{j\})}^{\tau, \mathfrak{I}}, x'_j \rangle} = \bigcap_{\tau' \subseteq \tau \setminus \{j\}} I_{A, (\mathfrak{I} \cup \{j\})}^{\tau', \mathfrak{I} \cup \{j\}}$$

which leads to the adjacencies of the $\hat{\Upsilon}_A^{\tau, \mathfrak{I}}$. The multiplicity $\nu_A^{A, \tau, \mathfrak{I}}$ can be measured on the chart $X_{\mathfrak{I}}$. Then the statement is a consequence of the fact that the ideal generated by $x'_j{}^2 \partial'_j$ has multiplicity two along $x'_j = 0$, combined with the formula for $\nu_A^{L, \tau}$ in Proposition 2.18. \square

This ends our discussion of the globalization of the toric ideal I_A . From now on we assume that $L_{x_j} + L_{\partial_j} = c > 0$ for all j and investigate the projectivized A -hypergeometric system $\mathcal{M}_A(\beta)$. In that case, $E_{(\mathfrak{P})}$ is L -homogeneous of L -degree c for every chart index \mathfrak{P} . We imitate Definition 3.7 as follows.

Definition 5.7. For $(\tau, \mathfrak{T}) \in \bar{\Phi}_A^{\tau, \mathfrak{T}}$, set $C_A^{\tau, \mathfrak{T}} := \dot{\Upsilon}_A^{\tau, \mathfrak{T}} \cap \text{Var}(E_{(\mathfrak{T})}) \subseteq T^*X_{\mathfrak{T}}$ and let $\mathcal{P}_{\tau, \mathfrak{T}} \subseteq \mathcal{W}$ be the prime ideal sheaf of $\bar{C}_A^{\tau, \mathfrak{T}}$.

The isomorphism of $\dot{\Upsilon}_A^{\tau, \mathfrak{T}}$ with $O_A^\tau \times \mathbb{C}^n$ via the partial Fourier transform (5.0.2) mentioned above identifies $E_{(\mathfrak{T})}$ with $E_{(\emptyset)}$. Thus $C_A^{\tau, \mathfrak{T}}$ is a quasi-affine variety isomorphic to C_A^τ . This yields the following estimate for the L -characteristic variety of $\mathcal{M}_A(\beta)$ in the spirit of Proposition 3.8.

Proposition 5.8. *The characteristic variety of $\mathcal{M}_A(\beta)$ is contained in the union of the closures of the Fourier twisted orbit conormals:*

$$\text{Ch}(\mathcal{M}_A(\beta)) \subseteq \bigcup_{(\tau, \mathfrak{T}) \in \bar{\Phi}_A^L} \bar{C}_A^{\tau, \mathfrak{T}}. \quad \square$$

We now wish to compute the actual components and their multiplicities in the characteristic cycle of $\mathcal{M}_A(\beta)$. We proceed exactly as in Section 4. Multiplicities along a candidate component $\bar{C}_A^{\tau, \mathfrak{T}}$ where $(\tau, \mathfrak{T}) \in \bar{\Phi}_A^L$ can be computed on the chart $X_{\mathfrak{T}}$. This means the \mathcal{D} -module $\mathcal{M}_A(\beta)$ can be replaced by the $D_{(\mathfrak{T})}$ -module $M_{A, (\mathfrak{P})}(\beta)$. Parallel to the affine case in Definition 4.6, one can introduce a global Euler–Koszul complex on \bar{X} .

Definition 5.9. Let N be a \mathbb{Z}^d -graded R -module and $\beta \in \mathbb{C}^d$. We consider \mathcal{D} as a \mathbb{Z}^d -graded right \mathcal{R} -module and N as a \mathbb{Z}^d -graded \mathcal{R} -module as in Notation 5.1. Then $\mathcal{D} \otimes_{\mathcal{R}} N$ is a \mathbb{Z}^d -graded left \mathcal{D} -module. On a \mathbb{Z}^d -graded left \mathcal{D} -module \mathcal{M} , the assignment (4.2.1) and \mathcal{D} -linear extension yield d commuting \mathcal{D} -linear endomorphisms $E - \beta$. We define the *global Euler–Koszul complex* $\mathcal{K}_{A, \bullet}(N; E - \beta)$ and the *global Euler–Koszul homology* $\mathcal{H}_{A, \bullet}(N; E - \beta)$ as the sheafified version of the objects in Definition 4.6. Note that $\mathcal{H}_{A, 0}(S_A; E - \beta) = \mathcal{M}_A(\beta)$.

A good behavior of this global Euler–Koszul functor such as a long exact sequence of Euler–Koszul homology requires flatness of \mathcal{D} over \mathcal{R} .

Lemma 5.10. *The sheaves of \mathcal{R} -modules \mathcal{D} and \mathcal{W} are locally free and hence flat.*

Proof. It suffices to consider the single variable case on a chart $X_{\mathfrak{P}}$. For $j \in \mathfrak{P}$, $\mathbb{C}[x'_j, \partial'_j]$ is a free $\mathbb{C}[-x'_j{}^2 \partial'_j]$ -module with basis $x'_j{}^k \partial'_j{}^l$ where $k \leq 1$ or $l = 0$. Since $\mathbb{C}[x'_j, \partial'_j] = \text{gr}^F(\mathbb{C}[x'_j] \langle \partial'_j \rangle)$ the same is true for $\mathbb{C}[x'_j] \langle \partial'_j \rangle$. \square

The Euler–Koszul homology sheaves of a toric R -module are holonomic for reasons similar to those given in [MMW05] for Euler–Koszul homology modules. Their L -characteristic variety is hence n -dimensional by [Smi01]. A spectral sequence argument like in Theorem 4.11 shows the possible components of $\text{Ch}^L(\mathcal{H}_{A, \bullet}(N; \beta))$ are of the form $\bar{C}_A^{\tau, \mathfrak{T}}$ with $(\tau, \mathfrak{T}) \in \bar{\Phi}_A^L$.

Definition 5.11. For $(\tau, \mathfrak{T}) \in \bar{\Phi}_A^L$ and $i \in \mathbb{N}$, we denote by $\mu_{A, i}^{L, \tau, \mathfrak{T}}(\beta)$ the multiplicity of $\text{gr}^L(\mathcal{H}_{A, i}(S_A; \beta))$ along $\bar{C}_A^{\tau, \mathfrak{T}}$ and put $\mu_A^{L, \tau, \mathfrak{T}} := \sum_{i=1}^d (-1)^i \mu_{A, i}^{L, \tau, \mathfrak{T}}(\beta)$.

By looking at the chart $X_{\bar{\mathfrak{X}}}$ it follows from the appendix that, as in the affine case, the alternating sum of the $\mu_{A,i}^{L,\tau,\bar{\mathfrak{X}}}(\beta)$ agrees with the intersection multiplicity of $I_{A,(\bar{\mathfrak{X}})}^L$ with $E_{(\bar{\mathfrak{X}})}$ in the local ring at $\bar{C}_A^{\tau,\bar{\mathfrak{X}}}$. By Theorem 5.6, this reduces to the intersection multiplicity of $I_{A,(\bar{\mathfrak{X}})}^{\tau,\bar{\mathfrak{X}}}$ with $E_{(\bar{\mathfrak{X}})}$, multiplied by $\nu_A^{L,\tau,\bar{\mathfrak{X}}} = 2^{|\bar{\mathfrak{X}}|}\nu_A^{L,\tau}$. From Theorem 4.11 we conclude the following statement which shows in particular that $\mu_A^{L,\tau,\bar{\mathfrak{X}}}$ is independent of β as suggested by the notation. For the sake of simplicity we concentrate on the case of interest $N = S_A$.

Theorem 5.12. *For all $(\tau, \bar{\mathfrak{X}}) \in \bar{\Phi}_A^L$ and all $\beta \in \mathbb{C}^d$,*

$$\mu_A^{L,\tau,\bar{\mathfrak{X}}} = \chi^C(\mathcal{W}/\mathcal{W}\langle E \rangle, \mathcal{W} \otimes_{\mathcal{R}} S_A^L) = 2^{|\bar{\mathfrak{X}}|}\mu_A^{L,\tau},$$

where $C = \bar{C}_A^{\tau,\bar{\mathfrak{X}}}$, is independent of β and positive. \square

In [MMW05, Thm. 6.3], a spectral sequence is constructed that is used to show good vanishing properties of Euler–Koszul homology. We leave it to the reader to verify that it generalizes to our global context. It follows that if β is generic then $\mathcal{K}_{A,\bullet}(S_A, E - \beta)$ has only homology in degree zero. In particular, $\mu_{A,0}^{L,\tau,\bar{\mathfrak{X}}}(\beta) = \mu_A^{L,\tau,\bar{\mathfrak{X}}} > 0$ for generic β , similar to Corollary 4.12.

In order to show non-vanishing for all β one can generalize Theorems 3.10 and 4.28 by following exactly the steps of our proof in the chart $X_{\bar{\mathfrak{X}}}$ corresponding to $\bar{C}_A^{\tau,\bar{\mathfrak{X}}}$, always twisting the toric input module by $\rho_{\bar{\mathfrak{X}}}$ from Equation (5.0.1).

Theorem 5.13. *The L -characteristic variety of $\mathcal{M}_A(\beta)$ is given by*

$$\text{Ch}^L(\mathcal{M}_A(\beta)) = \bigcup_{(\tau,\bar{\mathfrak{X}}) \in \bar{\Phi}_A^L} \bar{C}_A^{\tau,\bar{\mathfrak{X}}}.$$

For all $(\tau, \bar{\mathfrak{X}}) \in \bar{\Phi}_A^L$ and all $\beta \in \mathbb{C}^d$,

$$\mu_{A,0}^{L,\tau,\bar{\mathfrak{X}}}(\beta) \geq \mu_A^{L,\tau,\bar{\mathfrak{X}}} = 2^{|\bar{\mathfrak{X}}|}\mu_A^{L,\tau}$$

is positive, and equality holds for generic β . In particular, the L -characteristic variety of $\mathcal{M}_A(\beta)$ is entirely determined by the (A, L) -umbrella Φ_A^L . For $\tau \in \Phi_A^{L,n-1}$,

$$\mu_A^{L,\tau,\bar{\mathfrak{X}}} = 2^{|\bar{\mathfrak{X}}|}\nu_A^{L,\tau} = 2^{|\bar{\mathfrak{X}}|}[\mathbb{Z}^d : \mathbb{Z}\tau]. \quad \square$$

In order to study slopes of $\mathcal{M}_A(\beta)$ along coordinate varieties, we consider \mathfrak{Y} as a partitioned set $\mathfrak{Y} = \mathfrak{Y}_0 \sqcup \mathfrak{Y}_\infty \subseteq \{1, \dots, n\}$. As in (3.1.1), \mathfrak{Y} defines the coordinate variety

$$Y := \text{Var}(x_{\mathfrak{Y}_0}, x'_{\mathfrak{Y}_\infty}) \subseteq \bar{X}$$

which meets the chart $X_{\mathfrak{P}}$ exactly if $\mathfrak{Y}_0 \cap \mathfrak{P} = \emptyset$ and $\mathfrak{Y}_\infty \subseteq \mathfrak{P}$. On any such $X_{\mathfrak{P}}$, Y induces the family of filtrations

$$L = pF + qV, \quad p/q \in \mathbb{Q}_{>0} \cup \{\infty\},$$

on $D_{\mathfrak{P}}$ as in (3.1.2) where V is the V -filtration along $Y \cap X_{\mathfrak{P}}$ defined by the assignment $-V_{x_i} = 1 = V_{\partial_i}$ for $i \in \mathfrak{Y}_0$, $V_{x'_i} = 1 = -V_{\partial'_i}$ for $i \in \mathfrak{Y}_\infty$, and $V_{x'_i} = 0 = -V_{\partial'_i}$ for $i \notin \mathfrak{Y}$. One may view the patch $W_{\mathfrak{P}} = \text{gr}^L(D_{\mathfrak{P}})$ of \mathcal{W} as the ring of polynomial functions on $T^*(T_Y^*X_{\mathfrak{P}})$ and identify $T^*T_Y^*\bar{X} = T^*(\bar{X} \setminus \text{Var}(x'_{\mathfrak{Y}_0}, x_{\mathfrak{Y}_\infty}))$. By Definition 3.4, $L = p/q$ a slope of a finite \mathcal{D} -module \mathcal{M} on \bar{X} at $y \in Y$ along Y if the set of components of $\text{Ch}^L(\mathcal{M})$ which meet $T_y^*\bar{X}$ jumps at L .

Recall that all slopes of $M_A(\beta)$ along coordinate varieties are slopes at the origin in X by Lemma 3.14. But the following example shows that jumps of $\text{Ch}^L(\mathcal{M}_A(\beta))$ can be irrelevant for the slopes along Y .

Example 5.14. Let $A = \begin{pmatrix} 3 & 1 & 0 \\ 0 & 1 & 3 \end{pmatrix}$, pick $Y = \text{Var}(x'_1, x'_2)$ and $L = pF + qV$ with $p > 0, q \geq 0$ in \mathbb{Q} . Since I_A^L is Cohen–Macaulay, $\text{gr}^L(\mathcal{W}(I_A, E - \beta))$ equals $\langle \partial_2^3, 3x_1\partial_1 + x_2\partial_2, x_2\partial_2 + 3x_3\partial_3 \rangle$ if $p/q > 2$ but $\langle \partial_1\partial_3, 3x_1\partial_1 + x_2\partial_2, x_2\partial_2 + 3x_3\partial_3 \rangle$ if $p/q < 2$. It follows that $\text{Ch}^L(\mathcal{M}_A(\beta))$ is defined by $\langle \partial_2, x_1\partial_1, x_3\partial_3 \rangle$ for $p/q > 2$ and by $\langle x_2\partial_2, \partial_1\partial_3, x_3\partial_3, x_1\partial_1 \rangle$ if $p/q < 2$. Thus, 2 is the only possible slope value and, as p/q passes through 2 from above, the component of $\text{Ch}^L(\mathcal{M}_A(\beta))$ to $\langle x_1, \partial_2, x_3 \rangle$ is replaced by the components to $\langle x_2, \partial_1\partial_3, x_3\partial_3, x_1\partial_1 \rangle$. However, as Y is defined by $x_1 = x_2 = \infty$, none of these components meet Y and hence $\mathcal{M}_A(\beta)$ is regular along Y .

It follows that Theorem 5.13 does not give directly a combinatorial description of the slopes of $\mathcal{M}_A(\beta)$ along Y . We must select the components of $\text{Ch}^L(\mathcal{M}_A(\beta))$ whose projection meets Y . For this, we look at a chart $X_{\mathfrak{P}}$ which meets Y and hence $\mathfrak{P} \supseteq \mathfrak{V}_\infty$ and $\mathfrak{P} \cap \mathfrak{V}_0 = \emptyset$. Consider the component $\bar{C}_A^{\tau, \mathfrak{T}}$ of $\text{Ch}^L(\mathcal{M}_A(\beta))$ for some $(\tau, \mathfrak{T}) \in \bar{\Phi}_A^L$. Since $I_{A, (\mathfrak{P})}^{\tau, \mathfrak{T}}$ contains $x'_{\mathfrak{T}}$, $\bar{C}_A^{\tau, \mathfrak{T}}$ can be visible in $X_{\mathfrak{P}}$ only if $\mathfrak{P} \supseteq \mathfrak{T}$. On the other hand, if $\bar{C}_A^{\tau, \mathfrak{T}}$ is not visible in $X_{\mathfrak{P}}$ then (since $\bar{C}_A^{\tau, \mathfrak{T}}$ is irreducible) its defining ideal contains an x_i with $i \in \mathfrak{P}$. This may be the case even if $\mathfrak{P} \supseteq \mathfrak{T}$.

Definition 5.15. $\tau \in \bar{\Phi}_A^L$ is a *pyramid* with vertex $i \in \tau$ if $\dim(\tau \setminus \{i\}) < \dim(\tau)$.

Lemma 5.16. *Let $(\tau, \mathfrak{T}) \in \bar{\Phi}_A^L$. There is no chart $X_{\mathfrak{P}}$ that meets both Y and $\bar{C}_A^{\tau, \mathfrak{T}}$ if and only if either $\mathfrak{T} \cap \mathfrak{V}_0 \neq \emptyset$ or τ is a pyramid with vertex in \mathfrak{V}_∞ . Moreover, if $\bar{C}_A^{\tau, \mathfrak{T}}$ is a component of $\text{CC}^{L'}(\mathcal{M}_A(\beta))$ for L' near L and meets some chart $X_{\mathfrak{P}}$ which meets Y then its projection also meets Y .*

Proof. To prove the first statement, assume that $\mathfrak{T} \cap \mathfrak{V}_0 = \emptyset$ and $\bar{C}_A^{\tau, \mathfrak{T}}$ does not meet the chart $X_{\mathfrak{P}}$ with $\mathfrak{P} = \mathfrak{V}_\infty \cup \mathfrak{T}$. Then $C_A^{\tau, \mathfrak{T}} \subseteq \text{Var}(x_i)$ for some $i \in \mathfrak{V}_\infty$ by the preceding discussion. Since $C_A^{\tau, \mathfrak{T}}$ is the conormal to $O_A^{\tau, \mathfrak{T}}$, the ideal $I_A^{\tau, \mathfrak{T}}$ is hence independent of ∂_i . By definition of $I_A^{\tau, \mathfrak{T}}$ through the toric ideal I_τ , this is equivalent to τ being a pyramid with vertex i . The converse implication of the first statement is obvious.

Under the hypothesis of the second statement, the ideal $\mathcal{P}_{\tau, \mathfrak{T}}$ of $\bar{C}_A^{\tau, \mathfrak{T}}$ is an associated prime of an L - and an L' -homogeneous ideal for some L' near L . As such, $\mathcal{P}_{\tau, \mathfrak{T}}$ is (L, L') - and hence (F, V) -bihomogeneous. It follows that the non-trivial $W_{(\mathfrak{P})}$ -ideal $P' = P_{\tau, \mathfrak{T}, (\mathfrak{P})} + \langle \partial_{\mathfrak{P}}, \partial'_{\mathfrak{P}} \rangle$ is (F, V) -bihomogeneous, generated by $\langle \partial_{\mathfrak{P}}, \partial'_{\mathfrak{P}} \rangle$ and some ∂ -independent V -homogeneous elements. In particular, $P' + \langle x_{\mathfrak{V}_0}, x'_{\mathfrak{V}_\infty} \rangle$ is non-trivial and so $\bar{C}_A^{\tau, \mathfrak{T}}$ meets Y . \square

From the dimension one case, it is clear that non- (F, V) -bihomogeneous components are relevant for slopes. Indeed, the L -characteristic variety equals $\text{Var}(x_1\partial_1)$ for most L -homogeneous differential operators in one variable x_1 . However we believe that the non- (F, V) -bihomogeneous components are irrelevant in our case. More precisely, it seems that the conormal space of an L -homogeneous non- (F, V) -bihomogeneous pointed toric ring can not meet any point T_y^*X with $y \in Y$ unless

$\mathfrak{V}_\infty = \emptyset$. This would imply the following combinatorial description of slopes of projectivized A -hypergeometric systems along coordinate varieties.

Definition 5.17. Let $L = pF + qV$ with $p/q \in \mathbb{Q}_{>0} \cup \{\infty\}$ where V is the V -filtration along $Y = \text{Var}(x_{\mathfrak{V}_0}, x'_{\mathfrak{V}_\infty})$. By $\Phi_A^Y(p/q)$ we denote the subset of $\tau \in \Phi_A^L$ which are not a pyramid with vertex in \mathfrak{V}_∞ .

Conjecture 5.18. *The slopes of the projectivized A -hypergeometric system $\mathcal{M}_A(\beta)$ along a coordinate variety Y are the jump parameters $p/q \in \mathbb{Q}_{>0} \cup \{\infty\}$ of $\Phi_A^Y(p/q)$.*

APPENDIX A. ON SPECTRAL SEQUENCES

Let (K^\bullet, d) be a complex of groups equipped with a descending filtration F of subgroups such that $d(F^p K^\bullet) \subseteq F^p K^{\bullet+1}$. We stress that there are no boundedness conditions on the filtration. We abbreviate $\text{gr} = \text{gr}_F$ and freely use induced filtrations on sub- and quotient groups.

By [God58, Thm. I.4.2.1] and [God58, Thm. I.4.2.2] there is an associated spectral sequence involving the following data:

$$(A.1) \quad \begin{aligned} Z_r^{p,q} &= F^p K^{p+q} \cap d^{-1}(F^{p+r} K^{p+q+1}), \\ B_r^{p,q} &= F^p K^{p+q} \cap d(F^{p-r} K^{p+q-1}), \\ E_r^{p,q} &= \text{gr}^p \frac{d^{-1}(F^{p+r} K^{p+q+1})}{d(F^{p-r+1} K^{p+q-1})} = \text{gr}^p \frac{Z_r^{p,q}}{B_{r-1}^{p,q}} = Z_r^{p,q} / (Z_{r-1}^{p+1,q-1} + B_{r-1}^{p,q}). \end{aligned}$$

Note that $Z_r^{p,q} \cap F^{p+1} K^{p+q} = Z_{r-1}^{p+1,q-1}$, $d(Z_r^{p-r,q+r-1}) = B_r^{p,q}$, $E_0^{p,q} = \text{gr}^p K^{p+q}$, and $E_1^{p,q} = H^{p+q}(\text{gr}^p(K^\bullet), \text{gr}^p(d))$.

For all p, q and r , d induces a group homomorphism $d_r^{p,q} : E_r^{p,q} \rightarrow E_r^{p+r,q-r+1}$ such that $d_r^{p,q} \circ d_r^{p-r,q+r+1} = 0$. More explicitly,

$$(A.2) \quad d_r^{p,q} \left(z \bmod (Z_{r-1}^{p+1,q-1} + B_{r-1}^{p,q}) \right) = d(z) \bmod (Z_{r-1}^{p+1+r,q-r} + B_{r-1}^{p+r,q-r+1})$$

and $E_r^{p,q} = (\ker d_r^{p,q}) / (\text{im } d_r^{p-r,q+r-1})$. Let

$$(A.3) \quad \begin{aligned} Z_\infty^{p,q} &= F^p K^{p+q} \cap d^{-1}(0), \\ B_\infty^{p,q} &= F^p K^{p+1} \cap d(K^{p+q-1}), \\ E_\infty^{p,q} &= \text{gr}^p H^{p+q}(K^\bullet) = \text{gr}^p \frac{Z_\infty^{p,q}}{B_\infty^{p,q}} = Z_\infty^{p,q} / (Z_\infty^{p+1,q-1} + B_\infty^{p,q}). \end{aligned}$$

In general, there is no relation between the terms $E_r^{p,q}$ for finite r and $E_\infty^{p,q}$. The purpose of this section is to prove the following theorem and its corollary below.

Theorem A.1. *Let D be a ring and (K^\bullet, d) be a complex of D -modules, both equipped with a descending filtration F of subgroups subject to the hypotheses:*

- (a) D is a filtered ring: $(F^p D) \cdot (F^{p'} D) \subseteq F^{p+p'} D$.
- (b) K^\bullet is a filtered D -module: $(F^p D) \cdot (F^{p'} K^n) \subseteq F^{p+p'} K^n$.
- (c) d is compatible with F : $d(F^p K^{p+q}) \subseteq F^p K^{p+q+1}$.
- (d) $E_0 = \text{gr}(K^\bullet)$ is a Noetherian $\text{gr}(D)$ -module.
- (e) F is exhaustive and separated on K^\bullet : $\bigcup_p F^p K^n = K^n$, $\bigcap_p F^p K^n = 0$.

Then one has the consequences:

- (1) $d_r : E_r \rightarrow E_r$ are graded $\text{gr}(D)$ -linear maps of graded $\text{gr}(D)$ -modules with $\text{gr}^{p'}(D) \cdot E_r^{p,q} \subseteq E_r^{p+p',q-p'}$.
- (2) E_r is Noetherian and d_r is the zero map for all sufficiently large r .

(3) The stable value of E_r is isomorphic to E_∞ .

Proof. We abbreviate $W = \text{gr}(D)$.

(1) follows from (a), (b), and (c): E_0 is a graded W -module by (a) and (b). By induction, we may assume that E_r is a graded W -module and d_{r-1} is graded W -linear. Let $\delta \in F^{p'}D$ and $c \in Z_r^{p,q}$ represent non zero elements $\bar{\delta} \in W^{p'}$ and $\bar{c} \in E_r^{p,q}$. By (A.2) and (c),

$$\begin{aligned} d_r(\bar{\delta} \cdot \bar{c}) &= d_r\left((\delta \bmod F^{p'+1}D) \cdot (c \bmod (Z_{r-1}^{p+1,q-1} + B_{r-1}^{p,q}))\right) \\ &= d_r\left(\delta c \bmod (Z_{r-1}^{p+p'+1,q-p'-1} + B_{r-1}^{p+p',q-p'})\right) \\ &= d(\delta c) \bmod (Z_{r-1}^{p+p'+r+1,q-p'-r-1} + B_{r-1}^{p+p'+r,q-r-p'}) \\ &= \delta d(c) \bmod (Z_{r-1}^{p+p'+r+1,q-p'-r-1} + B_{r-1}^{p+p'+r,q-r-p'}) \\ &= (\delta \bmod F^{p'+1}D) \cdot \left(d(c) \bmod (Z_{r-1}^{p+r+1,q-r-1} + B_{r-1}^{p+r,q-r})\right) \\ &= (\delta \bmod F^{p'+1}D) \cdot d_r\left(c \bmod (Z_{r-1}^{p+1,q-1} + B_{r-1}^{p,q})\right) \\ &= \bar{\delta} \cdot d_r(\bar{c}) \in E_r^{p+p',q-p'}. \end{aligned}$$

Thus, d_r is graded W -linear and $\ker(d_r)$, $\text{im}(d_r)$, and $E_r = \ker(d_r)/\text{im}(d_r)$ are graded W -modules.

(2) follows from (d): As a subquotient of E_r , E_{r+1} is Noetherian by induction. Let $\pi_r: \ker(d_r) \rightarrow \ker(d_r)/\text{im}(d_r) = E_{r+1}$ denote the natural projection. Note that $\pi_r^{-1}(\text{im}(d_{r+1})) \supseteq \pi_r^{-1}(0) = \text{im}(d_r)$. Then the chain

$$\text{im}(d_0) \subseteq \pi_0^{-1}(\text{im}(d_1)) \subseteq \pi_0^{-1}(\pi_1^{-1} \text{im}(d_2)) \subseteq \dots$$

of submodules of E_0 must stabilize at some r_0 : $\pi_r^{-1}(\text{im}(d_{r+1})) = \text{im}(d_r) = \pi_r^{-1}(0)$ and hence $\text{im}(d_{r+1}) = 0$ for all $r \geq r_0$.

(3) follows from (e):

$$\begin{aligned} B_\infty^{p,q} &= F^p K^{p+q} \cap d(K^{p+q-1}) = F^p K^{p+q} \cap d\left(\bigcup_r F^{p-r} K^{p+q-1}\right) \\ &= \bigcup_r (F^p K^{p+q} \cap d(F^{p-r} K^{p+q-1})) = \bigcup_r B_r^{p,q}, \\ Z_\infty^{p,q} &= F^p K^{p+q} \cap d^{-1}(0) = F^p K^{p+q} \cap d^{-1}\left(\bigcap_r F^{p+r} K^{p+q+1}\right) \\ &= \bigcap_r (F^p K^{p+q} \cap d^{-1}(F^{p+r} K^{p+q+1})) = \bigcap_r Z_r^{p,q}. \end{aligned}$$

By (A.2), $d_r = 0$ means that

$$(A.4) \quad B_r^{p,q} = d(Z_r^{p-r,q+r-1}) \subseteq Z_{r-1}^{p+1,q-1} + B_{r-1}^{p,q} \subseteq B_{r-1}^{p,q} + F^{p+1}K^{p+q}$$

and also

$$d(Z_r^{p,q}) \subseteq Z_{r-1}^{p+r+1,q-r} + B_{r-1}^{p+r,q-r+1} = Z_{r-1}^{p+r+1,q-r} + d(Z_{r-1}^{p+1,q-1})$$

which implies that

$$(A.5) \quad Z_r^{p,q} \subseteq Z_{r+1}^{p,q} + Z_{r-1}^{p+1,q-1} \subseteq Z_{r+1}^{p,q} + F^{p+1}K^{p+q}.$$

From (A.4) and (A.5) follows $\mathrm{gr}^p B_r^{p,q} = \mathrm{gr}^p B_{r-1}^{p,q}$ and $\mathrm{gr}^p Z_r^{p,q} = \mathrm{gr}^p Z_{r+1}^{p,q}$ for $r \geq r_0$. By (A.1) and (A.3) and exactness of gr for induced filtrations, we conclude that, for $r \geq r_0$,

$$E_r^{p,q} = \mathrm{gr}^p \frac{Z_r^{p,q}}{B_{r-1}^{p,q}} = \frac{\mathrm{gr}^p Z_r^{p,q}}{\mathrm{gr}^p B_{r-1}^{p,q}} = \frac{\mathrm{gr}^p (\bigcap_r Z_r^{p,q})}{\mathrm{gr}^p (\bigcup_r B_{r-1}^{p,q})} = \frac{\mathrm{gr}^p Z_\infty^{p,q}}{\mathrm{gr}^p B_\infty^{p,q}} = \mathrm{gr}^p \frac{Z_\infty^{p,q}}{B_\infty^{p,q}} = E_\infty^{p,q}.$$

□

Corollary A.2. *Under the hypotheses of Theorem A.1 assume that $W = \mathrm{gr}(D)$ is commutative, let $P \in \mathrm{Spec}(W)$ be a prime ideal and denote by W_P the local ring of $\mathrm{Spec}(W)$ at P . Then there is a spectral sequence*

$$H^p(\mathrm{gr}(K^\bullet) \otimes_W W_P) = {}'E_1 \implies {}'E_\infty = (\mathrm{gr}(H^p(K^\bullet))) \otimes_W W_P.$$

This spectral sequence collapses eventually and converges: $'E_r = {}'E_{r+1} = {}'E_\infty$ for all large r .

Proof. The spectral sequence (E_r, d_r) of Theorem A.1 is a sequence of W -modules and W -morphisms. Since localization commutes with the formation of kernels and cokernels, the objects and morphisms $({}'E_r, {}'d_r) = (E_r \otimes_W W_P, d_r \otimes_W W_P)$ form a spectral sequence. Since (E_r, d_r) collapses, so does $({}'E_r, {}'d_r)$, converging to the localization of E_∞ . □

Remark A.3. As far as we can see, the collapse of the sequence is essential for the corollary. If in general (E_r, d_r) is a spectral sequence of W -modules and W -morphisms converging to E_∞ in the sense that $E_\infty = \bigcap_r Z_r / \bigcup_r B_r$ then while $(\bigcup_r B_r) \otimes_W W_P = \bigcup_r (B_r \otimes_W W_P)$, it is not necessarily the case that $(\bigcap_r Z_r) \otimes_W W_P = \bigcap_r (Z_r \otimes_W W_P)$. The problem is that infinite intersections (in contrast to infinite unions) and homomorphisms (such as the localization morphism) are not necessarily interchangeable.

In the situation of the theorem it is probably not sufficient to replace condition (d) by “ K^\bullet consists of Noetherian D -modules”. For example, in [ST04] some rings appear in the theory of hypergeometric systems that are Noetherian, but the order filtration leads to non-Noetherian associated graded rings.

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