Given name:____________________ **Family name:**____________________

Student number:____________________ **Signature:**____________________________

OKLAHOMA STATE UNIVERSITY
Department of Mathematics

Calculus III (MATH 2163)

Instructor: Mathias Schulze

FINAL EXAM
December 11/15, 2006

Duration: 110 minutes

No aids allowed.

This examination paper consists of **5** pages and **8** questions. Please bring any discrepancy to the attention of an invigilator. The number in brackets at the start of each question is the number of points the question is worth.

Answer **7 of 8** questions.

To obtain credit, you must explicitly state your result and give arguments to support your answer.

For graders' use:

	Score
1 (10)	
2 (10)	
3 (10)	
4 (10)	
5 (10)	
6 (10)	
7 (10)	
8 (10)	
Total (80)	

1. [10] Find the linearization of the function $f(x,y) = \sqrt{20 - x^2 - 7y^2}$ at $(2,1)$ and write it in the form $L(x,y) = a + bx + cy$. Use your result to approximate $f(1.95, 1.08)$. Write down the differential of f explicitly.

   Solution: In the review session I recommended to review linearization and differential at home. The first part was even a homework problem for Lecture 10.

   First compute the partial derivatives,

$$f_x = -\frac{x}{\sqrt{20 - x^2 - 7y^2}}, \quad f_y = -\frac{7y}{\sqrt{20 - x^2 - 7y^2}},$$

   and evaluate at $(2,1)$, $f(2,1) = 3$, $f_x(2,1) = -\frac{2}{3}$, $f_y(2,1) = -\frac{7}{3}$. Then the linearization of f at $(2,1)$ is given by the formula

$$\begin{aligned} L(x,y) &= f(2,1) + f_x(2,1)(x-2) + f_y(2,1)(y-1) \\ &= 3 - \frac{2}{3}(x-2) - \frac{7}{3}(y-1) = 3 + \frac{4}{3} + \frac{7}{3} - \frac{2}{3}x - \frac{7}{3}y = \frac{20}{3} - \frac{2}{3}x - \frac{7}{3}y. \end{aligned}$$

   The linear approximation of f at $(1.95, 1.08)$ is

$$f(1.95, 1.08) \approx L\left(2 - \frac{5}{100}, 1 + \frac{8}{100}\right) = 3 + \frac{2 \cdot 5}{3 \cdot 100} - \frac{7 \cdot 8}{3 \cdot 100} = \frac{427}{150} \approx 2.847.$$

   The total differential of f is

$$df = f_x\,dx + f_y\,dy = -\frac{x\,dx + 7y\,dy}{\sqrt{20 - x^2 - 7y^2}}.$$

2. [10] Compute the gradient of $f(x,y) = \frac{y^2}{x}$. Determine the maximal and minimal rates of change of f at $(2,4)$ and unit vectors in the directions in which these rates of change occur. Find a unit vector in a direction in which f has zero rate of change at $(2,4)$.

   Solution: In the review session I recommended to review gradient, directional derivative, and rate of change at home.

   The gradient equals

$$\vec{\nabla} f(x,y) = \left\langle -\frac{y^2}{x^2}, 2\frac{y}{x} \right\rangle.$$

   The maximal and minimal/minimal rate of change of f at $(2,4)$ equals $\pm|\vec{\nabla} f(2,4)| = \pm|\langle -4, 4\rangle| = \pm 4\sqrt{2}$ and occurs in direction of

$$\pm\frac{\vec{\nabla} f(2,4)}{|\vec{\nabla} f(2,4)|} = \pm\frac{\langle -4, 4\rangle}{4\sqrt{2}} = \left\langle \mp\frac{1}{\sqrt{2}}, \pm\frac{1}{\sqrt{2}} \right\rangle.$$

   Orthogonally to these directions the rate of change of f at $(2,4)$ is zero. The corresponding unit vectors are simply $\pm\left\langle \frac{1}{\sqrt{2}}, \frac{1}{\sqrt{2}} \right\rangle$.

3. [10] Use Lagrange multipliers to find the maximal and minimal values of the function $f(x,y) = x^2y^2$ subject to the constraint $x^2+y^2 = 1$. (Hint: There are 8 critical points.)

Solution: This is essentially a simplified version of a problem discussed in the review session and of a homework problem for Lecture 19.

The gradients of f and g are

$$\vec{\nabla} f = 2\big\langle xy^2, x^2y\big\rangle, \quad \vec{\nabla} g = 2\langle x, y\rangle.$$

By the method of Lagrange multipliers, we have to solve the system

$$xy^2 = \lambda x, \quad x^2y = \lambda y, \quad x^2 + y^2 = 1.$$

By the last equation and symmetry we may assume that $x \neq 0$. Then the first equation gives $\lambda = y^2$ and substituting into the second yields $(x^2 - y^2)y = 0$. So either $y = 0$, and hence $x = \pm 1$ by the third equation, or $0 = x^2 - y^2 = (x-y)(x+y)$. In this latter case $x = \pm y$ and the third equality gives $|x| = |y| = \frac{1}{\sqrt{2}}$. Reconsidering or symmetry argument, we have found 8 points: $(\pm 1, 0)$, $(0, \pm 1)$, $(\pm\frac{1}{\sqrt{2}}, \pm\frac{1}{\sqrt{2}})$, $(\pm\frac{1}{\sqrt{2}}, \mp\frac{1}{\sqrt{2}})$. The function f is zero at the former 4 points and equals $\frac{1}{4}$ at the latter 4 points. So 0 is the only minimal value and $\frac{1}{4}$ is the only maximal value of f subject to the given constraint.

4. [10] Use spherical coordinates to find the volume of the solid that lies above the cone $z = \sqrt{x^2+y^2}$ and below the sphere $x^2 + y^2 + z^2 = z$.

Solution: This is Example 4 in Section 16.8 and has been discussed in class. In the review session I recommended to review volume computations in spherical coordinates at home.

5. [10] Integrate the vector field

$$\vec{F}(x,y) = \left\langle y - \frac{x}{\sqrt{x^2+y^2}^3}, -x - \frac{y}{\sqrt{x^2+y^2}^3} \right\rangle$$

along a clockwise oriented unit circle C centered at the origin. (Hint: Write $\vec{F}$ as a sum of two vector fields.) Explain why $\vec{F}$ is not conservative.

Solution: All ideas to solve this problem have been discussed in the review session. The vector field decomposes as $\vec{F} = \vec{G} + \vec{H}$ where $\vec{G}(x,y) = \langle y, -x\rangle$ and

$$\vec{H}(x,y) = \left\langle -\frac{x}{\sqrt{x^2+y^2}^3}, -\frac{y}{\sqrt{x^2+y^2}^3} \right\rangle.$$

Up to constants, $\vec{H}$ is just the gravitational/electric force known from the lecture. Hence $\vec{H} = \vec{\nabla} h$ is conservative where, again up to constants, $h(x,y) = \frac{1}{\sqrt{x^2+y^2}}$ is the gravitational/electric potential. By the fundamental theorem of line integrals $\int_C \vec{H} \cdot d\vec{r} = 0$ and so

$$\int_C \vec{F} \cdot d\vec{r} = \int_C \vec{G} \cdot d\vec{r}.$$

A clockwise parametrization of C is given by $\vec{r}(t) = \langle \cos t, -\sin t\rangle$ where $0 \le t \le 2\pi$ and hence

$$\begin{aligned}\int_C \vec{G} \cdot d\vec{r} &= \int_0^{2\pi} \vec{G}(\vec{r}(t)) \cdot \vec{r}'(t)\, dt \\ &= \int_0^{2\pi} \langle -\sin t, -\cos t\rangle \cdot \langle -\sin t, -\cos t\rangle\, dt = \int_0^{2\pi} dt = 2\pi.\end{aligned}$$

In particular, this result tells us that $\vec{F}$ can not be conservative because integrals of conservative vector fields along closed curves are always zero.

6. [10] Evaluate the line integral $\int_C z\,dx + x\,dy + y\,dz$ where C is the curve parametrized by $x(t) = t, y(t) = t^2, z(t) = t^3$ for $0 \le t \le 1$.

Solution: This is equivalent to a homework problem for Lecture 41. Moreover, line integral w.r.t. coordinates have been discussed in the review session.

One computes

$$\begin{aligned}\int_C z\,dx + x\,dy + y\,dz &= \int_0^1 z(t)x'(t) + x(t)y'(t) + y(t)z'(t)\, dt \\ &= \int_0^1 t^3 \cdot 1 + t \cdot 2t + t^2 \cdot 3t^2\, dt \\ &= \frac{1}{4} + \frac{2}{3} + \frac{3}{5} = \frac{15+40+36}{60} = \frac{91}{60}.\end{aligned}$$

7. [10] Compute the mass center of the circular cone C with constant density ρ defined by the inequalities $0 \leq z \leq 1-r$.

Solution: A slightly modified version of this problem has been discussed in the review session.

By symmetry reasons the x- and y-coordinates of the mass center equal 0. Its z-coordinate is computed by the integral

$$\bar{z} = \frac{\int_C z\rho \, dV}{\int_C \rho \, dV} = \frac{\int_C z \, dV}{\int_C dV}.$$

To compute each of these integrals we use polar coordinates. There are values of z such that $0 \leq z \leq 1-r$ exactly if $r \leq 1$. Therefore

$$\int_C dV = \int_0^{2\pi} \int_0^1 \int_0^{1-r} dz \, r dr \, d\theta = 2\pi \int_0^1 r - r^2 \, dr = 2\pi \left(\frac{1}{2} - \frac{1}{3} \right) = \frac{\pi}{3}$$

and similarly

$$\int_C z \, dV = \pi \int_0^1 (1-r)^2 r \, dr = \pi \int_0^1 r - 2r^2 + r^3 \, dr = \pi \left(\frac{1}{2} - \frac{2}{3} + \frac{1}{4} \right) = \frac{\pi}{12}.$$

Thus $\bar{z} = \frac{1}{4}$ and the mass center is $(0, 0, \frac{1}{4})$.

8. [10] Determine whether or not the vector field $\vec{F} = \langle 6x + 5y, 5x + 4y \rangle$ is conservative. If it is, find a potential for $\vec{F}$. How many potentials for $\vec{F}$ are there?

Solution: This example is known from the review session.

Set $P = 6x + 5y$ and $Q = 5x + 4y$ such that $\vec{F} = \langle P, Q \rangle$. Then $\frac{\partial P}{\partial y} = 5 = \frac{\partial Q}{\partial y}$ and hence $\vec{F}$ is conservative by the criterion from the lecture.

To find f we have to solve $\langle P, Q \rangle = \vec{F} = \vec{\nabla} f = \langle f_x, f_y \rangle$ and hence the system of linear partial differential equations

$$6x + 5y = f_x, \quad 5x + 4y = f_y.$$

Integrating the first equation gives $3x^2 + 5xy + g = f$ for some function $g = g(y)$. Substituting into the second equation we find $5x + g' = f_y = 5x + 4y$ and then $g = 2y^2 + c$ for any constant c.

By computing the gradient, we easily verify that $f = 3x^2 + 5xy + 2y^2 + c$ is really a potential for $\vec{F}$ for any constant c. In particular, there is an infinite number of such potentials, a different one for each real number c.

End of examination
Total pages: 5
Total marks: 80