

3.3.7 | Let  $S \subseteq \mathbb{R}^n$  be compact with  $\overset{\circ}{S} \neq \emptyset$ , and  
 $f, g \in C^0(S) \cap C^1(\overset{\circ}{S})$  such that  $f|_{\partial S} = g|_{\partial S}$ .  
Then  $Df(p_0) = Dg(p_0)$  for some  $p_0 \in \overset{\circ}{S}$ .

proof: By subtracting  $g$  from both  $f$  and  $g$ ,  
we reduce the problem to the case where  $g = 0$ .  
By Theorem 2.4.11,  $f$  takes a maximum  
and a minimum value on  $S$ . If both  
equal zero, then  $f = 0$  and any  $p_0 \in \overset{\circ}{S}$  works.  
Otherwise,  $f$  takes an extremum at some  
 $p_0 \in S \setminus \partial S = \overset{\circ}{S}$ , since  $f|_{\partial S} = 0$ . Then  
 $Df(p_0) = 0$  by Theorem 3.3.11.