1.8.1 Let $S \subseteq \mathbb{R}^n$ be compact. Then the open covering $S = \bigcup_{k \in \mathbb{N}} B(0,k) \cap S$ has a finite subcovering, say with indices $k_1, \dots, k_r$. Setting $K := \max\{k_1, \dots, k_r\}$ gives $S = B(0,K) \cap S$, so S is bounded.

1.8.2 Let $S \subseteq \mathbb{R}^n$ be compact. If S is not closed then $\exists\, p \in \overline{S} \setminus S$. Hence, the open covering $S = \bigcup_{\varepsilon > 0} S - \overline{B(p,\varepsilon)}$ can not have a finite subcovering by definition of a cluster point. Therefore, $S = \overline{S}$.

1.8.3 Let S be compact and $T \subset S$ closed. If $T = \bigcup_{i \in I} U_i$ is an open covering of T then $U_i = V_i \cap T$ for some $V_i \subset S$ open and $S = S \setminus T \cup \bigcup_{i \in I} V_i$ is an open covering of S. By compactness of S, $\exists$ finite $J \subset I$ s.th. $S = S \setminus T \cup \bigcup_{j \in J} V_j$ and hence $T = \bigcup_{j \in J} U_j$ by intersecting with T.

1.8.5 $(\Rightarrow)$ Let $S \subset \mathbb{R}^n$ be compact, i.e. closed and bounded, and let $\{p_n\} \subset S$ be a sequence in S. By Bolzano-Weierstrass, there is a convergent subsequence $\{p_{\alpha(n)}\}$. Then $p = \lim_{k \to \infty} p_{\alpha(n)}$ is a limit point of

$\{p_n\}$ by Theorem 1.6.6., and hence a limit point of S. By closedness of S, we have $p \in \bar{S} \subset S$.

($\Leftarrow$) Assume that any sequence in S has a limit point in S.

If S is not closed, pick $p \in \bar{S} \setminus S$ and, for each $n \in \mathbb{N}$, a point $p_n \in S \cap B(p, 1/n)$. Then $S \ni p_n \to p \notin S$. $\lightning$

If S is not bounded, there is, for each $n \in \mathbb{N}$, a point $p_n \in S - B(0, n)$. Then $\{p_n\}$ has no limit point $\lightning$ (Indeed, for any $q \in \mathbb{R}^n$ and $\varepsilon > 0$, pick $N > |q| + \varepsilon$. Then, for all $n \geq N$, $|p_n - q| \geq |p_n| - |q| \geq n - N + \varepsilon \geq \varepsilon$.)

Thus, S is closed and bounded and hence compact by Heine-Borel.