First Name: ________________ **Last Name:** ________________

OSU Number: ________________ **Signature:** ________________

OKLAHOMA STATE UNIVERSITY
Department of Mathematics

MATH 2144 (Calculus I)
Instructor: Dr. Mathias Schulze

MIDTERM 2
October 29, 2008

Duration: 50 minutes

No aids allowed.

This examination paper consists of **7** pages and **6** questions. Please bring any discrepancy to the attention of an invigilator. The number in brackets at the start of each question is the number of points the question is worth.

Answer **5 of 6** questions.

To obtain credit, you must give arguments to support your answers.

For graders' use:

	Score
1 (10)	
2 (10)	
3 (10)	
4 (10)	
5 (0)	
6 (0)	
Total (60)	

1. [10] Compute $y' = \frac{dy}{dx}$.

   (a) $y = 5^{4^{x^2}}$

   (b) $\frac{5}{x} + \frac{5}{y} = 3$

   (c) $y = \frac{\sin^2(x)\tan^6(x)}{(x^2+1)^2}$

   (d) $y = \sinh^{-1}(x)$

Solution:

(a) $y' = \ln 5 \cdot 5^{4^{x^2}} \cdot \ln 4 \cdot 4^{x^2} \cdot 2 \cdot x = 2 \cdot \ln 4 \cdot \ln 5 \cdot x \cdot 4^{x^2} \cdot 5^{4^{x^2}}$

(b) $-5x^{-2} - 5y^{-2}y' = 0 \Rightarrow y' = -\frac{y^2}{x^2} = -\frac{1}{x^2}\left(\frac{5}{3-\frac{5}{x}}\right)^2 = -\frac{25}{(3x-5)^2}$

(c) $y'/y = (\ln y)' = (2\ln\sin x + 6\ln\tan x - 2\ln(x^2+1))' = 2\frac{\cos x}{\sin x} + 6\frac{\sec^2 x}{\tan x} - \frac{4x}{x^2+1}$

(d) $y' = \frac{1}{\sinh'(y)} = \frac{1}{\cosh(\sinh^{-1}(x))} = \frac{1}{\sqrt{1+\sinh^2(\sinh^{-1}(x))}} = \frac{1}{\sqrt{1+x^2}}$

2. [10] Compute the limits.

   (a) $\lim_{x\to\infty} \frac{\ln(5x)}{\sqrt{5x}}$

   (b) $\lim_{x\to-\infty} (x^2 \cdot e^{2x})$

   (c) $\lim_{x\to 1} \left(\frac{x}{x-1} - \frac{1}{\ln x}\right)$

   (d) $\lim_{x\to\infty} \left(1 + \frac{1}{x}\right)^x$

Solution:

(a) $\lim_{x\to\infty} \frac{\ln(5x)}{\sqrt{5x}} = \lim_{x\to\infty} \frac{\frac{1}{x}}{\frac{5}{2\sqrt{5x}}} = 2\lim_{x\to\infty} \frac{\sqrt{5x}}{5x} = 2\lim_{x\to\infty} \frac{1}{\sqrt{5x}} = 0$

(b) $\lim_{x\to-\infty} (x^2 \cdot e^{2x}) = \lim_{x\to-\infty} \frac{x^2}{e^{-2x}} = \lim_{x\to-\infty} \frac{2}{4e^{-2x}} = \frac{1}{2}\lim_{x\to-\infty} e^{2x} = 0$

(c) $\lim_{x\to 1} \left(\frac{x}{x-1} - \frac{1}{\ln x}\right) = \lim_{x\to 1} \frac{x\ln x - (x-1)}{(x-1)\ln x} = \lim_{x\to 1} \frac{\ln x + 1 - 1}{\ln x + 1 - \frac{1}{x}} = \lim_{x\to 1} \frac{\frac{1}{x}}{\frac{1}{x} + \frac{1}{x^2}} = \frac{1}{2}$

(d) $\lim_{x\to\infty} \left(1 + \frac{1}{x}\right)^x = e^{\lim_{x\to\infty} \frac{\ln(1+\frac{1}{x})}{x^{-1}}} = e^{\lim_{x\to\infty} \frac{\frac{-x^{-2}}{1+\frac{1}{x}}}{-x^{-2}}} = e^{\lim_{x\to\infty}(1+\frac{1}{x})} = e$

3. [10]

(a) Find the number c that satisfies the conclusion of the Mean Value Theorem:

$$f(x) = x^3 + x - 7, \quad [0, 2].$$

(b) Suppose that $3 \leq f'(x) \leq 4$ for all values of x. Find numbers a and b such that

$$a \leq f(5) - f(3) \leq b.$$

Solution:

(a) $3c^2 + 1 = f'(c) = \frac{f(2)-f(0)}{2-0} = \frac{8+2}{2} = 5$ gives $c = \frac{2}{\sqrt{3}}$.

(b) $f(5) - f(3) = (5-3) \cdot f'(c)$ for some $c \in (3, 5)$ and hence $2 \cdot 3 \leq f(5) - f(3) \leq 2 \cdot 4$. So $a = 6$ and $b = 8$ works.

4. [10] A particle moves along the curve $y = \sqrt{1 + x^3}$. As it reaches the point $(2, 3)$, the y-coordinate is increasing at a rate of 12 cm/s. How fast is the x-coordinate of the point changing at that instant?

Solution: Differentiating $y^2 = 1 + x^3$ with respect to time gives $2y\dot{y} = 3x^2\dot{x}$ and hence $\dot{x} = \frac{2}{3}\frac{y\dot{y}}{x^2} = \frac{2}{3}\frac{3 \cdot 12}{2^2}\frac{\text{cm}}{\text{s}} = 2 \cdot 3\,\frac{\text{cm}}{\text{s}} = 6\,\frac{\text{cm}}{\text{s}}$.

5. [10] A piece of wire, 10 m long, is cut into 2 pieces. One piece is bent into a square and the other is bent into an equiliteral triangle. How should the wire be cut so that the total area enclosed is a maximum?

Solution:

- Draw a diagram (omitted here).
- Introduce notation: $l = 10$ – length of the wire, x – length of first piece, y – length of the second piece, a – width/height of the square, b – length of each side of the triangle, h – height of the triangle, A – area of the square, B – area of the triangle, C – total area.
- Write down relations:

$$l = x + y,\ x = 4a,\ A = a^2,\ y = 3b,\ b^2 = h^2 + \frac{1}{4}b^2,\ B = \frac{1}{2}bh,\ C = A + B.$$

- Eliminate variables:

$$C = A + B = a^2 + \frac{1}{2}bh = a^2 + \frac{1}{2}\frac{\sqrt{3}}{2}b^2 = \frac{x^2}{4} + \frac{\sqrt{3}}{4 \cdot 9}y^2 = \frac{1}{4}(x^2 + \frac{\sqrt{3}}{9}(l - x)^2).$$

- Compute citical numbers: $0 = \frac{dC}{dx} \Leftrightarrow 0 = x - \frac{\sqrt{3}}{9}(l - x) \Leftrightarrow x = \frac{\sqrt{3}}{9+\sqrt{3}} \cdot l.$
- Determine the maximum (closed interval method): $0 \le x \le l$

$$C\left(\frac{\sqrt{3}}{9 + \sqrt{3}} \cdot l\right) = \frac{1}{4}\frac{\sqrt{3}}{9 + \sqrt{3}} \cdot l^2 < C(0) = \frac{1}{4}\frac{\sqrt{3}}{9} \cdot l^2 < C(l) = \frac{1}{4} \cdot l^2.$$

- Result: Use the entire wire for the square to get the maximal total area of 25 square meters.

6. [10] Discuss the properties of the function $f(x) = \frac{2x^2}{x^2-1}$ and sketch its graph. Make sure that your discussion covers steps A-H from Section 4.5.

Solution:

- Domain: $x \neq \pm 1$
- Intercepts: $x = 0 \Leftrightarrow y = 0$, so $(0, 0)$ is the only intercept.
- Symmetry: even function, as only squares of x appear.
- Periodicity: none
- Assymptotes: $x = \pm 1$, $y = 2$.
- Derivatives: $f'(x) = -\frac{4x}{(x^2-1)^2}$, $f''(x) = 4\frac{3x^2+1}{(x^2-1)^3}$.
- Critical points: $f'(x) = 0 \Rightarrow x = 0$, so $(0, 0)$ is the only critical point.
- Extrema: $f''(0) = -4$, so $(0, 0)$ is a local maximum.
- Growth: increasing on $(-\infty, -1) \cup (-1, 0)$, increasing on $(0, 1) \cup (1, \infty)$.
- Concavity: concave down on $(-1, 1)$, concave up on $(-\infty, -1) \cup (1, \infty)$.
- Inflection points: none
- Sketch graph (omitted here).

End of examination
Total pages: 7
Total marks: 60