First Name:____________________ **Last Name:**____________________

OSU Number:____________________ **Signature:**__________________________

OKLAHOMA STATE UNIVERSITY
Department of Mathematics

MATH 4143/5043 (Advanced Calculus)

Instructor: Dr. Mathias Schulze

MIDTERM 1
September 21, 2009

Duration: 50 minutes

No aids allowed.

This examination paper consists of **3** pages and **3** questions. Please bring any discrepancy to the attention of an invigilator. The number in brackets at the start of each question is the number of points the question is worth.

Answer 3 questions.

To obtain credit, you must give arguments to support your answers.

For graders' use:

	Score
1 (12)	
2 (12)	
3 (15)	
Total (39)	

1. [12] Formulate definitions (describe the data involved and list the defining properties). Pick 4 out of 5 questions.

   (a) What is a *topological space*?

   (b) What is a *cluster point* of a subset of real n-space?

   (c) What is a *Cauchy sequence*?

   (d) What is a *connected* topological space?

   (e) What is a *continuous* function? What is a *uniformly continuous* function?

   Solution:

   (a) A set X together with a set $\mathcal{U}$ of subsets of X is called a topological space if
      (i) $X, \emptyset \in \mathcal{U}$,
      (ii) $\bigcap_{V \in \mathcal{V}} V \in \mathcal{U}$ for any finite subset $\mathcal{V} \subset \mathcal{U}$, and
      (iii) $\bigcup_{V \in \mathcal{V}} V \in \mathcal{U}$ for any subset $\mathcal{V} \subset \mathcal{U}$.

   (b) For a general topological space X, $p \in X$ is a cluster point of a subset $S \subset X$ if any open neighborhood U of p meets $S \setminus \{p\}$, i.e. $U \cap (S \setminus \{p\}) \neq \emptyset$. For $X = \mathbb{R}^n$, the open neighborhoods of p can be replaced by $B(p, \epsilon)$ for all $\epsilon > 0$.

   (c) A sequence $\{p_n\} \subset \mathbb{R}^n$ is called a Cauchy sequence if $\forall \epsilon > 0 \colon \exists N \in \mathbb{N} \colon \forall m, n \geq N \colon |p_n - p_m| < \epsilon$.

   (d) A topological space is connected if it is not the disjoint union of two non-empty open sets.

   (e) A function $f \colon X \to Y$ between topological spaces is continuous if the preimage under f of any open set in Y is open in X. A function $f \colon D \to \mathbb{R}$, $U \subset \mathbb{R}^n$, is uniformly continuous if $\forall \epsilon > 0 \colon \exists \delta > 0 \colon \forall p, q \in D \colon |p - q| < \delta \Rightarrow |f(p) - f(q)| < \epsilon$.

2. [12] Formulate theorems (list all hypotheses and formulate the statement). Pick 4 out of 5 questions.

   (a) What does the *Cauchy-Schwarz* inequality say?

   (b) What does the *Bolzano–Weiherstraß Theorem* say?

   (c) What does the *Heine–Borel Theorem* say?

   (d) What does the *Nested Interval Theorem* say?

   (e) What is the main result on *Cauchy sequences* in real n-space?

   Solution:

   (a) For any $p, q \in \mathbb{R}^n$, $|p \bullet q| \leq |p| \cdot |q|$.

   (b) Every bounded sequence in $\mathbb{R}^n$ has a convergent subsequence.

   (c) A subset of $\mathbb{R}^n$ is compact if it is closed and bounded.

   (d) The intersection of any nested sequence $I_1 \supset I_2 \supset I_3 \supset \cdots$ of closed bounded intervals $I_n \subset \mathbb{R}$ is non-empty, i.e. $\bigcap_{n \in \mathbb{N}} I_n \neq \emptyset$.

(e) Any Cauchy sequence in real n-space is convergent.

3. [15] Prove statements (give rigorous arguments based on the definitions). Pick 3 out of 4 statements.

   (a) The sequence $p_n = \left(\frac{n+1}{n}, (-1)^n\right)$ has exactly two limit points.

   (b) The function $M(x, y) = xy$ is continuous.

   (c) Any convergent sequence is a Cauchy sequence.

   (d) Any compact subset of real n-space is bounded.

Solution:

(a) For a given $\epsilon > 0$, we choose $N \in \mathbb{N}$ such that $N \geq 1/\epsilon$; then for all $n \geq N$ we have $|\frac{n+1}{n} - 1| = 1/n \leq 1/N \leq \epsilon$. Writing $p_n = (x_n, y_n)$, this shows that $x_n \to 1$, and hence, by Theorem 1.6.7, that $p_{2n} \to (1, 1)$ and $p_{2n+1} \to (1, -1)$. So $p_\pm = (1, \pm 1)$ are two limit points of $\{p_n\}$. For any other point $p \neq p_\pm$, set $\epsilon = \min\{2, |p - p_+|, |p - p_-|\}/2$; then the ϵ-balls centered at p, p_+, and p_- do not meet. As $B(p_+, \epsilon) \cup B(p_-, \epsilon)$ contain all but finitely many p_n, p can not be a limit point of $\{p_n\}$.

(b) For $\epsilon > 0$ and $p = (a, b)$, set $\delta = \min\{|a| + |b|, \frac{\epsilon}{2(|a|+|b|)}\}$ if $p \neq 0$, and $\delta = \sqrt{\epsilon}$ otherwise. Then, for any $q = (x, y)$ with $|p - q| < \delta$, we have $|y| - |b| \leq |b - y| < \delta$, and hence $|M(p) - M(q)| = |ab - xy| = |a(b - y) + y(a - y)| \leq |a||b - y| + |y||a - x| \leq (|a| + |y|)|p - q| \leq (|a| + |b| + \delta)|p - q| < \epsilon$.

(c) Let $p_n \to p$, and $\epsilon > 0$ arbitrary. Then, by convergence, there exists an $N \in \mathbb{N}$ such that, for all $n \geq N$, we have $|p_n - p| \leq \epsilon/2$. For $n, m \geq N$, we conclude that $|p_n - p_m| \leq |p_n - p| + |p - p_m| < \epsilon/2 + \epsilon/2 = \epsilon$.

(d) By compactness of S, the open covering $\bigcup_{n \in \mathbb{N}} B(0, n) \cap S$ of S has a finite subcovering $\bigcup_{n=1}^{N} B(0, n) \cap S$ and hence $S \subset B(0, N)$ and S is bounded.

End of examination
Total pages: 3
Total marks: 39