Solutions X3

53.a) I assume you can do it.

53.b) $n \cdot f(x,y) = \frac{\partial}{\partial t}\left(t^n f(x,y)\right)(t=0) = \frac{\partial}{\partial t}\left(f(tx,ty)\right)(t=0)$

$\overset{\text{chain}}{\underset{\text{rule}}{=}} f_x(tx,ty)\frac{\partial(tx)}{\partial t}(t=0) + f_y(tx,ty)\frac{\partial(ty)}{\partial t}(t=0)$

$= f_x(x,y)x + f_y(x,y)y$

$= x\frac{\partial f}{\partial x}(x,y) + y\frac{\partial f}{\partial y}(x,y).$

54) By 53 we know that

$$n^2 f = \left(x\frac{\partial}{\partial x} + y\frac{\partial}{\partial y}\right)^2 f$$

$$= \Big(\underbrace{x\frac{\partial}{\partial x}x\frac{\partial}{\partial x}}_{=x^2\frac{\partial^2}{\partial x^2}+x\frac{\partial}{\partial x}} + \underbrace{x\frac{\partial}{\partial x}y\frac{\partial}{\partial y} + y\frac{\partial}{\partial y}x\frac{\partial}{\partial x}}_{=2xy\frac{\partial}{\partial x}\frac{\partial}{\partial y}} + \underbrace{y\frac{\partial}{\partial y}y\frac{\partial}{\partial y}}_{=y^2\frac{\partial^2}{\partial y^2}+y\frac{\partial}{\partial y}}\Big)f$$

$$= \left(x^2\frac{\partial^2}{\partial x^2} + 2xy\frac{\partial}{\partial x}\frac{\partial}{\partial y} + y^2\frac{\partial^2}{\partial y^2}\right)f + \underbrace{\left(x\frac{\partial}{\partial x} + y\frac{\partial}{\partial y}\right)f}_{=n\cdot f \text{ by } 53}$$

Hence,

$$n(n-1)f = (n^2-n)f = x^2\frac{\partial^2 f}{\partial x^2} + 2xy\frac{\partial^2 f}{\partial x \partial y} + y^2\frac{\partial^2 f}{\partial y^2}$$

55) $t^n f_x(x,y) = \frac{\partial}{\partial x}\left(t^n f(x,y)\right) = \frac{\partial}{\partial x}\left(f(tx,ty)\right)$

$\overset{\text{chain}}{\underset{\text{rule}}{=}} f_x(tx,ty)\underbrace{\frac{\partial(tx)}{\partial x}}_{=t} + f_y(tx,ty)\underbrace{\frac{\partial(ty)}{\partial x}}_{=0}$

$= t f_x(tx,ty)$

Divide by t to obtain:

$$t^{n-1}\frac{\partial f}{\partial x}(x,y) = \frac{\partial f}{\partial x}(tx,ty).$$

*) If $f(t^2x, t^3y) = t^n f(x,y)$ the computation in 53.b gives $n \cdot f(x,y) = 2x\frac{\partial f}{\partial x}(x,y) + 3y\frac{\partial f}{\partial y}(x,y)$. Such a function can be homogeneous, e.g. $f(x,y) = x$.